Well-posedness in anisotropic electrical conductivity reconstruction from three sets of internal current data

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Abstract.

1. Introduction

2. Well-posedness of isotropic conductivity reconstruction problem

3. Existence and Uniqueness of anisotropic conductivity reconstruction problem

Here, Ω is a simply connected bounded region with smooth boundary.

Lemma 1. *Let $A(x)$ be a smooth symmetric matrix field on $\bar{\Omega}$. Assume $A(x) \neq 0$ and $\det A(x) < 0$ for all $x \in \bar{\Omega}$. Then there are two smooth vector field v and w such that $v \neq 0$, $w \neq 0$ and $< Av, v >=< Aw, w >= 0$ for all $x \in \bar{\Omega}$. Furthermore, if $< A(x)V, V >= 0$ for some V at x, then V is a scalar multiple of $v(x)$ or $w(x)$.*

Proof. Let's write $A(x)$ with entries $\begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix}$, then $a(x)c(x) - b(x)^2 < 0$ by assumption.

If $V = (V_1, V_2)$ and $< A(x)V, V >= 0$ then

$$a(x)V_1^2 + 2a(x)b(x)V_1V_2 + c(x)V_2^2 = 0. \tag{1}$$

Let's define two smooth vector field v and w as

$$v := \Big(-b - b\sqrt{1 - \frac{ac}{b^2}}, a \Big), \quad w := \Big(c, -b - b\sqrt{1 - \frac{ac}{b^2}} \Big). \tag{2}$$

Then we can check $v(x)$ and $w(x)$ are two non-trivial linearly independent solutions of (1) for each x. If $a(x) \neq 0$,

$$\left(\frac{-b + b\sqrt{1 - \frac{ac}{b^2}}}{a} \right) v(x) = \Big(c, -b + b\sqrt{1 - \frac{ac}{b^2}} \Big) \neq w(x).$$

Unless if $c(x) \neq 0$,

$$\left(\frac{-b + b\sqrt{1 - \frac{ac}{b^2}}}{c} \right) w(x) = \Big(-b + b\sqrt{1 - \frac{ac}{b^2}}, a \Big) \neq v(x).$$

If both $a(x) = c(x) = 0$, by assumption $b(x) \neq 0$ and

$$v(x) = (-2b, 0), \quad w(x) = (0, -2b).$$

Therefore $v(x) \neq 0$ and $w(x) \neq 0$ for all $x \in \bar{\Omega}$ and are linearly independent. Finally, suppose any V satisfies (1). If $a(x) \neq 0$, then $V_1/V_2 = \frac{-b \pm \sqrt{b^2 - ac}}{a}$, so we have only two possible ratios of components that are the ratios of $v(x)$ and $w(x)$. $c(x) \neq 0$ or $a(x) = c(x) = 0, b(x) \neq 0$ case can be similarly proved.

Definition 1 (Admissibility). *The data of three smooth vector fields* $(\mathbf{J}_1, \mathbf{J}_2,$ *on* $\bar{\Omega}$ *is admissible if following properties are satisfied.*

1. $(\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)$ *are all divergence free so that there exist three smooth stream-function* (ψ_1, ψ_2, ψ_3).
2. *The map* $(\xi(x,y), \eta(x,y)) := (\psi_2(x,y), -\psi_1(x,y))$ *are globally defined diffeomorphism between* $\bar{\Omega}$ *and its image.*
3. *The hessian of* ψ_3 *in* (ξ, η) *coordinate chart, the* $D^2\psi_3 \neq 0$ *and* $\det D^2\psi_3 < 0$ *for all* (ξ, η).
4. *Two vector fields defined by matrix field* $D^2\psi_3$ *as in Lemma 1 are admissible in the sense of definition @@@.*

Definition 2. $\Gamma_v^- := \Gamma^-$ *for vector field* v *in the definition @@@ and* $\Gamma_w^- := \Gamma^-$ *for vector field* w.

Theorem 1. *blabla*

From Ohm's Law,

$$\begin{pmatrix} -\psi_y \\ \psi_x \end{pmatrix} = \sigma \begin{pmatrix} u_x \\ u_y, \end{pmatrix}$$

$$\begin{pmatrix} \partial_y\eta & -\partial_y\xi \\ -\partial_x\eta & \partial_x\xi \end{pmatrix} \begin{pmatrix} -\psi_\eta \\ \psi_\xi \end{pmatrix} = \sigma \begin{pmatrix} \partial_x\xi \; \partial_x\eta \\ \partial_y\xi \; \partial_y\eta \end{pmatrix} \begin{pmatrix} u_\xi \\ u_\eta, \end{pmatrix}$$

$$\begin{pmatrix} -\psi_\eta \\ \psi_\xi \end{pmatrix} = \frac{1}{\xi_x\eta_y - \eta_x\xi_y} \begin{pmatrix} \xi_x \; \xi_y \\ \eta_x \; \eta_y \end{pmatrix} \sigma \begin{pmatrix} \xi_x \; \eta_x \\ \xi_y \; \eta_y \end{pmatrix} \begin{pmatrix} u_\xi \\ u_\eta, \end{pmatrix}$$

$$\begin{pmatrix} -\psi_\eta \\ \psi_\xi \end{pmatrix} = \frac{W^t\sigma W}{\det W} \begin{pmatrix} u_\xi \\ u_\eta, \end{pmatrix}$$

Lemma 2. *Let (ξ,η) be a diffeomorphism as in the definition for an admissible data $(\mathbf{J}^1,\mathbf{J}^2,\mathbf{J}^3)$. Let's define a matrix field $S := \frac{W\sigma W^t}{\det W}$, where $W := \begin{pmatrix} \xi_x \; \xi_y \\ \eta_x \; \eta_y \end{pmatrix}$ is a jacobian matrix of the diffeomorphism. Then in this chart, we have another set of Ohm's laws,*

$$S\begin{pmatrix} u^1_\xi \\ u^1_\eta \end{pmatrix} = \begin{pmatrix} -\psi^1_\eta \\ \psi^1_\xi \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$S\begin{pmatrix} u^2_\xi \\ u^2_\eta \end{pmatrix} = \begin{pmatrix} -\psi^2_\eta \\ \psi^2_\xi \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$S\begin{pmatrix} u^3_\xi \\ u^3_\eta \end{pmatrix} = \begin{pmatrix} -\psi^3_\eta \\ \psi^3_\xi \end{pmatrix}.$$

Lemma 3. *For an admissible data $(\mathbf{J}^1,\mathbf{J}^2,\mathbf{J}^3)$ define S as in the Lemma 2 and define $R := S^{-1}$. Then there is a scalar $\phi(\xi,\eta)$ such that $R = D'^2\phi$ and ϕ is a solution of following 2nd order linear equation*

$$\psi_{\eta\eta}\phi_{\xi\xi} - 2\psi_{\xi\eta}\phi_{\xi\eta} + \psi_{\xi\xi}\phi_{\eta\eta} = 0 \tag{3}$$

and the equation is hyperbolic.

Proof. By Lemma 2,

$$S\begin{pmatrix} u^1_\xi \; u^2_\xi \\ u^1_\eta \; u^2_\eta \end{pmatrix} = \begin{pmatrix} 1 \; 0 \\ 0 \; 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} u^1_\xi \; u^2_\xi \\ u^1_\eta \; u^2_\eta \end{pmatrix} = R.$$

Since S and R is symmetric, $u^2_\xi = u^1_\eta$, i.e. the vector field (u^1,u^2) is a gradient field. Therefore there is a ϕ such that $u^1 = \phi_\xi$ and $u^2 = \phi_\eta$ and hence $R = D'^2\phi$.

Applying this R into third Ohm's law, we have

$$\begin{pmatrix} u^3_\xi \\ u^3_\eta \end{pmatrix} = D'^2\phi \begin{pmatrix} -\psi_\eta \\ \psi_\xi \end{pmatrix}.$$

By applying $\nabla\times$ to above equation we have

$$\begin{aligned} 0 &= \psi_{\eta\eta}\phi_{\xi\xi} - 2\psi_{\xi\eta}\phi_{\xi\eta} + \psi_{\xi\xi}\phi_{\eta\eta} + \psi_{\eta}(\phi_{\xi\xi\eta} - \phi_{\xi\eta\xi}) + \psi_{\xi}(-\phi_{\xi\eta\eta} + \phi\eta\eta\xi) \\ &= \psi_{\eta\eta}\phi_{\xi\xi} - 2\psi_{\xi\eta}\phi_{\xi\eta} + \psi_{\xi\xi}\phi_{\eta\eta}. \end{aligned}$$

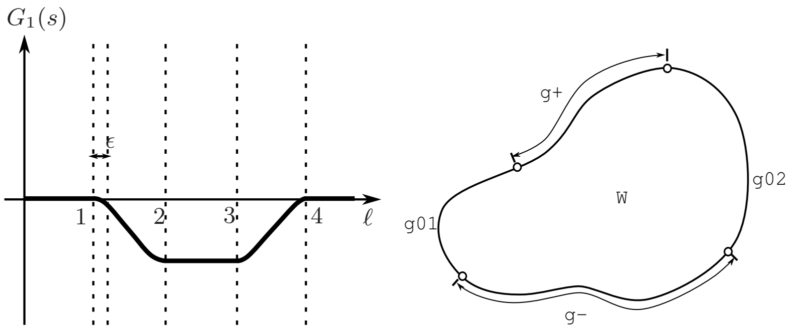
This equation for ϕ is 2nd order hyperbolic because $\det D'^2\psi = \psi_{\xi\xi}\psi\eta\eta - \psi_{\xi\eta}^2 < 0$ by assumption.

Theorem 2. *Suppose the entries of the symmetric positive matrix field $\sigma(x)$ are in $C^{2,\alpha}(\bar{\Omega})$. Then there are explicit choices of three Neumann condition g_1, g_2, g_3 on $\partial\Omega$ such that the current density $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ generated from the solution of equation @@@ with the boundary conditions are admissible.*

Lemma 4. *Let $\gamma(\ell) : [0,4] \longrightarrow \partial\Omega$ be an embedding of $\partial\Omega$ and assume ℓ is an arc length. Define smooth functions $G_1(\ell), G_2(\ell)$ such that*

$$G_1 = \begin{cases} 0, & \epsilon \le \ell \le 1-\epsilon, \\ -\ell, & 1 \le \ell \le 2, \\ -1, & 2+\epsilon \le \ell \le 3-\epsilon, \\ \ell - 4, & 3 \le \ell \le 4, \end{cases} \qquad G_2 = \begin{cases} \ell, & 0 \le \ell \le 1, \\ 0, & 1+\epsilon \le \ell \le 2-\epsilon, \\ 3-\ell, & 2 \le \ell \le 3, \\ 0, & 3+\epsilon \le \ell \le 4-\epsilon. \end{cases}.$$

G_1 and G_2 are monotone where it is not defined by above. Let ψ_1 and ψ_2 be



(a) $G1(\ell)$ (b) $G2(\ell)$

a solution of the equation @@@ with boundary condition G_1 and G_2 each. Let $B := [0,1] \times [0,1]$. Then the map $W := (\xi(x,y), \eta(x,y)) := (\psi_2, -\psi_1) : \bar{\Omega} \longrightarrow \bar{B}$ is a bi-Lipchitz, and $W|_{\Omega} : \Omega \longrightarrow B$ is a C^1-diffeomorphism.

Proof. First we claim that $\nabla\psi_1 \neq 0$, $\nabla\psi_2 \neq 0$ and $\nabla\psi_1 \times \nabla\psi_2 \neq 0$.

Lemma 5. *Let $G_3 = \xi^2 - \eta^2$ on $\bar{B}$ and ψ be a solution of the equation @@@ with boundary condition $\psi = G_3$ on ∂B. Then $D^2\psi \neq 0$ in $\bar{B}$ and $\det D^2\psi < 0$ in $\bar{B}$.*

Lemma 6. *Let* ψ *be a function defined in lemma @@@ and* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *be two vector fields associated with* $D^2\psi$ *as in lemma @@@. Then* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *are admissible in the sense of Def @@@.*

Proof. a

4. Numerical reconstruction strategies : Network method and Potential method