Well-posedness of the conductivity reconstruction from an interior current density in terms of Schauder theory

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Abstract. We show the well-posedness of the conductivity image reconstruction problem with a single set of interior electrical current data and boundary conductivity data. Isotropic conductivity is considered in two space dimensions. Uniqueness for similar conductivity reconstruction problems has been known for several cases. However, the existence and the stability are obtained in this paper for the first time. The main tool of the proof is the method of characteristics of a related curl equation.

1. Introduction

We consider an inverse problem of a second order linear elliptic equation involving interior information. Inverse problems make use of pieces of information that typically are boundary data but there are some cases which employ internal data.

Before going further we introduce a common setting on these problems in terms of electromagnetism. We are concerned with the following linear elliptic equation in a bounded domain $\Omega \subset \boldsymbol{R}^n$

$$-\frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial}{\partial x_j}u\big) = f, \quad \text{in } \Omega, \tag{1}$$

$$-\big(a_{ij}(x)\frac{\partial}{\partial x_j}u\big)\cdot \mathbf{n} = g, \quad \text{on } \partial\Omega. \tag{2}$$

We assume the ellipticity with bounded coefficients such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \xi \in \boldsymbol{R}^n \backslash \{0\},$$

where $0 < \lambda \leq \Lambda < \infty$. In order for (1) and (2) have a solution, one also imposes the condition

$$\int_\Omega f(x)\, dx = \int_{\partial\Omega} g\, dS.$$

When $a_{ij}(x) = a(x)\delta_{ij}$, $a(x)$ is a real-valued function, we say the coefficient is isotropic, when $a_{ij}(x)$ is a diagonal matrix we say the coefficient is orthotropic and for the remaining cases we say the coefficient is anisotropic. In all cases $a_{ij}(x)$ is symmetric and positive definite. In this paper, $\sigma(x)$ also denotes coefficient matrix for all cases and is called the conductivity. Solution u of (1) and (2) is called the voltage and $\mathbf{J} := -\sigma\nabla u$ is called the current density.

Among these kinds of problems we are mainly motivated from MREIT(Magnetic Resonance Electrical Impedance Tomography) problems. The MREIT is a thread of research stemmed from the classical EIT. In chronological order, the area is reviewed in [18],[21] and [19]. In MREIT, one runs an MRI machine and obtains internal magnetic field data. By collecting 3 spatial magnetic field data one can basically acquire internal current density data $\mathbf{J}$ using Ampere's Law

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B}.$$

Hence constructing internal conductivity can be sought using internal current density data. MREIT also focuses on inverse problems starting from magnetic field data especially its z-directional component B^z but this paper discusses only a problem with an internal current density. Other than MREIT, the aquifer identification problem is more classical and vast. See [8] for a mathematical introduction. The problem makes use of u data for reconstruction. Though u in that problem is not the voltage, let us call this problem a voltage problem in terms of electromagnetics. There are two different mathematical approaches for the voltage problem. One seeks a solution by optimizing an energy functional which is often regularized. When this process is numerically implemented, the algorithm typically uses an iterative structure. The other approach locally solves (1) for $a_{ij}(x)$ treating ∇u as coefficients. This approach uses a non-iterative procedure and a local numerical algorithm and is also more straightforward in analysis. Papers [2] and [17] made important progress in each approach for an isotropic problem. Recently, there also are active discussions on inverse problems from internal power density $P := \mathbf{J} \cdot \nabla u$. See [3, 15, 13, 14]

Also MREIT has been developed beyond these papers since 1992. Uniqueness for an isotropic problem with one piece of current density and a conditional stability on the given data are given in [7, 9, 16]. In addition, many numerical algorithms have been suggested, see review papers [18, 21, 19]. For an anisotropic problem, there is also a recent paper [4].

In this paper, we introduce a curl-based local approach and this is closely connected to an above voltage problem. We replace (1) and (2) with

$$\nabla \times (r\mathbf{J}) = 0, \quad \text{in } \Omega, \tag{3}$$
$$\nabla \cdot \mathbf{J} = f, \quad \text{in } \Omega, \tag{4}$$
$$\mathbf{J} \cdot \mathbf{n} = g, \quad \text{on } \partial\Omega, \tag{5}$$

as our governing equations. Here $r(x) := \sigma^{-1}(x)$ is the resistivity. For sufficiently regular class of functions, (1),(2) and (3),(4),(5) are equivalent. Both can be deduced from

$$\nabla \times \mathbf{E} = 0, \tag{6}$$
$$\nabla \cdot \mathbf{J} = f, \tag{7}$$
$$\sigma\mathbf{E} = \mathbf{J}, \quad \text{or} \quad \mathbf{E} = r\mathbf{J} \quad \text{(Ohm's Law)}, \tag{8}$$

by introducing a potential u so that $\mathbf{E} = -\nabla u$. We assumed Ω is a simply connected domain. Note that we do not need f to be zero in (4). If $f = 0$, at least in 2-dimensions, we will see in the next section that the problem is reduced to the equivalent voltage problem. In other words, many theorems on isotropic problems in MREIT actually can be deduced directly from results of [17]. In higher dimensions, they are different however. We will explain this in detail in the next section.

Curl-related formulas appear in the MREIT literature, see for example [10], but a complete use of (3) as an equivalent governing equation is first applied in this paper. This approach provides a more natural framework for current problems. It enables a local and linear analysis as was done in a voltage problem in [17] and thus we establish well-posedness. Furthermore it does not involve any problem regularization or optimization.

Although there are the results of [17] which imply many facts on an isotropic MREIT problem and theorems are already established in MREIT literature, we provide our own theorem because we have additional progress. First, typically an MREIT problem arises in a compact domain and requires us to reconstruct the conductivity in the exact whole domain. In [17], conductivity is sought on characteristic lines starting from the inflow boundary. We provided more detailed lemmas and a boundary control method to achieve the whole domain in our coverage. Second, our theorems are set exactly to answer an inverse question on a solvability of (1) and (2). The forward elliptic problems are studied conventionally according to a regularity class of f and a_{ij}. Typically for a source term and coefficients in Hölder continuous or more differentiable class, a solvability is sought in those $C^{k,\alpha}$ spaces which is called a Schauder theory. For a source term and coefficients such as p-th power integrable or weakly differentiable class, a solvability is sought in appropriate sobolev spaces $W^{k,p}$ and is called the Calderon-Zygmund theory. In this paper, we answer the exact mathematical question; an inverse Schauder solvability theory on σ. Thus we have existence, uniqueness, stability and regularity of solution $\sigma(x)$ for given data in the Schauder setting . We will show we lose one derivative of σ internally and even Hölder continuity on boundary with prepared counter examples.

Though we only considered a 2-dimensional problem for an isotropic conductivity in this paper, this is a first step toward an orthotropic and an anisotropic problem with a same curl-based linear approach which is the basis for current research by the authors.

2. Preliminaries and Problem Description

2.1. Voltage problems versus current problems

As was mentioned, there are inverse problems with u or $\mathbf{J}$ given. Before we define our problem, we first show how they are equivalent in 2-dimensions but different in higher dimensions under a condition $f = 0$ in (4). The equivalence in 2-dimensions is a well-known fact as shown in [1] for example . We include the discussion here for completeness.

For a divergence free $\mathbf{J}$, one introduces a potential. The potential is a scalar, called a stream function of $\mathbf{J}$,

$$\mathbf{J} = \nabla^{\perp}\psi, \quad \text{where } \nabla^{\perp} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix}$$

and $\nabla^{\perp}\psi$ is automatically divergence free. From (6) and (8), we have

$$\begin{aligned} \nabla \times \left(r\nabla^{\perp}\psi\right) &= 0, \quad \text{in } \Omega, \\ \nabla^{\perp}\psi \cdot \mathbf{n} &= g, \quad \text{on } \partial\Omega, \end{aligned}$$

but

$$0 = \nabla \times \left(r\nabla^{\perp}\psi\right) = (\partial_x\ \partial_y) \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{:=S} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \psi = \nabla \cdot \left(S\nabla\psi\right),$$

$$g = \nabla^{\perp}\psi \cdot \mathbf{n} = (\partial_y\psi\ -\partial_x\psi) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T(x) = \nabla\psi \cdot T(x), \tag{9}$$

where $T(x)$ is a counterclockwise unit tangent vector on $\partial\Omega$. If ℓ is an arc length parameter on $\partial\Omega$, (9) becomes a Dirichlet boundary condition,

$$\psi = G \quad \text{on } \partial\Omega, \quad \text{where } G := \psi(x(0)) + \int_0^{\ell} g(x(\ell'))d\ell'.$$

Therefore we have a voltage problem with Dirichlet boundary condition,

$$\nabla \cdot (S\nabla\psi) = 0, \quad \text{in } \Omega, \tag{10}$$

$$\psi = G, \quad \text{on } \partial\Omega. \tag{11}$$

Once S is known, r is known.

A potential for a divergence free vector field in n-dimension is an $\binom{n}{2}$-dimensional quantity. This is the dimension of a space of 2-forms. For 3-dimensions, we know it is a vector potential,

$$\mathbf{J} = \nabla \times \mathbf{B}.$$

Then (8) becomes

$$\nabla \times \mathbf{B} = -\sigma\nabla u. \tag{12}$$

We take a divergence on (12) and obtain a single and linear equation for σ. This applies in any dimension. There are no obstacles in applying what one can do in 2-dimensions and indeed, [17] dealt with arbitrary dimensions.

However for a current problem, (12) gives us 3 equations for unknowns u and σ. Thus for a real-valued σ, this is an over-determined problem. If we restrict ourselves to know only one component of $\mathbf{J}$ or $\mathbf{B}$, we will have a non-linear problem since the unknown σ and the unknown components of $\mathbf{J}$ or $\mathbf{B}$ will be multiplied together. Taking curl to have $\nabla \times (r\mathbf{J}) = 0$ does not help. Hence the properties of the voltage problems and the current problems are different in dimension $n \geq 3$.

In all cases, if $\mathbf{J}$ is given, it is straightforward to consider (3) as a governing equation. (1) does not give any information other than

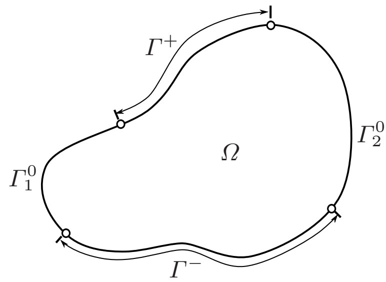


Fig. 1.

divergence of $\mathbf{J}$ which we have no control. Considering (1) as a governing equation is a main reason making the inverse problem non-linear. One seeks a σ so that with the σ one can solve (1) and (2) for u and make $-\sigma\nabla u$ match a given $\mathbf{J}$. Note that this process has a pde solving step as its non-linearity. Due to this non-linearity, an algorithm implementing this matching procedure might need an iteration. However one can see the pde (3) for r is just a linear equation. The authors and Kwon et al. developed a non-iterative algorithm based on this approach in [11] and [12].

2.2. Problem description

The vector field $\mathbf{J}$ in MREIT is assumed to be divergence free in general but we did not exclude the general case to prove the unique existence and other results. $\mathbf{J}$ may contain a certain noise and $\nabla\cdot\mathbf{J}$ might not be zero but still we will have a compatible solution. To be clear, we will now write $\mathbf{F}$ instead of $\mathbf{J}$ from now on. However, $\mathbf{F}$ cannot be an arbitrary vector field. We will first introduce a notion of an admissibility which is a sufficient condition for a unique solvability for conductivity.

Definition 1. *Consider a two dimensional vector field* $\mathbf{F}=(f^1,f^2)\in C^{1,\alpha}(\overline{\Omega})$ *for* $0<\alpha<1$. *Denote* $\Gamma^+ := \{\mathbf{x}\in\partial\Omega \,|\, \mathbf{F}^\perp\cdot\mathbf{n}(\mathbf{x})>0\}$, $\Gamma^- := \{\mathbf{x}\in\partial\Omega \,|\, \mathbf{F}^\perp\cdot\mathbf{n}(\mathbf{x})<0\}$, $\Gamma^0 := \{\mathbf{x}\in\partial\Omega \,|\, \mathbf{F}^\perp\cdot\mathbf{n}(\mathbf{x})=0\}$ *and* $\Omega' := \overline{\Omega}\setminus\Gamma^0$, *where* $\mathbf{F}^\perp := (-f^2,f^1)$. *The vector field* $\mathbf{F}$ *is called admissible in this paper if* $\mathbf{F}\neq 0$ *in* $\overline{\Omega}$ *and* $\Gamma^\pm$ *are connected.*

If the conductivity σ is $C^{1,\alpha}(\overline{\Omega})$ and the source f is $C^{0,\alpha}(\overline{\Omega})$, then it is well known that the voltage u is $C^{2,\alpha}(\overline{\Omega})$ and the current $\mathbf{F}$ is $C^{1,\alpha}(\overline{\Omega})$ (see Theorem 6.19 [6]). The regularity of $\mathbf{F}$ in the definition is to be consistent with classical Schauder theory. The part of boundary, Γ^0, consists of two components, $\Gamma^0=\Gamma_1^0\cup\Gamma_2^0$ and each of them can be a single point. However, in general, it can be more than a single point and we include such a case in our analysis (see Figure 1). The well-posedness of the conductivity reconstruction is stated in the following theorem using the notion of Definition 1:

Theorem 1. *Let Ω be a bounded simply connected open set with $C^{2,\alpha}$ boundary. Suppose that an admissible vector field $\mathbf{F} \in C^{1,\alpha}(\overline{\Omega})$ and a boundary resistivity $r_0 \in C^{0,\alpha}(\overline{\Gamma^-})$ are given. Then,*

(i) There exists a unique $r \in C^{0,\alpha}_{loc}(\Omega') \cap C^0(\overline{\Omega})$ that satisfies

$$\nabla \times (r\mathbf{F}) = 0 \quad in \quad \Omega, \tag{13}$$

$$r = r_0 \quad on \quad \overline{\Gamma^-} \subset \partial\Omega. \tag{14}$$

(ii) Let $\tilde{r}$ be the solution for an admissible vector field $\tilde{\mathbf{F}}$ with $\tilde{\Gamma}^- = \Gamma^-$ and a $\tilde{r}_0 \in C^{0,\alpha}(\overline{\Gamma^-})$. Then, for any compact set $K \subset \Omega'$,

$$\|r - \tilde{r}\|_{L^\infty(K)} \le C\left(\|r_0 - \tilde{r}_0\|_{L^\infty(\Gamma^-)} + \|\mathbf{F} - \tilde{\mathbf{F}}\|^\alpha_{C^1(\overline{\Omega})}\right), \tag{15}$$

where $C = C\big(K, \|F\|_{C^{1,\alpha}(\overline{\Omega})}, \|\tilde{F}\|_{C^{1,\alpha}(\overline{\Omega})}, \|r_0\|_{C^{0,\alpha}(\overline{\Omega})}, \|\tilde{r}_0\|_{C^{0,\alpha}(\overline{\Omega})}\big)$.

Uniqueness has been shown for several reconstruction methods. However as far as authors know, the existence and the stability are obtained for the first time. One can find conditional stability in [16] for a equipotential line method, which contains certain stability structure obtained in the theorem. The proof of Theorem 1 is given in Section 3. The main technique of its proof is the method of characteristics because (3) will be a hyperbolic equation.

3. Existence, uniqueness, stability and regularity of the solution

3.1. Preliminary lemmas

The construction of the resistivity r is based on an analysis of integral curves of the vector field $\mathbf{F}^\perp$. For any given $\mathbf{x}_0 \in \overline{\Omega}$, the integral curve is a solution of the ordinary differential equation (or ODE for brevity)

$$\frac{d}{dt}\mathbf{x}(t) = F^\perp(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad -\infty < t < \infty. \tag{16}$$

In the following lemma we quickly summarize elementary properties of integral curves of a smooth vector field such that $\mathbf{F} \neq 0$ in $\overline{\Omega}$.

Lemma 1. *If $\mathbf{F} \in C^1(\overline{\Omega})$ and $\mathbf{F} \neq 0$ in $\overline{\Omega}$, then*

(i) Integral curves of $\mathbf{F}^\perp$ do not touch other ones nor themselves.
(ii) The length of an integral curves of $\mathbf{F}^\perp$ is uniformly bounded.

(iii) Both ends of an integral curve of $\mathbf{F}^\perp$ are extendable to the boundary.

Proof. Let $\mathbf{x}_0$ be a tangential or intersection point of different integral curves. This implies that there exist two solutions of (16) locally at $\mathbf{x}_0$. However, F is assumed to be smooth and hence it contradicts the existence of unique solutions to such ODEs and hence we obtained the first assertion.

The second assertion depends on the assumption $\mathbf{F}^\perp \neq 0$ in $\overline{\Omega}$. Suppose that there is an integral curve $\mathbf{x}(t)$ which is infinitely long. Then, since the domain Ω is bounded, there exist nonempty limit set $\omega(\mathbf{x})$. Since there is no critical point, Poincare-Bendixon implies that $\omega(\mathbf{x})$ is a periodic orbit. This implies that there exists a critical point in the interior of the orbit, which contradicts to the assumption $\mathbf{F}^\perp \neq 0$ in $\overline{\Omega}$. Therefore, all the integral curves are finitely long. Since $\overline{\Omega}$ is compact, they are uniformly bounded.

Since $\overline{\Omega}$ is compact and $|\mathbf{F}^\perp| > 0$ on $\overline{\Omega}$, there exists a lower bound $a > 0$ such that

$$|\mathbf{F}^\perp| \geq a > 0.$$

Suppose that an integral curve $\mathbf{x}(t)$ converges to an interior point $\mathbf{y} \in \Omega$ as $t \to \infty$. One can easily see that this is not possible since the speed of the curve is uniformly bounded from below, i.e., $|\mathbf{x}'(t)| = |\mathbf{F}^\perp(\mathbf{x}(t)| \geq a$, the curve cannot stay in a small neighborhood of $\mathbf{y}$ forever. Therefore, the integral curve $\mathbf{x}$ should connect two boundary points of $\partial\Omega$.

We will see in the following lemma that, if the vector field is admissible in the sense of Definition 1, integral curves should connect the boundaries Γ^- and Γ^+.

Lemma 2. *If $\mathbf{F}$ is admissible, then the integral curve of $\mathbf{F}^\perp$ that passes through an interior point $\mathbf{x}_0 \in \Omega$ starts from Γ^- and ends at Γ^+. Furthermore, there exists $T > 0$, a uniform upper bound of the domain size of integral curves.*

Proof. Since the vector field $\mathbf{F}$ is assumed to be admissible, the boundary $\partial\Omega$ is divided into four parts, $\partial\Omega = \Gamma^- \cup \Gamma_1^0 \cup \Gamma^+ \cup \Gamma_2^0$, where $\mathbf{F}^\perp \cdot \mathbf{n}(\mathbf{x}) = 0$ on Γ_i^0 (see Figure 1).

Note that each Γ_i^0 is a single point or is an integral curve of $\mathbf{F}^\perp$ by the definition of admissibility. From Lemma 1, we know that the integral curve that passes through an interior point $\mathbf{x}_0$ is unique and has two end points on $\partial\Omega$, i.e., there exist $t_- < 0 < t_+$ such that

$$\mathbf{x}'(t) = \mathbf{F}^\perp(\mathbf{x}(t)) \quad \text{for} \quad t_- < t < t_+, \qquad \mathbf{x}(t_-), \mathbf{x}(t_+) \in \partial\Omega.$$

Since $\mathbf{x}'(t_-) \cdot \mathbf{n} \leq 0$ and $\mathbf{x}'(t_+) \cdot \mathbf{n} \geq 0$, we have $\mathbf{x}(t_-) \in \Gamma^- \cup \Gamma_1^0 \cup \Gamma_2^0$ and $\mathbf{x}(t_+) \in \Gamma^+ \cup \Gamma_1^0 \cup \Gamma_2^0$. If any of Γ_i^0's is not a single point, then

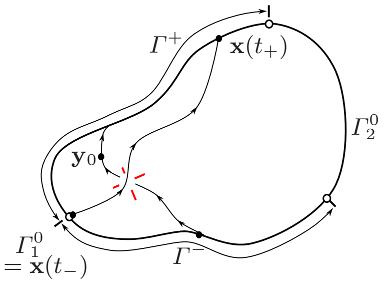


Fig. 2. An illustration for the proof of Lemma 2.

they are integral curves by definition. Since two integral curves do not intersect with each other for admissible vector fields, $\mathbf{x}(t_-) \in \Gamma^-$ and $\mathbf{x}(t_+) \in \Gamma^+$.

Suppose that Γ_1^0 is a single point and $\mathbf{x}(t_-) \in \Gamma_1^0$ as in Figure 2. (Notice that it is enough to show that this is not possible. Then, it implies $\mathbf{x}(t_-) \notin \Gamma_2^0$ by the same arguments and hence $\mathbf{x}(t_-) \in \Gamma^-$. The same arguments also give $\mathbf{x}(t_+) \in \Gamma^+$ and the first part of proof is complete.) Then, $\mathbf{x}(t_+) \in \Gamma^+ \cup \Gamma_2^0$. If Γ_2^0 is not an single point, then, by the same reason, $\mathbf{x}(t_+) \in \Gamma^+$. In any case, $\mathbf{x}(t_+) \in \overline{\Gamma^+} \setminus \Gamma_1^0$. Let $\mathbf{y}_0$ be an interior point of a region surrounded by the integral curve $\mathbf{x}(t)$, $t_- < t < t_+$, and Γ^+. The integral curve $\mathbf{y}(t)$ that passes through the point $\mathbf{y}_0$ should start from $\overline{\Gamma^-}$. Therefore, the integral curve $\mathbf{y}(t)$ should intersect the integral curve $\mathbf{x}(t)$, which contradicts to Lemma 1. Therefore $\mathbf{x}(t_-) \notin \Gamma_1^0$ even if Γ_1^0 is a single point. Similarly $\mathbf{x}(t_-) \notin \Gamma_2^0$ and hence $\mathbf{x}(t_-) \in \Gamma^-$. Similarly $\mathbf{x}(t_+) \in \Gamma^+$.

Since the $|\mathbf{F}^\perp|$ is uniformly bounded below away from zero and the length of an integral curve is uniformly bounded, there exists $T > 0$ such that the domain size of any integral curve is less than T, i.e.,

$$t_+ - t_- \le T,$$

which completes the proof

We will always consider an admissible vector field in Definition 1. The boundary Γ^- is assumed to be smooth, where the curve $\gamma : [0, L] \to \overline{\Gamma^-}$ is $C^{2,\alpha}$. We will write the whole set of integral curves appeared earlier into a mapping of two parameters, such that

$$\frac{\partial}{\partial t}\mathbf{x}(s,t) = \mathbf{F}^\perp(\mathbf{x}(s,t)), \quad \mathbf{x}(s,0) = \gamma(s), \quad 0 \le s \le L. \tag{17}$$

The domain of the mapping $\mathbf{x}$ is a the closure of a bounded open subset $E \subset [0, L] \times [0, T]$. In the following lemma we will see that the mapping $\mathbf{x}$ gives a new coordinate system of the problem.

Lemma 3. *Let* $\mathbf{F}$ *be admissible.* (*i*) *The mapping* $\mathbf{x} : \overline{E} \to \overline{\Omega}$ *defined by the relation* (17) *is a homeomorphism.* (*ii*) *Furthermore, its restriction* $\mathbf{x} : E' \to \Omega'$ *is a* C^1*-diffeomorphism, where* $E' = \mathbf{x}^{-1}(\Omega')$.

Proof. Lemma 1 implies that the mapping $\mathbf{x} : \overline{E} \to \overline{\Omega}$ is one-to-one. If not, $\mathbf{x}(s,t) = \mathbf{x}(s',t')$ for some $(s,t) \neq (s',t')$. This implies that an integral curve is touched by another one, if $s \neq s'$, or by itself, if $s = s'$. Then, it contradicts Lemma 1(i). Lemma 2 implies that $\Omega' \subset \mathbf{x}(\overline{E})$. To show $\mathbf{x}$ is a surjection, it is enough to show that Γ_1^0 and Γ_2^0 are actually integral curves $\mathbf{x}(0,\cdot)$ and $\mathbf{x}(L,\cdot)$. If each of them is a single point, there is nothing to prove. If not, we already know from Definition 1 that they are.

Now we show that $\mathbf{x}$ is continuous. In fact we will show that it is Lipschitz. Consider

$$|\mathbf{x}(s,t) - \mathbf{x}(s',t')| \leq |\mathbf{x}(s,t) - \mathbf{x}(s,t')| + |\mathbf{x}(s,t') - \mathbf{x}(s',t')|.$$

The first term is estimated by

$$|\mathbf{x}(s,t) - \mathbf{x}(s,t')| \leq \|\partial_t \mathbf{x}\|_\infty |t - t'| \leq \|F\|_\infty \; |t - t'|.$$

To estimate the second term, we first consider

$$\begin{aligned} \frac{\partial}{\partial t} |\mathbf{x}(s,t) - \mathbf{x}(s',t)| \Big|_{t=t'} &= |\mathbf{F}^\perp(\mathbf{x}(s,t')) - \mathbf{F}^\perp(\mathbf{x}(s',t'))| \\ &\leq \|D\mathbf{F}\|_\infty \; |\mathbf{x}(s,t') - \mathbf{x}(s',t')|. \end{aligned}$$

Therefore, Gronwall's inequality gives, for $C = e^{T\|D\mathbf{F}\|_\infty}$,

$$\begin{aligned} |\mathbf{x}(s,t) - \mathbf{x}(s',t)| &\leq C|\mathbf{x}(s,0) - \mathbf{x}(s',0)| \\ &= C|\gamma(s) - \gamma(s')| \\ &\leq C \; \|\gamma'\|_\infty \; |s - s'|. \end{aligned}$$

Combining these estimates, we have, for some constant $C > 0$,

$$|\mathbf{x}(s,t) - \mathbf{x}(s',t')| \leq C|(s,t) - (s',t')|. \tag{18}$$

Furthermore, since $\mathbf{x}$ is a continuous bijection from a compact set to a compact set, its inverse is also continuous and hence $\mathbf{x}$ is homeomorphism.

Differentiability of the mapping $\mathbf{x}(s,t)$ in s and t variables in E' is well-known from ODE theory (see Theorem 7.5 in [5] on pp.30 and remark on pp.23). We now show the differentiability of $\mathbf{x}^{-1}$ on Ω'. To do that it is enough to show that the determinant of the Jacobian matrix $D\mathbf{x}(s,t)$ is not zero on E'. Differentiation of (17) with respect to t and s gives

$$\begin{aligned} \partial_t \partial_s \mathbf{x}(s,t) &= D\mathbf{F}^\perp(\mathbf{x}(s,t)) \partial_s \mathbf{x}(s,t), \\ \partial_t \partial_t \mathbf{x}(s,t) &= D\mathbf{F}^\perp(\mathbf{x}(s,t)) \partial_t \mathbf{x}(s,t), \end{aligned}$$

which can be written in terms of Jacobian matrix as

$$\partial_t D\mathbf{x}(s,t) = D\mathbf{F}^{\perp}(\mathbf{x}(s,t))D\mathbf{x}(s,t).$$

Therefore, the determinant of the Jacobian matrix is given by

$$\big|D\mathbf{x}(s,t)\big| = \big|D\mathbf{x}(s,0)\big| \exp\Big(\int_0^t tr\big(D\mathbf{F}^{\perp}(\mathbf{x}(s,\tau))\big)\, d\tau\Big),$$

(see Theorem 7.3 in [5], pp.28). On the other hand,

$$\big|D\mathbf{x}(s,0)\big| = \big|[\partial_s\mathbf{x}(s,0), \partial_t\mathbf{x}(s,0)]\big| = \gamma'(s) \times \mathbf{F}^{\perp}(\gamma(s)).$$

Since $\mathbf{F}^{\perp}(\gamma(s)) \cdot \mathbf{n} < 0$ for $\gamma(s) \in \Gamma^-$ and $\gamma'(s) \cdot \mathbf{n} = 0$, $\mathbf{F}^{\perp}(\gamma(s))$ and $\gamma'(s)$ are not parallel to each other. Therefore, $\big|D\mathbf{x}(s,0)\big| \neq 0$ and hence $\big|D\mathbf{x}(s,t)\big| \neq 0$ for all $t > 0$ for all $(s,t) \in E'$.

3.2. Proof of Theorem 1

In this section we will show the well-posedness of the inverse problem of finding r that satisfies (13-14) for given $\mathbf{F}$ and r_0.

Proof (Proof of Theorem 1). Let $\mathbf{x} : \overline{E} \to \overline{\Omega}$ be the homeomorphism in Lemma 3. Then for any $\mathbf{x}_0 \in \overline{\Omega}$ there exist $0 \leq s_0 \leq L$ and $0 \leq t_0 \leq T$ such that $\mathbf{x}_0 = \mathbf{x}(s_0, t_0)$, i.e.,

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbf{x}(s_0,t) = \mathbf{F}^{\perp}(\mathbf{x}(s_0,t)), \quad 0 \leq t \leq T, \\ &\mathbf{x}(s_0,0) \in \Gamma^-, \quad \mathbf{x}(s_0,t_0) = \mathbf{x}_0. \end{aligned}$$

If r is smooth, then we have following equivalence relations.

$$\begin{aligned} \nabla \times (r\mathbf{F}) = 0 &\Longleftrightarrow (rf^2)_x - (rf^1)_y = -\mathbf{F}^{\perp} \cdot \nabla r + (f^2_x - f^1_y)r = 0 \\ &\Longleftrightarrow -\frac{d}{dt}r(\mathbf{x}(s,t)) + (\nabla \times \mathbf{F})r = 0 \qquad (19) \\ &\Longleftrightarrow \frac{\frac{d}{dt}r(\mathbf{x}(s,t))}{r(\mathbf{x}(s,t))} = \nabla \times \mathbf{F}(\mathbf{x}(s,t)). \end{aligned}$$

Therefore, the resistivity r at $\mathbf{x}_0 = \mathbf{x}(s_0, t_0)$ should be given by

$$r(\mathbf{x}_0) = r(\mathbf{x}(s_0,0)) \exp\Big(\int_0^{t_0} \nabla \times \mathbf{F}(\mathbf{x}(s_0,\tau))d\tau\Big). \qquad (20)$$

Since the relations are equivalent this is the unique weak solution.

In the following, we will first show that $\big(r \circ \mathbf{x}\big)(s,t)$ has the regularity of $C^{0,\alpha}(\overline{E})$. Then the Lemma 3 will imply $r(x,y) \in C^{0,\alpha}(\Omega') \cap$

$C^0(\overline{\Omega})$ as in statement of Theorem 1 because $\mathbf{x}^{-1}(x,y)$ is continuous in $\overline{\Omega}$ and differentiable in Ω'.

Let $\mathbf{x}_i \in \overline{\Omega}$ and $\mathbf{x}(s_i,t_i) = \mathbf{x}_i$ for $i = 1,2$. First $\big(r \circ \mathbf{x}\big)(s,t)$ is differentiable with respect to the variable t by (19). Also, $\mathbf{x}(s,t)$ is Lipschitz and $r_0(s)$ is Hölder continuous on the boundary Γ^- with respect to the variable s, hence their composition map $s \to r(\mathbf{x}(s,0))$ is also Hölder continuous with respect to s. Similarly, the map $s \to e^{\left(\int_0^{t_0} \nabla\times\mathbf{F}(\mathbf{x}(s,\tau))d\tau\right)}$ is Hölder continuous and hence r in (20) is Hölder continuous with respect to s because it is given by the product of those two maps. Therefore $r\circ\mathbf{x} \in C^{0,\alpha}(\overline{E})$ and hence $r = r\circ\mathbf{x}\circ\mathbf{x}^{-1} \in C^{0,\alpha}(\Omega') \cap C^0(\overline{\Omega})$.

Now we show stability, the second part of Theorem 1. Let $\tilde{\mathbf{F}}$ be another admissible vector field and $\tilde{\mathbf{x}} : \tilde{E} \to \Omega$ and $\tilde{r} : \overline{\Omega} \to \boldsymbol{R}$ be the corresponding diffeomorphism and resistivity, respectively. We assume $\Gamma^- = \tilde{\Gamma}^-$ and $\mathbf{x}(s,0) = \tilde{\mathbf{x}}(s,0)$ for $s \in [0,L]$ for a simpler representation. We will show (15), for a fixed compact subset $K \subset \Omega'$. Let $\mathbf{x}_0 \in K$ be fixed and $\mathbf{x}_0 = \mathbf{x}(s_0,t_0) = \tilde{\mathbf{x}}(\tilde{s}_0,\tilde{t}_0)$ where $\Delta t := \tilde{t}_0 - t_0 \geq 0$ (see Figure 3 for an illustration). Consider, for $t \in [0,t_0]$,

$$
\begin{aligned}
|\partial_t \mathbf{x}(s_0,&t_0-t) - \partial_t\tilde{\mathbf{x}}(\tilde{s}_0,\tilde{t}_0-t)| \\
&= |-\mathbf{F}^{\perp}(\mathbf{x}(s_0,t_0-t)) + \tilde{\mathbf{F}}^{\perp}(\tilde{\mathbf{x}}(\tilde{s}_0,\tilde{t}_0-t))| \\
&\leq |-\mathbf{F}^{\perp}(\mathbf{x}(s_0,t_0-t)) + \tilde{\mathbf{F}}^{\perp}(\mathbf{x}(s_0,t_0-t))| \\
&+ |-\tilde{\mathbf{F}}^{\perp}(\mathbf{x}(s_0,t_0-t)) + \tilde{\mathbf{F}}^{\perp}(\tilde{\mathbf{x}}(\tilde{s}_0,\tilde{t}_0-t))| \\
&\leq \|\mathbf{F}-\tilde{\mathbf{F}}\|_\infty + \|D\tilde{\mathbf{F}}\|_\infty |\mathbf{x}(s_0,t_0-t) - \tilde{\mathbf{x}}(\tilde{s}_0,\tilde{t}_0-t)|.
\end{aligned}
$$

Therefore, Gronwall's inequality gives, for $0 < t < t_0$,

$$
|\mathbf{x}(s_0,t_0-t) - \tilde{\mathbf{x}}(\tilde{s}_0,\tilde{t}_0-t)| \leq C\|\mathbf{F}-\tilde{\mathbf{F}}\|_\infty, \tag{21}
$$

where $C = t_0 e^{t_0\|D\tilde{F}\|_\infty}$.

Denote $\mathbf{x}_1 := \mathbf{x}(s_0,0) \in \Gamma^-$, $\tilde{\mathbf{x}}_1 := \tilde{\mathbf{x}}(\tilde{s}_0,\Delta t) \in \Omega$, $h(t) := \nabla \times \mathbf{F}(\mathbf{x}(s_0,t))$ and $\tilde{h}(t) := \nabla \times \tilde{\mathbf{F}}(\tilde{\mathbf{x}}(\tilde{s}_0,t+\Delta t))$. Then, from (20),

$$
r(\mathbf{x}_0) = r(\mathbf{x}_1)e^{\int_0^{t_0} h(t)\,dt}, \quad \tilde{r}(\mathbf{x}_0) = \tilde{r}(\tilde{\mathbf{x}}_1)e^{\int_0^{t_0} \tilde{h}(t+\Delta t)dt}.
$$

Hence,

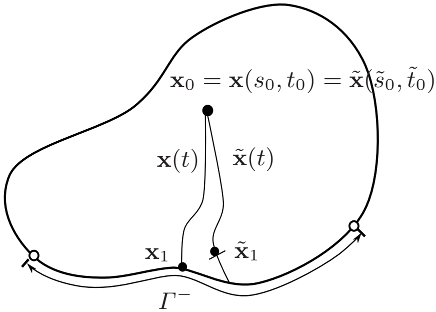


Fig. 3. This figure is used as an illustration in the stability proof.

$$\begin{aligned}
&|r(\mathbf{x}_0) - \tilde{r}(\mathbf{x}_0)| \\
&\leq \left| r(\mathbf{x}_1) e^{\int_0^{t_0} h\, dt} - r(\mathbf{x}_1) e^{\int_0^{t_0} \tilde{h}\, dt} \right| + \left| r(\mathbf{x}_1) e^{\int_0^{t_0} \tilde{h}\, dt} - \tilde{r}(\tilde{\mathbf{x}}_1) e^{\int_0^{t_0} \tilde{h}\, dt} \right| \\
&\leq \|r_0\|_{C^0(\Gamma^-)} \left| e^{\int_0^{t_0} h\, dt} - e^{\int_0^{t_0} \tilde{h}\, dt} \right| + |r(\mathbf{x}_1) - \tilde{r}(\tilde{\mathbf{x}}_1)| \left| e^{\int_0^{t_0} \tilde{h}\, dt} \right| \\
&\leq \|r_0\|_{C^0(\Gamma^-)} \max\left(e^{\int_0^{t_0} h\, dt}, e^{\int_0^{t_0} \tilde{h}\, dt} \right) \left| \int_0^{t_0} h - \tilde{h}\, dt \right| \\
&\qquad\qquad\qquad\qquad + |r(\mathbf{x}_1) - \tilde{r}(\tilde{\mathbf{x}}_1)| \left| e^{\int_0^{t_0} \tilde{h}\, dt} \right| \\
&\leq C\Big(\|h - \tilde{h}\|_\infty + |r(\mathbf{x}_1) - \tilde{r}(\tilde{\mathbf{x}}_1)| \Big),
\end{aligned}$$

where C depends on the same quantities that the coefficient in (15) does. Now we estimate the two terms separately.

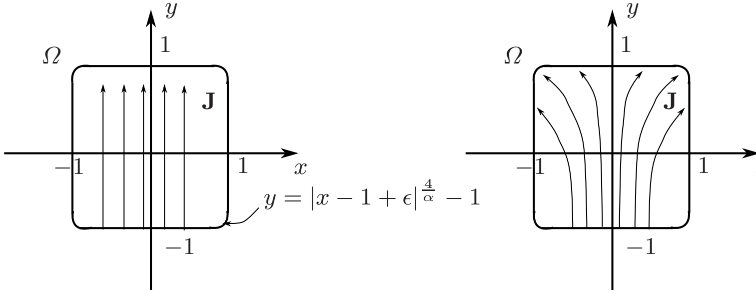
First, we have

$$\begin{aligned}
|r(\mathbf{x}_1) - \tilde{r}(\tilde{\mathbf{x}}_1)| &\leq |r(\mathbf{x}_1) - \tilde{r}(\mathbf{x}_1)| + |\tilde{r}(\mathbf{x}_1) - \tilde{r}(\tilde{\mathbf{x}}_1)|. \\
&\leq \|r_0 - \tilde{r}_0\|_\infty + [\tilde{r}]_{C^{0,\alpha}(K')} |\mathbf{x}(s_0, 0) - \tilde{\mathbf{x}}(\tilde{s}_0, \Delta t)|^\alpha \\
&\leq \|r_0 - \tilde{r}_0\|_\infty + [\tilde{r}]_{C^{0,\alpha}(K')} (C_1 \|\mathbf{F} - \tilde{\mathbf{F}}\|_\infty)^\alpha,
\end{aligned}$$

where, in the second inequality, K' is a compact set containing $\mathbf{x}_1$ and $\tilde{\mathbf{x}}_1$ and hence $[\tilde{r}]_{C^{0,\alpha}(K')}$ is bounded. Also we used the fact that $\mathbf{x}_1 = \mathbf{x}(s_0, 0) = \tilde{\mathbf{x}}(s_0, 0) \in \Gamma^-$. Equation (21) is used in the last inequality.

The other term is estimated by

$$\begin{aligned}
|h(t) - \tilde{h}(t)| &\leq |\nabla \times \mathbf{F}(\mathbf{x}(s_0, t)) - \nabla \times \tilde{\mathbf{F}}(\tilde{\mathbf{x}}(\tilde{s}_0, t + \Delta t))| \\
&\leq |\nabla \times \mathbf{F}(\mathbf{x}(s_0, t)) - \nabla \times \tilde{\mathbf{F}}(\mathbf{x}(s_0, t))| \\
&\quad + |\nabla \times \tilde{\mathbf{F}}(\mathbf{x}(s_0, t)) - \nabla \times \tilde{\mathbf{F}}(\tilde{\mathbf{x}}(\tilde{s}_0, t + \Delta t))| \\
&\leq \|\mathbf{F} - \tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})} + [D\tilde{\mathbf{F}}]_{C^{0,\alpha}(\overline{\Omega})} |\mathbf{x}(s_0, t) - \tilde{\mathbf{x}}(\tilde{s}_0, t + \Delta t)|^\alpha \\
&\leq \|\mathbf{F} - \tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})} + [D\tilde{\mathbf{F}}]_{C^{0,\alpha}(\overline{\Omega})} (C_1 \|\mathbf{F} - \tilde{\mathbf{F}}\|_\infty)^\alpha,
\end{aligned}$$



(a) domain of first example (b) domain of second example

Fig. 4. These illustrations are used to show the optimality in regularity theory.

where estimate (21) is used again. Therefore we have

$$
\begin{aligned}
&|r(\mathbf{x}_0) - \tilde{r}(\tilde{\mathbf{x}}_0)| \\
&\le C_4\Big((C_1^\alpha [D\tilde{\mathbf{F}}]_{C^{0,\alpha}(\overline{\Omega})} + 1) + (C_1^\alpha [\tilde{r}]_{C^{0,\alpha}(K')} + 1)\Big) \\
&\Big(\|r_0 - \tilde{r}_0\|_\infty + \|\mathbf{F} - \tilde{\mathbf{F}}\|_\infty^\alpha + \|\mathbf{F} - \tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})}\Big) \\
&\le C\Big(\|r_0 - \tilde{r}_0\|_\infty + \|\mathbf{F} - \tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})}^\alpha\Big).
\end{aligned}
\tag{22}
$$

$[\tilde{r}]_{C^{0,\alpha}(K')}$ depends on $\tilde{\mathbf{F}}$, $\tilde{r}_0$ and K. So $C = C(\mathbf{F}, \tilde{\mathbf{F}}, r_0, \tilde{r}_0, K)$. Here we assumed $\|\mathbf{F} - \tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})} < 1$ so that $\|\mathbf{F} - \tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})} < \|\mathbf{F} - \tilde{\mathbf{F}}\|_{C^1(\overline{\Omega})}^\alpha$. Note that $[\tilde{r}]_{C^{0,\alpha}(K')}$ in (22) is not bounded as $\mathbf{x}_0$ approaches to $\mathbf{x}(0,t)$ or $\mathbf{x}(L,t)$ in Γ^0 thus the estimate (22) holds only for $K \subset\subset \Omega'$.

3.3. The optimal regularity of r

We obtained in Theorem 1 that $r \in C^{0,\alpha}_{loc}(\Omega') \cap C^0(\overline{\Omega})$. The same regularity is true for σ if r is away from 0. If $r_0 > 0$, the exponential term in (20) does not alter the sign, hence $r > 0$ in $\overline{\Omega}$ and r has minimum in the compact domain thus is away from 0. Thus we will freely use r or σ for discussions.

We will show that the regularity cannot be improved. For a forward elliptic problem, $\sigma \in C^{1,\alpha}(\overline{\Omega})$ guarantees $\mathbf{J} \in C^{1,\alpha}(\overline{\Omega})$ and $\sigma \in C^{1,\alpha}(\Omega)$ guarantees $\mathbf{J} \in C^{1,\alpha}(\Omega)$ without a boundary estimate. Our theorem says that the above sufficient conditions are not necessary conditions. We could lose one derivative interior and even Hölder continuity on boundary because sometimes a less regular conductivity gives a regular $\mathbf{J}$. This is seen in the following examples.

First, we will show that we lose Hölder continuity of σ on boundary, i.e., $r \notin C^{0,\alpha}(\overline{\Omega})$ in general. Consider an example,

$$r(x,y) := f(y) > 0, \quad u(x,y) := -\int_0^y f(y')\,dy'.$$

This is an example of one dimensional electrical current in two space dimensions and one can easily check that the electrical current is

$$\mathbf{J} = -\sigma \nabla u = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which is real analytic function. Consider a domain given as in Figure 4(a), where a part of its boundary is along the line $y = -1$. According to Definition 1, this part of boundary belongs to Γ^0. Set $f(y) = 1 + |y+1|^{\frac{\alpha}{2}}$. This certainly does not belongs to $C^{0,\alpha}(\overline{\Omega})$ but belongs merely to $C^{0,\alpha}_{loc}(\Omega')$. Note that $r_0 \in C^{0,\alpha}(\Gamma^-)$, provided the curve at the corner of boundary is set as in Figure 4(a). One might even consider a discontinuous f, but this case is excluded by an assumption of Theorem 1 that $r_0 \in C^{0,\alpha}(\Gamma^-)$ since we are considering a classical Schauder theory and here $r \notin C^{0,\alpha}_{loc}(\Omega')$.

In the next example we will see we could lose one derivative inside of Ω, i.e., $r \notin C^{0,\beta}_{loc}(\Omega)$ for any $\beta > \alpha$. Let the domain be given as in Figure 4 (b) and let

$$r(x,y) := \frac{1}{\left(1 + |x|^{\frac{1}{2}}(1+y)\right)^3} > 0, \quad u(x,y) := \frac{-x}{\left(1 + |x|^{\frac{1}{2}}(1+y)\right)^2}.$$

Then, the electrical current is

$$\mathbf{J} = -\sigma \nabla u = \begin{pmatrix} 1 \\ -2x|x|^{\frac{1}{2}} \end{pmatrix},$$

which is $C^{1,\alpha}(\overline{\Omega})$. However $r \in C^{0,\alpha}(\Omega')$ but $r \notin C^{0,\beta}(\Omega')$ for any $\beta > \alpha$.

One might wonder if the assumption in theorem 1 that r_0 to $C^{0,\alpha}(\overline{\Gamma^-})$ is a source of lowering regularities. However

$$r(\mathbf{x}_0) = r(\mathbf{x}(s_0, 0)) \exp\left(\int_0^{t_0} \nabla \times \mathbf{F}(\mathbf{x}(s_0,\tau))d\tau\right),$$

and the regularity of r depends also on the the vector field $\mathbf{F}$, hence increasing the boundary regularity of r_0 to $C^{k,\alpha}(\overline{\Gamma^-})$ for $k \geq 1$ does not improve the regularity.

3.4. Voltage construction

Now let us construct the voltage u from constructed r. It is well-defined up to an addition of a constant. If $r \in C^1(\Omega)$, then the existence of u that satisfies

$$-\nabla u = r\mathbf{F} \quad \text{in} \quad \overline{\Omega} \tag{23}$$

is clear. Even if $r \in C^{0,\alpha}(\Omega)$ as in our case, the existence theory of such a $u \in H^1(\Omega)$ is classical (see Weyl [20]). Since $-\nabla u = r\mathbf{F}$ in Ω, we conclude $u \in C^{1,\alpha}(\Omega') \cap C^1(\overline{\Omega})$.

We can also directly construct u. Define $\tilde{u} : \overline{E} \to \boldsymbol{R}$ by

$$\begin{aligned}
\tilde{u}(s,0) &:= -\int_0^s r_0\big(\gamma(\tau)\big)\mathbf{F}\big(\gamma(\tau)\big) \cdot \gamma'(\tau)d\tau, \\
\tilde{u}(s,t) &:= \tilde{u}(s,0),
\end{aligned}$$

and $u : \overline{\Omega} \to \boldsymbol{R}$ by $u = \tilde{u} \circ \mathbf{x}^{-1}$. Then, one can easily see that $-r\mathbf{F} = \nabla u$ in $\overline{\Omega}$.

In summary, we have optimally answered the inverse Schauder solvability for $\sigma(x)$ and its by-product u.

4. Boundary control and admissibility

Theorem 2.8 in [1] exactly is saying that one can construct an admissible $\mathbf{J}$ by controlling the Neumann boundary condition. For a completeness we quote the theorem here.

Theorem 2 (Alessandrini et al.). *Let $g \in H^{-1/2}(\partial\Omega)$ be such that $\partial\Omega$ can be split into $2M$ closed arcs $\Gamma_1, ..., \Gamma_{2M}$ such that $(-1)^j g \geq 0$ on $\Gamma_j, j = 1, ..., 2M$, in the sense of distributions. Let $u \in W^{1,2}(\Omega)$ be a solution of* (1) *and satisfying the Neumann condition* (2) *on $\partial\Omega$. Then, the geometric critical points of u in Ω, when counted according to their indices, are at most $M - 1$.*

By considering a case $M = 1$, we can easily obtain the admissibility. For our $\mathbf{J} \in C^{1,\alpha}(\overline{\Omega})$ the geometric critical point is the usual critical point.

Appendix. Comparison between conductivity and resistivity

If $r \neq 0$ or r is invertible, the conductivity σ is given by $\sigma = r^{-1}$, the inverse of the resistivity. Then, one can easily see that

$$-\operatorname{div}(\sigma\nabla u) = \operatorname{div}(\mathbf{F}).$$

If $\mathbf{F}$ is an electrical current without noise, then $\operatorname{div}(\mathbf{F}) = 0$. If a noise is included, $\mathbf{F}$ is not divergence free in general. Therefore, the above relation is what we can expect and the curl equation for resistivity is naturally connected to the divergence equation for conductivity with a forcing term. Theorem 1 and previous discussion implies that the resistivity r and voltage u are well-defined if a boundary resistivity r_0 and an admissible $\mathbf{F}$ are given.

The curl-based resistivity formulation and the divergence-based conductivity formulation become different when one is considering degenerate elliptic operators. The cases $\sigma = 0$ or $r = 0$.

Remember that the positivity of r_0 is not assumed in Theorem 1. Even if r_0 changes its sign, r is well defined by the relation (20). However if $\sigma = 0$, or $r = \infty$, not bounded, then our resistivity formulation does not work. $\mathbf{J} = -\sigma \nabla u = 0$ for some point and hence the electrical current is not admissible and Theorem 1 is not applicable in this case. However, if $\sigma = \infty$ in a region, the curl equation $\nabla \times (r\mathbf{J}) = 0$ with a resistivity r handles the case.

The equivalence between (1),(2) and (3),(4),(5) gives an implication that, if $0 < r < \infty$, the conductivity and resistivity formulations are equivalent. However, if $\sigma = \infty$, then it will be a better choice to work with resistivity r or vice versa.

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