ABSOLUTELY CONTINUOUS SPECTRUM OF A POLYHARMONIC OPERATOR WITH A LIMIT PERIODIC POTENTIAL IN DIMENSION TWO.

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Abstract. We consider a polyharmonic operator $H=(-\Delta)^l+V(x)$ in dimension two with $l\geq 6$, l being an integer, and a limit-periodic potential $V(x)$. We prove that the spectrum contains a semiaxis of absolutely continuous spectrum.

1. Introduction.

We study an operator

$$H=(-\Delta)^l+V(x) \tag{1}$$

in two dimensions, where $l\geq 6$ is an integer and $V(x)$ is a limit-periodic potential

$$V(x)=\sum_{r=1}^{\infty}V_r(x). \tag{2}$$

Here $\{V_r\}_{r=1}^{\infty}$ is a family of periodic potentials with doubling periods and decreasing L_∞-norms, namely, V_r has orthogonal periods $2^{r-1}\vec{\beta}_1,\ 2^{r-1}\vec{\beta}_2$ and

$$\|V_r\|_\infty<\hat{C}exp(-2^{\eta r}) \tag{3}$$

for some $\eta>2+64/(2l-11)$. Without loss of generality, we assume that $\hat{C}=1$, $\vec{\beta}_1=(\beta_1,0)$, $\vec{\beta}_2=(0,\beta_2)$ and $\int_{Q_r}V_r(x)dx=0$, Q_r being the elementary cell of periods corresponding to $V_r(x)$.

The one-dimensional analog of (1), (2) with $l=1$ is already thoroughly investigated. It is proven in [1]–[7] that the spectrum of the operator $H_1u=-u''+Vu$ is generically a Cantor type set. It has positive Lebesgue measure [1, 6]. The spectrum is absolutely continuous [1, 2], [5]–[9]. Generalized eigenfunctions can be represented in the form of $e^{ikx}u(x)$, $u(x)$ being limit-periodic [5, 6, 7]. The case of a complex-valued potential is studied in [10]. Integrated density of states is investigated in [11]–[14]. Properties of eigenfunctions of discrete multidimensional limit-periodic Schrödinger operators are studied in [15]. As to the continuum multidimensional case, it is proved [14] that the integrated density of states for (1) is the limit of densities of states for periodic operators. A particular case of a periodic operator ($V_r=0$ when $r\geq 2$) for dimensions $d\geq 2$ and different l is already studied well, e.g., see [16] – [31]. Here we prove that the spectrum of (1), (2) contains a semiaxis of absolutely continuous spectrum. This paper is based on [32]. We proved the following results for the case $d=2$, $l\geq 6$ in [32].

(1) The spectrum of the operator (1), (2) contains a semiaxis $[\lambda_*(V),\infty)$. A proof of the analogous result by different means can be found in [33]. The more general case $8l>d+3$, $d\neq 1(\mathrm{mod}4)$, is considered in [33], however, under the additional

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restriction on the potential: the lattices of periods of all periodic potentials V_r have to contain a nonzero vector $\vec{\gamma}$ in common, i.e., $V(x)$ is periodic in one direction.

(2) There are generalized eigenfunctions $\Psi_\infty(\vec{k},\vec{x})$, corresponding to the semiaxis, which are close to plane waves: for every $\vec{k}$ in a subset $\mathcal{G}_\infty$ of $\mathbb{R}^2$, there is a solution $\Psi_\infty(\vec{k},\vec{x})$ of the equation $H\Psi_\infty = \lambda_\infty(\vec{k})\Psi_\infty$, which can be described by the formula

$$\Psi_\infty(\vec{k},\vec{x}) = e^{i\langle\vec{k},\vec{x}\rangle}\left(1+u_\infty(\vec{k},\vec{x})\right), \tag{4}$$

$$\|u_\infty\|_{L_\infty(\mathbb{R}^2)} \underset{|\vec{k}|\to\infty}{=} O\left(|\vec{k}|^{-\gamma_1}\right), \quad \gamma_1>0, \tag{5}$$

where $u_\infty(\vec{k},\vec{x})$ is a limit-periodic function

$$u_\infty(\vec{k},\vec{x}) = \sum_{r=1}^{\infty} u_r(\vec{k},\vec{x}), \tag{6}$$

$u_r(\vec{k},\vec{x})$ being periodic with periods $2^{r-1}\vec{\beta}_1,\ 2^{r-1}\vec{\beta}_2$. The eigenvalue $\lambda_\infty(\vec{k})$ corresponding to $\Psi_\infty(\vec{k},\vec{x})$ is close to $|\vec{k}|^{2l}$:

$$\lambda_\infty(\vec{k}) \underset{|\vec{k}|\to\infty}{=} |\vec{k}|^{2l} + O\left(|\vec{k}|^{-\gamma_2}\right), \quad \gamma_2>0. \tag{7}$$

The "non-resonance" set $\mathcal{G}_\infty$ of vectors $\vec{k}$, for which (4) – (7) hold, is a Cantor type set $\mathcal{G}_\infty = \bigcap_{n=1}^{\infty}\mathcal{G}_n$, where $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in $\mathbb{R}^2$. Each $\mathcal{G}_n$ has a finite number of holes in each bounded region. More and more holes appear as n increases; however, holes added at each step are of smaller and smaller size. The set $\mathcal{G}_\infty$ satisfies the estimate

$$|\mathcal{G}_\infty \cap \mathbf{B_R}| \underset{R\to\infty}{=} |\mathbf{B_R}|\big(1+O(R^{-\gamma_3})\big), \quad \gamma_3>0, \tag{8}$$

where $\mathbf{B_R}$ is the disk of radius R centered at the origin and $|\cdot|$ is Lebesgue measure in $\mathbb{R}^2$.

(3) The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\vec{k})$,

$$\mathcal{D}_\infty(\lambda) = \left\{\vec{k}\in\mathcal{G}_\infty : \lambda_\infty(\vec{k}) = \lambda\right\},$$

is shown to be a slightly distorted circle with an infinite number of holes. It can be described by the formula

$$\mathcal{D}_\infty(\lambda) = \left\{\vec{k} : \vec{k} = \varkappa_\infty(\lambda,\vec{\nu})\vec{\nu},\ \vec{\nu}\in\mathcal{B}_\infty(\lambda)\right\}, \tag{9}$$

where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_\infty(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (4). The set $\mathcal{B}_\infty(\lambda)$ has a Cantor type structure and an asymptotically full measure on S_1 as $\lambda\to\infty$:

$$L\big(\mathcal{B}_\infty(\lambda)\big) \underset{\lambda\to\infty}{=} 2\pi + O\left(\lambda^{-\gamma_3/2l}\right), \tag{10}$$

here and below $L(\cdot)$ is a length of a curve. The value $\varkappa_\infty(\lambda,\vec{\nu})$ in (9) is the "radius" of $\mathcal{D}_\infty(\lambda)$ in a direction $\vec{\nu}$. The function $\varkappa_\infty(\lambda,\vec{\nu})-\lambda^{1/2l}$ describes the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect circle of the radius $\lambda^{1/2l}$. It is shown that the deviation is small

$$\varkappa_\infty(\lambda,\vec{\nu}) \underset{\lambda\to\infty}{=} \lambda^{1/2l}+O\left(\lambda^{-\gamma_4}\right), \quad \gamma_4>0. \tag{11}$$

The set G_∞ is the union of isoenergetic curves $\mathcal{D}_\infty(\lambda)$ over all sufficiently large λ:

$$G_\infty=\bigcup_{\lambda>\lambda_*(V)}\mathcal{D}_\infty(\lambda). \tag{12}$$

In this paper, we use the results of [32] to prove absolute continuity of the branch of the spectrum (the semiaxis) corresponding to $\Psi_\infty(\vec{k},\vec{x})$.

The following is a brief review of the technique used in [32], where we develop a modification of the Kolmogorov-Arnold-Moser (KAM) method to prove the results listed above. The paper [32] is inspired by [34, 35, 36], where the method is used for periodic problems. In [34], KAM method is applied to classical Hamiltonian systems. In [35, 36], the technique developed in [34] is applied for semiclassical approximation for multidimensional periodic Schrödinger operators at high energies.

In [32], we consider a sequence of operators

$$H_0=(-\Delta)^l, \qquad H^{(n)}=H_0+\sum_{r=1}^{M_n}V_r, \quad n\geq 1,\ M_n\to\infty \text{ as } n\to\infty. \tag{13}$$

Obviously, $\|H-H^{(n)}\|\to 0$ as $n\to\infty$ and $H^{(n)}=H^{(n-1)}+W_n$, where $W_n=\sum_{r=M_{n-1}+1}^{M_n}V_r$. We treat each operator $H^{(n)}$, $n\geq 1$, as a perturbation of the previous operator $H^{(n-1)}$, $H^{(0)}=H_0$. Each operator $H^{(n)}$ is periodic; however, the periods go to infinity as $n\to\infty$. We show that there exists $\lambda_*=\lambda_*(V)$ such that the semiaxis $[\lambda_*,\infty)$ is contained in the spectra of all operators $H^{(n)}$. For every operator $H^{(n)}$, there is a set of eigenfunctions (corresponding to the semiaxis) close to plane waves: for every $\vec{k}$ in an extensive open subset $\mathcal{G}_n$ of $\mathbb{R}^2$, there is a solution $\Psi_n(\vec{k},\vec{x})$ of the differential equation $H^{(n)}\Psi_n=\lambda^{(n)}\Psi_n$, which can be represented by the formula

$$\Psi_n(\vec{k},\vec{x})=e^{i\langle\vec{k},\vec{x}\rangle}\left(1+\tilde{u}_n(\vec{k},\vec{x})\right), \quad \|\tilde{u}_n\|_{L_\infty(\mathbb{R}^2)} \underset{|\vec{k}|\to\infty}{=} O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1>0, \tag{14}$$

where $\tilde{u}_n(\vec{k},\vec{x})$ has periods $2^{M_n-1}\vec{\beta}_1, 2^{M_n-1}\vec{\beta}_2$.[1] The corresponding eigenvalue $\lambda^{(n)}(\vec{k})$ is close to $|\vec{k}|^{2l}$:

$$\lambda^{(n)}(\vec{k}) \underset{|\vec{k}|\to\infty}{=} |\vec{k}|^{2l}+O\left(|\vec{k}|^{-\gamma_2}\right), \quad \gamma_2>0. \tag{15}$$

The asymptotic is differentiable in $\vec{k}$:

$$\nabla\lambda^{(n)}(\vec{k}) \underset{|\vec{k}|\to\infty}{=} 2l|\vec{k}|^{2l-2}\vec{k}+O\left(|\vec{k}|^{-\gamma_2'}\right), \quad \gamma_2'>0. \tag{16}$$

The non-resonance set $\mathcal{G}_n$ is shown to be extensive in $\mathbb{R}^2$:

$$|\mathcal{G}_n\cap\mathbf{B_R}| \underset{R\to\infty}{=} |\mathbf{B_R}|\big(1+O(R^{-\gamma_3})\big). \tag{17}$$

Estimates (14) – (17) are uniform in n. The set $\mathcal{D}_n(\lambda)$ is defined as the level (isoenergetic) set for non-resonant eigenvalue $\lambda^{(n)}(\vec{k})$:

$$\mathcal{D}_n(\lambda)=\left\{\vec{k}\in\mathcal{G}_n : \lambda^{(n)}(\vec{k})=\lambda\right\}. \tag{18}$$

[1] $\tilde{u}_n(\vec{k},\vec{x})=\sum_{r=M_{n-1}+1}^{M_n}u_r(\vec{k},\vec{x})$, $u_r(\vec{k},\vec{x})$ being in (6).

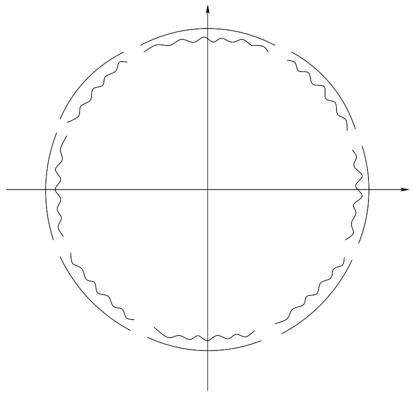

FIGURE 1. Distorted circle with holes, $\mathcal{D}_1(\lambda)$

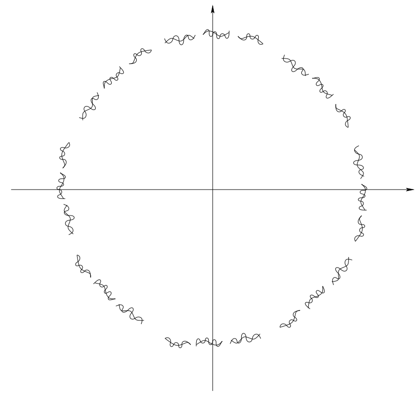

FIGURE 2. Distorted circle with holes, $\mathcal{D}_2(\lambda)$

This set is shown to be a slightly distorted circle with a finite number of holes (see Figs. 1, 2), the set $\mathcal{D}_1(\lambda)$ being strictly inside the circle of the radius $\lambda^{1/2l}$ for sufficiently large λ. The set $\mathcal{D}_n(\lambda)$ can be described by the formula

$$\mathcal{D}_n(\lambda) = \left\{ \vec{k} : \vec{k} = \varkappa_n(\lambda, \vec{\nu})\vec{\nu},\ \vec{\nu} \in \mathcal{B}_n(\lambda) \right\}, \tag{19}$$

where $\mathcal{B}_n(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_n(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (14). It has an asymptotically full measure on S_1 as $\lambda \to \infty$:

$$L\big(\mathcal{B}_n(\lambda)\big) \underset{\lambda\to\infty}{=} 2\pi + O\left(\lambda^{-\gamma_3/2l}\right). \tag{20}$$

The set $\mathcal{B}_n(\lambda)$ has only a finite number of holes; however, their number grows with n. More and more holes of a smaller and smaller size are removed at each step. The value $\varkappa_n(\lambda, \vec{\nu}) - \lambda^{1/2l}$ gives the deviation of $\mathcal{D}_n(\lambda)$ from the circle of the radius $\lambda^{1/2l}$ in the direction $\vec{\nu}$. It is shown that the deviation is asymptotically small:

$$\varkappa_n(\lambda, \vec{\nu}) = \lambda^{1/2l} + O\left(\lambda^{-\gamma_4}\right), \qquad \frac{\partial \varkappa_n(\lambda, \vec{\nu})}{\partial \varphi} = O\left(\lambda^{-\gamma_5}\right), \quad \gamma_4, \gamma_5 > 0, \tag{21}$$

φ being an angular variable,

$$\vec{\nu} = (\cos\varphi, \sin\varphi), \quad \varphi \in [0, 2\pi).$$

Estimates (20), (21) are uniform in n. The following relation holds:

$$\mathcal{G}_n = \bigcup_{\lambda > \lambda_*(V)} \mathcal{D}_n(\lambda). \tag{22}$$

At each step, more and more points are excluded from the non-resonance sets $\mathcal{G}_n$; thus, $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets. The set $\mathcal{G}_\infty$ is proven to be the limit set: $\mathcal{G}_\infty = \bigcap_{n=1}^{\infty} \mathcal{G}_n$. It has an infinite number of holes, but nevertheless satisfies the relation (8). For every $\vec{k} \in \mathcal{G}_\infty$ and every n, there is a generalized eigenfunction of $H^{(n)}$ of the type (14). It is shown that the sequence $\Psi_n(\vec{k}, \vec{x})$ has a limit in $L_\infty(\mathbb{R}^2)$ when $\vec{k} \in \mathcal{G}_\infty$. The function $\Psi_\infty(\vec{k}, \vec{x}) = \lim_{n\to\infty} \Psi_n(\vec{k}, \vec{x})$ is a generalized eigenfunction of H. It can be written in the form (4) – (6). Naturally, the corresponding eigenvalue $\lambda_\infty(\vec{k})$ is the limit of $\lambda^{(n)}(\vec{k})$ as $n \to \infty$.

It is shown that $\{\mathcal{B}_n(\lambda)\}_{n=1}^{\infty}$ is a decreasing sequence of sets at each step more and more directions being excluded. We consider the limit $\mathcal{B}_\infty(\lambda)$ of $\mathcal{B}_n(\lambda)$,

$$\mathcal{B}_\infty(\lambda) = \bigcap_{n=1}^{\infty} \mathcal{B}_n(\lambda).$$

This set has a Cantor type structure on the unit circle. It is shown that $\mathcal{B}_\infty(\lambda)$ has asymptotically full measure on the unit circle (see (10)). We prove that the sequence $\varkappa_n(\lambda,\vec{\nu})$, $n=1,2,...$, describing the isoenergetic curves $\mathcal{D}_n(\lambda)$, converges rapidly (super exponentially) as $n\to\infty$. Hence, $\mathcal{D}_\infty(\lambda)$ can be described as the limit of $\mathcal{D}_n(\lambda)$ in the sense of (9), where $\varkappa_\infty(\lambda,\vec{\nu}) = \lim_{n\to\infty}\varkappa_n(\lambda,\vec{\nu})$ for every $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$. It is shown that the derivatives $\frac{\partial\varkappa_n(\lambda,\vec{\nu})}{\partial\varphi}$ have a limit as $n\to\infty$ for every $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$. We denote this limit by $\frac{\partial\varkappa_\infty(\lambda,\vec{\nu})}{\partial\varphi}$. Using (21), we prove that

$$\frac{\partial\varkappa_\infty(\lambda,\vec{\nu})}{\partial\varphi} = O\left(\lambda^{-\gamma_5}\right). \tag{23}$$

Thus, the limit curve $\mathcal{D}_\infty(\lambda)$ has a tangent vector in spite of its Cantor type structure, the tangent vector being the limit of corresponding tangent vectors for $\mathcal{D}_n(\lambda)$ as $n\to\infty$. The curve $\mathcal{D}_\infty(\lambda)$ looks like a slightly distorted circle with infinite number of holes.

The main technical difficulty overcome in [32] is the construction of non-resonance sets $\mathcal{B}_n(\lambda)$ for every fixed sufficiently large λ, $\lambda>\lambda_*(V)$, where λ_* is the same for all n. The set $\mathcal{B}_n(\lambda)$ is obtained by deleting a "resonant" part from $\mathcal{B}_{n-1}(\lambda)$. The definition of $\mathcal{B}_{n-1}(\lambda)\setminus\mathcal{B}_n(\lambda)$ includes Bloch eigenvalues of $H^{(n-1)}$. To describe $\mathcal{B}_{n-1}(\lambda)\setminus\mathcal{B}_n(\lambda)$, one has to use not only non-resonant eigenvalues of type (7), but also resonant eigenvalues, for which no suitable formulas are known. The absence of formulas causes difficulties in estimating the size of $\mathcal{B}_{n-1}(\lambda)\setminus\mathcal{B}_n(\lambda)$. To deal with this problem we use angular variable φ. We show that the resonant set $\mathcal{B}_{n-1}(\lambda)\setminus\mathcal{B}_n(\lambda)$ can be described as the zero set of a determinant

$$\det(I+A_{n-1}(\varphi)), \tag{24}$$

$A_{n-1}(\varphi)$ being a trace type operator,

$$I+A_{n-1}(\varphi) = \Big(H^{(n-1)}\big(\vec{\varkappa}_{n-1}(\varphi)+\vec{b}\big)-(\lambda+\epsilon)I\Big)\Big(H_0\big(\vec{\varkappa}_{n-1}(\varphi)+\vec{b}\big)+\lambda I\Big)^{-1},$$

where $\vec{\varkappa}_{n-1}(\varphi)$ is a vector-valued function describing $\mathcal{D}_{n-1}(\lambda):\ \vec{\varkappa}_{n-1}(\varphi)=\varkappa_{n-1}(\lambda,\vec{\nu})\vec{\nu}$. To obtain $\mathcal{B}_{n-1}(\lambda)\setminus\mathcal{B}_n(\lambda)$ we take all the zeros of (24) for all values of ϵ in a small interval and vectors $\vec{b}$ in a finite set, $\vec{b}\neq 0$. To estimate the size of $\mathcal{B}_{n-1}(\lambda)\setminus\mathcal{B}_n(\lambda)$ we extend our considerations to a complex neighborhood $\mathit{\Phi}_0$ of $[0,2\pi)$. We show that the determinants are analytic functions of φ in $\mathit{\Phi}_0$, and, thus, reduce the problem of estimating the size of the resonance set to a problem in complex analysis. We use theorems for analytic functions to count the zeros of the determinants and to investigate how far zeros move when ϵ changes. This enables us to estimate the size of the zero set of the determinants and hence the size of the non-resonance set $\mathit{\Phi}_n\subset\mathit{\Phi}_0$, which is defined as a non-zero set for the determinants. Proving that the non-resonance set $\mathit{\Phi}_n$ is sufficiently large, we obtain estimates (17) for $\mathcal{G}_n$ and (20) for $\mathcal{B}_n$, the set $\mathcal{B}_n$ being the intersecton of $\mathit{\Phi}_n$ with the real line. To obtain $\mathit{\Phi}_n$ we delete from $\mathit{\Phi}_0$ more and more holes of smaller and smaller radii at each step. Thus, the non-resonance set $\mathit{\Phi}_n\subset\mathit{\Phi}_0$ has the structure of Swiss Cheese. We call deleting the resonance set from $\mathit{\Phi}_0$ at each step of the recurrent procedure "Swiss Cheese Method". The essential difference of our method from those applied earlier in similar situations (see, e.g., [34, 35, 36]) is that we construct a non-resonance set not only in the whole space of a parameter ($\vec{k} \in \mathbb{R}^2$ here), but also on the isoenergetic curves $\mathcal{D}_n(\lambda)$ in the space of parameter when λ is sufficiently large. Estimates for the size of non-resonant sets on a curve require more subtle technical considerations than those sufficient for description of a non-resonant set in the whole space of the parameter.

In the next section, using information obtained in [32], we prove absolute continuity of the branch of the spectrum $[\lambda_*(V), \infty)$ corresponding to the functions $\Psi_\infty(\vec{k}, \vec{x})$, $\vec{k} \in \mathcal{G}_\infty$. Absolute continuity, roughly speaking, follows from the fact that the area between isoenergetic curves $\mathcal{D}_\infty(\lambda)$ and $\mathcal{D}_\infty(\lambda+\epsilon)$ (integrated density of states) is proportional to ϵ.

Note that generalization of results from the case $l \geq 6$, l being an integer, to the case of rational l satisfying the same inequality is relatively simple; it requires just slightly more careful technical considerations. The restriction $l \geq 6$ is also technical, though it is more difficult to lift. The condition $l \geq 6$ is needed only for the first two steps of the recurrent procedure in [32]. The requirement for super-exponential decay of $\|V_r\|$ as $r \to \infty$ is more essential than $l \geq 6$ since it is needed to ensure convergence of the recurrent procedure. It is not essential that potentials V_r have doubling periods; periods of the type $q^{r-1}\vec{\beta}_1,\ q^{r-1}\vec{\beta}_2$, $q \in \mathbb{N}$, can be treated in the same way.

The periodic case ($V_r = 0$, when $r \geq 2$) is already carefully investigated for dimensions $d \geq 2$ and different l [16]–[31]. For briefness, we mention here only results for dimension two. Absolute continuity of the whole spectrum is proven in [16] for $l = 1$, however the proof can be extended for higher integers l. Bethe-Sommerfeld conjecture is first proved for $d = 2$, $l = 1$ in [17], [18] and for $l \geq 1$ in [21]. The perturbation formulas for eigenvalues are constructed in [20]. The formulas for eigenfunctions and the corresponding isoenergetic surfaces are obtained in [21].

2. Proof of Absolute Continuity of the Spectrum

2.1. **Projections $E_n(\mathcal{G}_n')$, $\mathcal{G}_n' \subset \mathcal{G}_n$.** Let us consider the open sets $\mathcal{G}_n$ given by (22). There is a family of Bloch eigenfunctions $\Psi_n(\vec{\varkappa}, x)$, $\vec{\varkappa} \in \mathcal{G}_n$,[2] of the operator $H^{(n)}$, which are described by the perturbation formulas (14). Let $\mathcal{G}_n'$ be a Lebesgue measurable subset of $\mathcal{G}_n$. We consider the spectral projection $E_n\left(\mathcal{G}_n'\right)$ of $H^{(n)}$, corresponding to functions $\Psi_n(\vec{\varkappa}, x)$, $\vec{\varkappa} \in \mathcal{G}_n'$. By [38], $E_n\left(\mathcal{G}_n'\right) : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ can be presented by the formula:

$$E_n\left(\mathcal{G}_n'\right) F = \frac{1}{4\pi^2} \int_{\mathcal{G}_n'} \big(F, \Psi_n(\vec{\varkappa})\big) \Psi_n(\vec{\varkappa}) d\vec{\varkappa} \tag{25}$$

for any $F \in C_0^\infty(\mathbb{R}^2)$, here and below $\big(\cdot,\cdot\big)$ is the canonical scalar product in $L_2(\mathbb{R}^2)$, i.e.,

$$\big(F, \Psi_n(\vec{\varkappa})\big) = \int_{\mathbb{R}^2} F(x) \overline{\Psi_n(\vec{\varkappa}, x)} dx.$$

The above formula can be rewritten in the form:

$$E_n\left(\mathcal{G}_n'\right) = S_n\left(\mathcal{G}_n'\right) T_n\left(\mathcal{G}_n'\right), \tag{26}$$

$$T_n : C_0^\infty(\mathbb{R}^2) \to L_2\left(\mathcal{G}_n'\right), \qquad S_n : L_\infty\left(\mathcal{G}_n'\right) \to L_2(\mathbb{R}^2),$$

$$T_n F = \frac{1}{2\pi}\big(F, \Psi_n(\vec{\varkappa})\big) \ \text{ for any } F \in C_0^\infty(\mathbb{R}^2), \tag{27}$$

[2] We use $\vec{\varkappa}$ in this section instead of $\vec{k}$ (see (4) and further) to be consistent with notations in [32].

T_nF being in $L_\infty(\mathcal{G}'_n)$, and,

$$S_nf = \frac{1}{2\pi}\int_{\mathcal{G}'_n} f(\vec{\varkappa})\Psi_n(\vec{\varkappa},x)d\vec{\varkappa} \quad \text{for any } f \in L_\infty(\mathcal{G}'_n). \tag{28}$$

By [38],

$$\|T_nF\|_{L_2(\mathcal{G}'_n)} \le \|F\|_{L_2(\mathbb{R}^2)}$$

on $C_0^\infty(\mathbb{R}^2)$ and

$$\|S_nf\|_{L_2(\mathbb{R}^2)} \le \|f\|_{L_2(\mathcal{G}'_n)}$$

on $L_\infty(\mathcal{G}'_n)$. Hence T_n, S_n can be extended by continuity from $C_0^\infty(\mathbb{R}^2)$, $L_\infty(\mathcal{G}'_n)$ to $L_2(\mathbb{R}^2)$ and $L_2(\mathcal{G}'_n)$, respectively. Thus the operator $E_n(\mathcal{G}'_n)$ is described by (26) in the whole space $L_2(\mathbb{R}^2)$.

Let us introduce new coordinates (λ_n, φ) in $\mathcal{G}_n$: $\lambda_n = \lambda^{(n)}(\vec{\varkappa})$, $(\cos\varphi, \sin\varphi) = \frac{\vec{\varkappa}}{|\vec{\varkappa}|}$.

Lemma 1. *Every point $\vec{\varkappa}$ in $\mathcal{G}_n$ is represented by a unique pair (λ_n,φ), $\lambda_n > \lambda_*$, $\varphi \in [0, 2\pi)$,*

$$\vec{\varkappa}(\lambda_n,\varphi) = \varkappa_n(\lambda_n,\vec{\nu})\vec{\nu}, \quad \vec{\nu} = (\cos\varphi, \sin\varphi), \tag{29}$$

$\varkappa_n(\lambda_n,\vec{\nu})$ being the "radius" of the isoenergetic curve $\mathcal{D}_n(\lambda_n)$ in the direction $\vec{\nu}$.

Proof. Obviously, for every $\vec{\varkappa}$ in $\mathcal{G}_n$, there exists a pair (λ_n,φ) such that $\lambda_n = \lambda^{(n)}(\vec{\varkappa})$ and that $(\cos\varphi,\sin\varphi) = \frac{\vec{\varkappa}}{|\vec{\varkappa}|}$. For uniqueness, suppose there are two points $\vec{\varkappa}_1, \vec{\varkappa}_2$ corresponding to (λ_n,φ), i.e., $\lambda^{(n)}(\vec{\varkappa}_1) = \lambda^{(n)}(\vec{\varkappa}_2) = \lambda_n$ and $\frac{\vec{\varkappa}_1}{|\vec{\varkappa}_1|} = \frac{\vec{\varkappa}_2}{|\vec{\varkappa}_2|} = \vec{\nu}$. The former means that both $\vec{\varkappa}_1$ and $\vec{\varkappa}_2$ belong to $\mathcal{D}_n(\lambda_n)$. The curve $\mathcal{D}_n(\lambda_n)$ is parameterized by φ, therefore, $\vec{\varkappa}_1 = \vec{\varkappa}_2$. Formula (29) follows from the relation $\lambda^{(n)}(\vec{\varkappa}) = \lambda_n$, which is the definition of the curve $\mathcal{D}_n(\lambda_n)$, and formula (19). □

For any function $f(\vec{\varkappa})$ integrable on $\mathcal{G}_n$, we use the new coordinates and write

$$\begin{aligned}\int_{\mathcal{G}_n} f(\vec{\varkappa})d\vec{\varkappa} &= \int_{\mathbb{R}^2}\chi(\mathcal{G}_n,\vec{\varkappa})f(\vec{\varkappa})d\vec{\varkappa}\\ &= \int_{\lambda_*}^{\infty}\int_0^{2\pi}\chi\big(\mathcal{G}_n,\vec{\varkappa}(\lambda_n,\varphi)\big)f\big(\vec{\varkappa}(\lambda_n,\varphi)\big)\frac{\varkappa_n(\lambda_n,\vec{\nu})}{\frac{\partial\lambda_n}{\partial\varkappa}}d\varphi\, d\lambda_n,\end{aligned}$$

where $\chi(\mathcal{G}_n,\vec{\varkappa})$ is the characteristic function on $\mathcal{G}_n$, $\vec{\varkappa}(\lambda_n,\varphi)$ is given by (29) and $\frac{\partial\lambda_n}{\partial\varkappa} = \big(\nabla\lambda^{(n)}(\vec{\varkappa}),\vec{\nu}\big)\big|_{\vec{\varkappa}=\vec{\varkappa}_n(\lambda_n,\vec{\nu})}$. Let

$$\mathcal{G}_{n,\lambda} = \{\vec{\varkappa} \in \mathcal{G}_n : \lambda^{(n)}(\vec{\varkappa}) < \lambda\}. \tag{30}$$

This set is Lebesgue measurable, since $\mathcal{G}_n$ is open and $\lambda^{(n)}(\vec{\varkappa})$ is continuous on $\mathcal{G}_n$.

Lemma 2. *$|\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda}| \le 2\pi\lambda^{-(l-1)/l}\varepsilon$ when $0 \le \varepsilon \le 1$.*

Proof. Considering that $\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda} = \{\vec{\varkappa} \in \mathcal{G}_n : \lambda \le \lambda_n(\vec{\varkappa}) < \lambda + \epsilon\}$, we get

$$|\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda}| = \int_{\mathcal{G}_n}\chi(\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda},\vec{\varkappa})\,d\vec{\varkappa} \quad = \int_{\lambda}^{\lambda+\varepsilon}\int_{\Theta_n(\lambda_n)}\frac{\varkappa_n(\lambda_n,\vec{\nu})}{\frac{\partial\lambda_n}{\partial\varkappa}}d\varphi d\lambda_n,$$

where $\Theta_n(\lambda_n) \subset [0, 2\pi)$ is the set of φ corresponding to $\mathcal{B}_n(\lambda_n)$. Using perturbation formulas (16), (21) we easily arrive at the required inequality. □

By (25), $E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda}) = E_n(\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda})$. Let us obtain an estimate for this projection.

Lemma 3. *For any $F \in C_0^\infty(\mathbb{R}^2)$ and $0 \le \varepsilon \le 1$,*

$$\left\| \big(E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda})\big)F \right\|^2_{L_2(\mathbb{R}^2)} \le C(F)\lambda^{-(l-1)/l}\epsilon, \tag{31}$$

where $C(F)$ is uniform with respect to n and λ.

Proof. Considering formula (25), we easily see that

$$\big(\big(E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda})\big)F, F\big) = \int_{\mathcal{G}_{n,\lambda+\varepsilon}\setminus\mathcal{G}_{n,\lambda}} \left|\big(F, \Psi_n(\vec{\varkappa})\big)\right|^2 d\vec{\varkappa}.$$

Using estimate (14) uniform in n for every cell of periods covering the support of F and summing up over such cells, we readily obtain

$$\left|\big(F, \Psi_n(\vec{\varkappa})\big)\right|^2 < C(F)$$

for all n and $\vec{\varkappa} \in \mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda}$. Hence, by Lemma 2,

$$\big(\left(E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda})\right)F, F\big) \le C(F)\left|\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda}\right| \le C(F)\lambda^{-(l-1)/l}\varepsilon.$$

Estimate (31) follows since $E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda})$ is a projection. □

2.2. **Sets $\mathcal{G}_\infty$ and $\mathcal{G}_{\infty,\lambda}$.** The sets $\mathcal{G}_\infty$, $\mathcal{G}_n$ are given by (12), (22). It is proven in [32] (Theorem 6.10) that

$$\mathcal{G}_{n+1} \subset \mathcal{G}_n. \tag{32}$$

$$\mathcal{G}_\infty = \bigcap_{n=1}^{\infty} \mathcal{G}_n. \tag{33}$$

Therefore, the perturbation formulas for $\lambda^{(n)}(\vec{\varkappa})$ and $\Psi_n(\vec{\varkappa})$ hold in $\mathcal{G}_\infty$ for all n. Moreover, coordinates (λ_n, φ) can be used in $\mathcal{G}_\infty$ for every n.

The following formulas, proven in [32], show that $\lambda^{(n)}(\vec{\varkappa})$ and $\Psi_n(\vec{\varkappa})$ approach $\lambda_\infty(\vec{\varkappa})$ and $\Psi_\infty(\vec{\varkappa})$ super-exponentially fast when $\vec{\varkappa} \in \mathcal{G}_\infty$. Indeed, let $\vec{\varkappa}$ belongs to a ring

$$R_{k,2k} = \left\{\vec{\varkappa} \in \mathbb{R}^2 : k < |\vec{\varkappa}| < 2k\right\} \tag{34}$$

for some $k : k^{2l} > \lambda_*$. Then,

$$\left|\lambda_\infty(\vec{\varkappa}) - \lambda^{(n)}(\vec{\varkappa})\right| < 24\epsilon_n^4, \quad n \ge 1, \tag{35}$$

$$\|\Psi_\infty - \Psi_n\|_{L_\infty(\mathbb{R}^2)} < Clk^{2l}\epsilon_n^3|Q_{n+1}|^{1/2}, \quad n \ge 1, \tag{36}$$

where

$$\epsilon_n = \exp\left(-\frac{1}{4}k^{\eta s_n}\right), \quad s_n = 2s_{n-1}, \quad s_1 = (2l-11)/32 \tag{37}$$

and Q_n is the elementary cell of periods of the operator H_n. It is formed by the periods $\tilde{N}_n\beta_1, \tilde{N}_n\beta_2$ of $W_n(x)$ and $\tilde{N}_n \approx k^{s_n}$, i.e.,

$$Q_n = [0, \tilde{N}_n\beta_1) \times [0, \tilde{N}_n\beta_1), \quad |Q_n| < C\beta_1\beta_2k^{2s_n}. \tag{38}$$

Let

$$\mathcal{G}_{\infty,\lambda} = \{\vec{\varkappa} \in \mathcal{G}_\infty : \lambda_\infty(\vec{\varkappa}) < \lambda\}. \tag{39}$$

The function $\lambda_\infty(\vec{\varkappa})$ is a Lebesgue measurable function, since it is a limit of the sequence of measurable functions. Hence, the set $\mathcal{G}_{\infty,\lambda}$ is measurable.

Lemma 4. *The measure of the symmetric difference of two sets $\mathcal{G}_{\infty,\lambda}$ and $\mathcal{G}_{n,\lambda}$ converges to zero as $n \to \infty$ uniformly in λ in every bounded interval:*

$$\lim_{n\to\infty} |\mathcal{G}_{\infty,\lambda}\Delta\mathcal{G}_{n,\lambda}| = 0,$$

where $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

Proof. Using the relation $\mathcal{G}_\infty \subset \mathcal{G}_n$ and estimate (35), we readily check that $\mathcal{G}_{\infty,\lambda} \subset \mathcal{G}_{n,\lambda+\delta_n}$, $\delta_n = 24\epsilon_n^4$, where ϵ_n is given by (37) with $k = \lambda^{1/2l}$. Therefore,

$$\mathcal{G}_{\infty,\lambda} \setminus \mathcal{G}_{n,\lambda} \subset \mathcal{G}_{n,\lambda+\delta_n} \setminus \mathcal{G}_{n,\lambda}.$$

Similarly, $\mathcal{G}_{\infty,\lambda} \supset \mathcal{G}_{n,\lambda-\delta_n} \cap \mathcal{G}_\infty$. Hence,

$$\mathcal{G}_{n,\lambda} \setminus \mathcal{G}_{\infty,\lambda} \subset \mathcal{G}_{n,\lambda} \cap \left(\mathcal{G}_{n,\lambda-\delta_n} \cap \mathcal{G}_\infty\right)^c \subset \left(\mathcal{G}_{n,\lambda} \setminus \mathcal{G}_{n,\lambda-\delta_n}\right) \cup \left(\mathcal{G}_{n,\lambda} \setminus \mathcal{G}_\infty\right).$$

Combining the two, we get

$$\mathcal{G}_{\infty,\lambda} \Delta \mathcal{G}_{n,\lambda} \subset \left(\mathcal{G}_{n,\lambda+\delta_n} \setminus \mathcal{G}_{n,\lambda-\delta_n}\right) \cup \left(\mathcal{G}_{n,\lambda} \setminus \mathcal{G}_\infty\right),$$

hence,

$$\left|\mathcal{G}_{\infty,\lambda} \Delta \mathcal{G}_{n,\lambda}\right| \leq \left|\mathcal{G}_{n,\lambda-\delta_n} \setminus \mathcal{G}_{n,\lambda+\delta_n}\right| + \left|\mathcal{G}_{n,\lambda} \setminus \mathcal{G}_\infty\right|.$$

Let us consider the first term of the right hand side. Using Lemma 2 with $\varepsilon = 2\delta_n$, we obtain $|\mathcal{G}_{n,\lambda-\delta_n} \setminus \mathcal{G}_{n,\lambda+\delta_n}| < 96\pi\lambda^{-(l-1)/l}\epsilon_n^4$. Using (37) for ϵ_n, we conclude easily that the first term goes to zero uniformly in λ. Obviously, $\mathcal{G}_{n,\lambda}$ is bounded uniformly in n when $n \to \infty$ and λ in every bounded interval. By (32) and (33) the second term goes to zero uniformly in λ in every bounded interval. □

2.3. Spectral Projections $E(\mathcal{G}_{\infty,\lambda})$.

In this section, we show that spectral projections $E_n(\mathcal{G}_{\infty,\lambda})$ have a strong limit $E_\infty(\mathcal{G}_{\infty,\lambda})$ in $L_2(\mathbb{R}^2)$ as n tends to infinity. The operator $E_\infty(\mathcal{G}_{\infty,\lambda})$ is a spectral projection of H. It can be represented in the form $E_\infty(\mathcal{G}_{\infty,\lambda}) = S_\infty T_\infty$, where S_∞ and T_∞ are strong limits of $S_n(\mathcal{G}_{\infty,\lambda})$ and $T_n(\mathcal{G}_{\infty,\lambda})$, respectively. For any $F \in C_0^\infty(\mathbb{R}^2)$, we show:

$$E_\infty\left(\mathcal{G}_{\infty,\lambda}\right) F = \frac{1}{4\pi^2} \int_{\mathcal{G}_{\infty,\lambda}} \big(F, \Psi_\infty(\vec{\varkappa})\big) \Psi_\infty(\vec{\varkappa}) d\vec{\varkappa}, \tag{40}$$

$$H E_\infty\left(\mathcal{G}_{\infty,\lambda}\right) F = \frac{1}{4\pi^2} \int_{\mathcal{G}_{\infty,\lambda}} \lambda_\infty(\vec{\varkappa}) \big(F, \Psi_\infty(\vec{\varkappa})\big) \Psi_\infty(\vec{\varkappa}) d\vec{\varkappa}. \tag{41}$$

Using properties of $E_\infty\left(\mathcal{G}_{\infty,\lambda}\right)$, we prove absolute continuity of the branch of the spectrum corresponding to functions $\Psi_\infty(\vec{\varkappa})$.

Now we consider the sequence of operators $T_n(\mathcal{G}_{\infty,\lambda})$ which are given by (27) and act from $L_2(\mathbb{R}^2)$ to $L_2(\mathcal{G}_{\infty,\lambda})$. We prove that the sequence has a strong limit and describe its properties.

Lemma 5. *The sequence $T_n(\mathcal{G}_{\infty,\lambda})$ has a strong limit $T_\infty(\mathcal{G}_{\infty,\lambda})$. The operator $T_\infty(\mathcal{G}_{\infty,\lambda})$ satisfies $\|T_\infty\| \leq 1$ and can be described by the formula $T_\infty F = \frac{1}{2\pi}\big(F, \Psi_\infty(\vec{\varkappa})\big)$ for any $F \in C_0^\infty(\mathbb{R}^2)$. The convergence of $T_n(\mathcal{G}_{\infty,\lambda})F$ to $T_\infty(\mathcal{G}_{\infty,\lambda})F$ is uniform in λ for every $F \in L_2(\mathbb{R}^2)$.*

Proof. Let $F \in C_0^\infty(\mathbb{R}^2)$. We consider $T_\infty F = \frac{1}{2\pi}\big(F, \Psi_\infty(\vec{\varkappa})\big)$. It follows from (36) and (27) that

$$\big|(T_\infty - T_n)F(\vec{\varkappa})\big| < C(F) g_n(\vec{\varkappa}), \quad g_n(\vec{\varkappa}) = l k^{2l} \epsilon_n^3 |Q_{n+1}|^{1/2}, \quad k = |\vec{\varkappa}|.$$

It is easy to see from (37), (38) that $g_n(\vec{\varkappa}) \in L_2(\mathcal{G}_\infty)$ for all n and $g_n(\vec{\varkappa})$ tends to zero in $L_2(\mathcal{G}_\infty)$ as $n \to \infty$. Therefore, $g_n(\vec{\varkappa})$ tends to zero in $L_2(\mathcal{G}_{\infty,\lambda})$ uniformly in λ. Hence, $\big\|(T_\infty - T_n)F\big\|_{L_2(\mathcal{G}_{\infty,\lambda})}$ tends to zero uniformly in λ for every $F \in C_0^\infty(\mathbb{R}^2)$ as $n \to \infty$. Considering $\|T_n\| \leq 1$, we obtain that $T_n F$ has a limit for every $F \in L_2(\mathbb{R}^2)$ uniformly in λ. The estimate $\|T_\infty\| \leq 1$ is now obvious. □

Now we consider the sequence of operators $S_n(\mathcal{G}_{\infty,\lambda})$ which are given by (28) with $\mathcal{G}_n' = \mathcal{G}_{\infty,\lambda}$:

$$S_n(\mathcal{G}_{\infty,\lambda}):\ L_2(\mathcal{G}_{\infty,\lambda}) \to L_2(\mathbb{R}^2).$$

We prove that the sequence has a strong limit and describe its properties.

Lemma 6. *The sequence of operators $S_n(\mathcal{G}_{\infty,\lambda})$ has a strong limit $S_\infty(\mathcal{G}_{\infty,\lambda})$. The operator $S_\infty(\mathcal{G}_{\infty,\lambda})$ satisfies $\|S_\infty\| \le 1$ and can be described by the formula*

$$(S_\infty f)(x) = \frac{1}{2\pi}\int_{\mathcal{G}_{\infty,\lambda}} f(\vec{\varkappa})\Psi_\infty(\vec{\varkappa},x)d\vec{\varkappa} \tag{42}$$

for any $f \in L_\infty(\mathcal{G}_{\infty,\lambda})$. The convergence of $S_n(\mathcal{G}_{\infty,\lambda})f$ to $S_\infty(\mathcal{G}_{\infty,\lambda})f$ is uniform in λ for every $f \in L_2(\mathcal{G}_\infty)$.

Proof. We start with proving that $S_n(\mathcal{G}_{\infty,\lambda})f$ is a Cauchy sequence in $L_2(\mathbb{R}^2)$ for every $f \in L_\infty(\mathcal{G}_{\infty,\lambda})$. Since Q_n is the cell of periods of the operator $H^{(n)}$, see (38), the function $\Psi_n(\vec{\varkappa},x)$ is quasiperiodic in Q_n. It can be represented as a combination of plane waves:

$$\Psi_n(\vec{\varkappa},x) = \frac{1}{2\pi}\sum_{r\in\mathbb{Z}^2} c_r^{(n)}(\vec{\varkappa})\exp i\langle\vec{\varkappa}+\vec{p}_r(0)/\tilde{N}_n, x\rangle, \tag{43}$$

where $c_r^{(n)}(\vec{\varkappa})$ are Fourier coefficients and $\vec{p}_r(0) = (\frac{2\pi r_1}{\beta_1}, \frac{2\pi r_2}{\beta_2})$. The Fourier transform of $\widehat{\Psi}_n$ is a combination of δ-functions:

$$\widehat{\Psi}_n(\vec{\varkappa},\vec{\xi}) = \sum_{r\in\mathbb{Z}^2} c_r^{(n)}(\vec{\varkappa})\delta\big(\vec{\xi}+\vec{\varkappa}+\vec{p}_r(0)/\tilde{N}_n\big).$$

From this, we compute easily the Fourier transform of $S_n f$

$$(\widehat{S_n f})(\vec{\xi}) = \frac{1}{2\pi}\sum_{r\in\mathbb{Z}^2} c_r^{(n)}\big(-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big)f\big(-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big)\chi\big(\mathcal{G}_{\infty,\lambda},-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big),$$

where $\chi(\mathcal{G}_{\infty,\lambda},\cdot)$ is the characteristic function on $\mathcal{G}_{\infty,\lambda}$. Since $\mathcal{G}_{\infty,\lambda}$ is bounded, the series contains only a finite number of non-zero terms for every $\vec{\xi}$. By Parseval's identity, triangle inequality and a parallel shift of the variable,

$$\|S_n f\|_{L_2(\mathbb{R}^2)} = \|\widehat{S_n f}\|_{L_2(\mathbb{R}^2)} \le$$

$$\frac{1}{2\pi}\sum_{r\in\mathbb{Z}^2}\Big\|c_r^{(n)}\big(-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big)f\big(-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big)\chi\big(\mathcal{G}_{\infty,\lambda},-\vec{\xi}-\vec{p}_r(0)/\tilde{N}_n\big)\Big\|_{L_2(\mathbb{R}^2)} =$$

$$\frac{1}{2\pi}\sum_{r\in\mathbb{Z}^2}\|c_r^{(n)}(\vec{\varkappa})f(\vec{\varkappa})\|_{L_2(\mathcal{G}_{\infty,\lambda})}.$$

Assume first that the support of f belongs to a ring $R_{k,2k}$ in (34) for some k such that $k^{2l} > \lambda_*(V)$. Then, the last inequality yields:

$$\|S_n f\|_{L_2(\mathbb{R}^2)} \le \frac{1}{2\pi}\|f\|_{L_\infty(R_{k,2k})}\sum_{r\in\mathbb{Z}^2}\|c_r^{(n)}\|_{L_2(R_{k,2k})} \le$$

$$\frac{1}{2\pi}\|f\|_{L_\infty(R_{k,2k})}\left(\sum_{r\in\mathbb{Z}^2} p_r^{4l}(0)\|c_r^{(n)}\|^2_{L_2(R_{k,2k})}\right)^{1/2}\left(\sum_{r\in\mathbb{Z}^2} p_r^{-4l}(0)\right)^{1/2},$$

where we used Cauchy-Schwarz inequality. By (43), Fourier coefficients $c_r^{(n)}(\vec{\varkappa})$ can be estimated as follows:

$$\sum_{r\in\mathbb{Z}^2} p_r^{4l}(0)|c_r^{(n)}(\vec{\varkappa})|^2 \leq 4\pi^2\|\Psi_n(\vec{\varkappa},\cdot)\exp\left(-i\langle\vec{\varkappa},\cdot\rangle\right)\|^2_{W_2^{2l}(Q_n)}|Q_n|^{-1}\tilde{N}_n^{4l} \leq$$

$$8\pi^2|\vec{\varkappa}|^{4l}\|\Psi_n(\vec{\varkappa},\cdot)\|^2_{W_2^{2l}(Q_n)}|Q_n|^{-1}\tilde{N}_n^{4l}.$$

Integrating the last inequality over $R_{k,2k}$, we arrive at

$$\sum_{r\in\mathbb{Z}^2} p_r^{4l}(0)\|c_r^{(n)}\|^2_{L_2(R_{k,2k})} \leq ck^{4l+2}|Q_n|^{-1}\tilde{N}_n^{4l}\sup_{\vec{\varkappa}\in R_{k,2k}}\|\Psi_n(\vec{\varkappa},\cdot)\|^2_{W_2^{2l}(Q_n)}.$$

Considering that $\sum_r p_r^{-4l}(0) < c$, we obtain

$$\|S_nf\|_{L_2(\mathbb{R}^2)} < ck^{2l+1}\|f\|_{L_\infty(R_{k,2k})}|Q_n|^{-1/2}\tilde{N}_n^{2l}\sup_{\vec{\varkappa}\in R_{k,2k}}\|\Psi_n(\vec{\varkappa},\cdot)\|_{W_2^{2l}(Q_n)}.$$

Similarly,

$$\|(S_{n+1}-S_n)f\|_{L_2(\mathbb{R}^2)}$$
$$< ck^{2l+1}\|f\|_{L_\infty(R_{k,2k})}|Q_{n+1}|^{-1/2}\tilde{N}_n^{2l}\sup_{\vec{\varkappa}\in R_{k,2k}}\|\big(\Psi_{n+1}(\vec{\varkappa},\cdot)-\Psi_n(\vec{\varkappa},\cdot)\big)\|_{W_2^{2l}(Q_{n+1})}.$$

It is proven in [32] (Section 6.2) that

$$\|\Psi_{n+1}(\vec{\varkappa},\cdot)-\Psi_n(\vec{\varkappa},\cdot)\|_{W_2^{2l}(Q_{n+1})} < ck^{2l}\epsilon_n^3|Q_{n+1}|^{1/2}, \quad n\geq 1, \quad \text{when} \quad \vec{\varkappa}\in R_{k,2k}. \quad (44)$$

Using the last estimate, we obtain

$$\|(S_n-S_{n+1})f\|_{L_2(\mathbb{R}^2)} \leq ck^{4l+1}\|f\|_{L_\infty(R_{k,2k})}\tilde{N}_n^{2l}\epsilon_n^3. \quad (45)$$

Considering that ϵ_n decays super-exponentially with n (see (37)) and the estimate $\tilde{N}_n \approx k^{s_n}$, we conclude that S_nf is a Cauchy sequence in $L_2(\mathbb{R}^2)$ for every $f\in L_\infty(R_{k,2k})$.

If $f\in L_\infty(\mathcal{G}_{\infty,\lambda})$, then we can express it as a sum of functions f_k such that f_k has the support in $R_{k,2k}$ and $\|f\|_{L_\infty(R_{k,2k})}\leq\|f\|_{L_\infty(\mathcal{G}_{\infty,\lambda})}$. Summing up estimates (45) over all k, we easily see that S_nf is a Cauchy sequence in $L_2(\mathbb{R}^2)$. We denote the limit of $S_n(\mathcal{G}_{\infty,\lambda})f$ by $S_\infty(\mathcal{G}_{\infty,\lambda})f$.

We see from formula (28) and estimate (36) that

$$\lim_{n\to\infty}\big(S_n(\mathcal{G}_{\infty,\lambda})f\big)(x) = \frac{1}{2\pi}\int_{\mathcal{G}_{\infty,\lambda}} f(\vec{\varkappa})\Psi_\infty(\vec{\varkappa},x)d\vec{\varkappa},$$

for all $x\in\mathbb{R}^2$ when $f\in L_\infty(\mathcal{G}_{\infty,\lambda})$. Hence, (42) holds.

Since $\|S_n\|\leq 1$, the limit $S_\infty(\mathcal{G}_{\infty,\lambda})f$ exists for all $f\in L_2(\mathcal{G}_{\infty,\lambda})$. Note that convergence is uniform in λ for every $f\in L_2(\mathcal{G}_\infty)$. It is obvious now that $\|S_\infty\|\leq 1$. □

Lemma 7. *Spectral projections $E_n(\mathcal{G}_{\infty,\lambda})$ have a strong limit $E_\infty(\mathcal{G}_{\infty,\lambda})$ in $L_2(\mathbb{R}^2)$, the convergence being uniform in λ for every element. For any $F\in C_0^\infty(\mathbb{R}^2)$ the operator $E_\infty(\mathcal{G}_{\infty,\lambda})$ is a projection given by (40) and formula (41) holds.*

Proof. By (26), $E_n = S_nT_n$. By lemmas 5 and 6 both S_n and T_n have strong limits S_∞, T_∞ and $\|S_n\|\leq 1$, $\|T_n\|\leq 1$. It follows easily that E_n has the strong limit $E_\infty = S_\infty T_\infty$. Since E_n is a sequence of projections, its strong limit satisfies the relations: $E_\infty = E_\infty^*$, $E_\infty^2 = E_\infty$. Hence E_∞ is a projection, see e.g. [37]. Using last two lemmas and considering that $T_\infty(\mathcal{G}_{\infty,\lambda})F\in L_\infty(\mathcal{G}_{\infty,\lambda})$ for any $F\in C_0^\infty(\mathbb{R}^2)$, we arrive at (40). Applying differential equation $H\Psi_\infty = \lambda_\infty(\vec{k})\Psi_\infty$, we obtain (41). It remains to prove that convergence of E_nF is uniform in λ for every $F \in L_2(\mathbb{R}^2)$. First, let $F \in C_0^\infty(\mathbb{R}^2)$. By the triangle inequality,

$$\|(E_\infty - E_n)F\| \le \|(S_\infty - S_n)T_\infty F\| + \|S_n(T_\infty - T_n)F\|.$$

Since T_nF converges to $T_\infty F$ uniformly in λ and $\|S_n\| \le 1$, the second term goes to zero uniformly in λ. We see easily from (40) that $T_\infty F \in L_\infty(\mathcal{G}_\infty)$. Then, by Lemma 6, $S_nT_\infty F$ converges to $E_\infty(\mathcal{G}_{\infty,\lambda})F$ uniformly in λ. This mean that $E_n(\mathcal{G}_{\infty,\lambda})F$ converges to $E_\infty(\mathcal{G}_{\infty,\lambda})F$ uniformly in λ for $F \in C_0^\infty(\mathbb{R}^2)$. Using $\|E_n\| = 1$, we obtain that uniform convergence holds for all $F \in L_2(\mathbb{R}^2)$.

□

Lemma 8. *There is a strong limit $E_\infty(\mathcal{G}_\infty)$ of the projections $E_\infty(\mathcal{G}_{\infty,\lambda})$ as λ goes to infinity.*

Corollary 9. *The operator $E_\infty(\mathcal{G}_\infty)$ is a projection.*

Proof. Considering that $\lim_{n\to\infty} E_n(\mathcal{G}_{\infty,\lambda}) = E_\infty(\mathcal{G}_{\infty,\lambda})$ and $E_n(\mathcal{G}_{\infty,\lambda})$ is a monotone in λ, we conclude that $E_\infty(\mathcal{G}_{\infty,\lambda})$ is monotone too. It is well-known that a monotone sequence of projections has a strong limit. □

Lemma 10. *Projections $E_\infty(\mathcal{G}_{\infty,\lambda})$, $\lambda \in \mathbb{R}$, and $E_\infty(\mathcal{G}_\infty)$ reduce the operator H.*

Proof. Let us show $E_\infty(\mathcal{G}_{\infty,\lambda})$ reduces H, i.e., $E_\infty(\mathcal{G}_{\infty,\lambda})Dom(H) \subset Dom(H)$ and $E_\infty(\mathcal{G}_{\infty,\lambda})H = HE_\infty(\mathcal{G}_{\infty,\lambda})$ on $Dom(H)$ (e.g., see Theorem 40.2 in [39]). For any $F,\ G \in Dom(H) = Dom(H^{(n)})$,

$$\begin{aligned}\big(F, E_\infty(\mathcal{G}_{\infty,\lambda})HG\big) &= \big(E_\infty(\mathcal{G}_{\infty,\lambda})F, HG\big) = \lim_{n\to\infty}\big(E_n(\mathcal{G}_{\infty,\lambda})F, H^{(n)}G\big)\\ &= \lim_{n\to\infty}\big(H^{(n)}E_n(\mathcal{G}_{\infty,\lambda})F, G\big) = \lim_{n\to\infty}\big(E_n(\mathcal{G}_{\infty,\lambda})H^{(n)}F, G\big)\\ &= \lim_{n\to\infty}\big(H^{(n)}F, E_n(\mathcal{G}_{\infty,\lambda})G\big) = \big(HF, E_\infty(\mathcal{G}_{\infty,\lambda})G\big) = \big(E_\infty(\mathcal{G}_{\infty,\lambda})HF, G\big).\end{aligned}$$

Hence, $E_\infty(\mathcal{G}_{\infty,\lambda})H$ is symmetric. Since $E_\infty(\mathcal{G}_{\infty,\lambda})$ is bounded, $(E_\infty(\mathcal{G}_{\infty,\lambda})H)^* = HE_\infty(\mathcal{G}_{\infty,\lambda})$ (e.g., see §115 in [37]). Therefore, $E_\infty(\mathcal{G}_{\infty,\lambda})H \subset HE_\infty(\mathcal{G}_{\infty,\lambda})$ which means that for every $F \in Dom(H)$, $E_\infty(\mathcal{G}_{\infty,\lambda})F \in Dom(H)$ and $E_\infty(\mathcal{G}_{\infty,\lambda})HF = HE_\infty(\mathcal{G}_{\infty,\lambda})F$. Thus, $E_\infty(\mathcal{G}_{\infty,\lambda})$ reduces H.

Now we show that $E_\infty(\mathcal{G}_\infty)$ reduces H. Noting that $E_\infty(\mathcal{G}_\infty)$ is the strong limit of $E_\infty(\mathcal{G}_{\infty,\lambda})$ as $\lambda \to \infty$, for any $F,\ G \in Dom(H)$,

$$\begin{aligned}\big(F, E_\infty(\mathcal{G}_\infty)HG\big) &= \lim_{\lambda\to\infty}\big(F, E_\infty(\mathcal{G}_{\infty,\lambda})HG\big) = \lim_{\lambda\to\infty}\big(HE_\infty(\mathcal{G}_{\infty,\lambda})F, G\big)\\ &= \lim_{\lambda\to\infty}\big(E_\infty(\mathcal{G}_{\infty,\lambda})HF, G\big) = \big(E_\infty(\mathcal{G}_\infty)HF, G\big),\end{aligned}$$

i.e., $E_\infty(\mathcal{G}_\infty)H$ is symmetric. Considering $(E_\infty(\mathcal{G}_\infty)H)^* = HE_\infty(\mathcal{G}_\infty)$ as before, we obtain $E_\infty(\mathcal{G}_\infty)H \subset HE_\infty(\mathcal{G}_\infty)$ which means that for every $F \in Dom(H)$, $E_\infty(\mathcal{G}_\infty)F \in Dom(H)$ and $E_\infty(\mathcal{G}_\infty)HF = HE(\mathcal{G}_\infty)F$. Thus, $E_\infty(\mathcal{G}_\infty)$ reduces H too. □

Lemma 11. *The family of projections $E_\infty(\mathcal{G}_\infty, \lambda)$ is the resolution of the identity of the operator $HE_\infty(\mathcal{G}_\infty)$ acting in $E_\infty(\mathcal{G}_\infty)L_2(\mathbb{R}^2)$.*

Proof. First, we show that $\lim_{\lambda\to-\infty} E_\infty(\mathcal{G}_{\infty,\lambda}) = 0$. It is enough to check that $\mathcal{G}_{\infty,\lambda} = \emptyset$ for every $\lambda < \lambda_*$. We see from the definitions (22) and (30) of $\mathcal{G}_n$ and $\mathcal{G}_{n,\lambda}$, respectively, that $\mathcal{G}_{n,\lambda_*} = \emptyset$. It follows from (35) and (39) that $\mathcal{G}_{\infty,\lambda_*-\delta_n} \subset \mathcal{G}_{n,\lambda_*}$, here $\delta_n = 24\epsilon_n^4$, $n \ge 2$. Hence, $\mathcal{G}_{\infty,\lambda} = \emptyset$ for every $\lambda < \lambda_*$.

Second, $\lim_{\lambda\to\infty} E_\infty(\mathcal{G}_{\infty,\lambda}) = E_\infty(\mathcal{G}_\infty)$ by Lemma 8.

Third, the family $E_\infty(\mathcal{G}_{\infty,\lambda})$ is left-continuous since each $E_n(\mathcal{G}_{\infty,\lambda})$ is left-continuous and $E_n(\mathcal{G}_{\infty,\lambda})F$ converges to $E_\infty(\mathcal{G}_{\infty,\lambda})F$ uniformly in λ for every F (Lemma 7).

Fourth, let $\lambda > \mu$. Then,

$$\big(E_\infty(\mathcal{G}_{\infty,\lambda})E_\infty(\mathcal{G}_{\infty,\mu})F, G\big) = \big(E_\infty(\mathcal{G}_{\infty,\mu})F, E_\infty(\mathcal{G}_{\infty,\lambda})G\big) = \lim_{n\to\infty}\big(E_n(\mathcal{G}_{\infty,\mu})F, E_n(\mathcal{G}_{\infty,\lambda})G\big)$$
$$= \lim_{n\to\infty}\big(E_n(\mathcal{G}_{\infty,\lambda})E_n(\mathcal{G}_{\infty,\mu})F, G\big) = \lim_{n\to\infty}\big(E_n(\mathcal{G}_{\infty,\mu})F, G\big) = \big(E_\infty(\mathcal{G}_{\infty,\mu})F, G\big).$$

This means that $E_\infty(\mathcal{G}_{\infty,\lambda})E_\infty(\mathcal{G}_{\infty,\mu}) = E_\infty(\mathcal{G}_{\infty,\mu})$.

Last, we check that for any $g \in [E_\infty(\mathcal{G}_{\infty,\lambda}) - E_\infty(\mathcal{G}_{\infty,\mu})]D(H),\ \lambda > \mu$,

$$\mu\|g\|^2 \le \big(Hg, g\big) \le \lambda\|g\|^2. \tag{46}$$

In fact, let

$$g = [E_\infty(\mathcal{G}_{\infty,\lambda}) - E_\infty(\mathcal{G}_{\infty,\mu})]F, \quad F \in C_0^\infty(\mathbb{R}^2). \tag{47}$$

By (40) and (41),

$$g(x) = \frac{1}{4\pi^2}\int_{\mathcal{G}_{\infty,\lambda}\setminus\mathcal{G}_{\infty,\mu}} \big(F, \Psi_\infty(\vec{\varkappa})\big)\Psi_\infty(x)d\vec{\varkappa},$$

$$Hg(x) = \frac{1}{4\pi^2}\int_{\mathcal{G}_{\infty,\lambda}\setminus\mathcal{G}_{\infty,\mu}} \lambda_\infty(\vec{\varkappa})\big(F, \Psi_\infty(\vec{\varkappa})\big)\Psi_\infty(x)d\vec{\varkappa},$$

$$\|g\|^2_{L_2(R^2)} = \big(g, F\big) = \frac{1}{4\pi^2}\int_{\mathcal{G}_{\infty,\lambda}\setminus\mathcal{G}_{\infty,\mu}} \left|\big(F, \Psi_\infty(\vec{\varkappa})\big)\right|^2 d\vec{\varkappa}, \tag{48}$$

$$\big(Hg, g\big) = \big(Hg, F\big) = \frac{1}{4\pi^2}\int_{\mathcal{G}_{\infty,\lambda}\setminus\mathcal{G}_{\infty,\mu}} \lambda_\infty(\vec{\varkappa})\left|\big(F, \Psi_\infty(\vec{\varkappa})\big)\right|^2 d\vec{\varkappa}. \tag{49}$$

By the definitions of $\mathcal{G}_{\infty,\mu}$ and $\mathcal{G}_{\infty,\lambda}$, the inequality $\mu \le \lambda_\infty(\vec{\varkappa}) < \lambda$ holds when $\vec{\varkappa} \in \mathcal{G}_{\infty,\lambda} \setminus \mathcal{G}_{\infty,\mu}$. Using the last equality in (49) and considering (48), we obtain (46) for all g given by (47). Since $C_0^\infty(\mathbb{R}^2)$ is dense in $Dom(H)$ with respect to $\|F\|_{L_2(\mathbb{R}^2)} + \|HF\|_{L_2(\mathbb{R}^2)}$ norm, inequality (46) can be extended to all $g = [E_\infty(\mathcal{G}_{\infty,\lambda}) - E_\infty(\mathcal{G}_{\infty,\mu})]F$, $F \in Dom(H)$.

From five properties of $E_\infty(\mathcal{G}_{\infty,\lambda})$ proved above, it follows that $E_\infty(\mathcal{G}_{\infty,\lambda})$ is the resolution of identity belonging to $HE_\infty(\mathcal{G}_\infty)$ [39]. □

2.4. **Proof of Absolute Continuity.**

Now we show that the branch of spectrum (semi-axis) corresponding to $\mathcal{G}_\infty$ is absolutely continuous.

Theorem 12. *For any $F \in C_0^\infty(\mathbb{R}^2)$ and $0 \le \varepsilon \le 1$,*

$$\left|\big((E_\infty(\mathcal{G}_{\infty,\lambda+\varepsilon}) - E_\infty(\mathcal{G}_{\infty,\lambda}))F, F\big)\right| \le C_F\lambda^{-(l-1)/l}\varepsilon. \tag{50}$$

Corollary 13. *The spectrum of the operator $HE_\infty(\mathcal{G}_\infty)$ is absolutely continuous.*

Proof. By formula (40),

$$\left|\big((E_\infty(\mathcal{G}_{\infty,\lambda+\varepsilon}) - E(\mathcal{G}_{\infty,\lambda}))F, F\big)\right| \le C_F\left|\mathcal{G}_{\infty,\lambda+\varepsilon} \setminus \mathcal{G}_{\infty,\lambda}\right|.$$

Applying Lemmas 2 and 4, we immediately get (50).

□

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