Moments determine long-time asymptotics for diffusion equations

정재환 (鄭在桓 Chung, Jaywan)
수리과학과
Department of Mathematical Sciences

KAIST

2011
Moments determine long-time asymptotics for diffusion equations
Moments determine long-time asymptotics for diffusion equations

Advisor : Professor Kim, Yong Jung

by
Chung, Jaywan
Department of Mathematical Sciences
KAIST

A thesis submitted to the faculty of KAIST in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematical Sciences. The study was conducted in accordance with Code of Research Ethics\textsuperscript{1}.

2010. 11. 25.

Approved by
Professor Kim, Yong Jung
[Advisor]

\textsuperscript{1}Declaration of Ethical Conduct in Research: I, as a graduate student of KAIST, hereby declare that I have not committed any acts that may damage the credibility of my research. These include, but are not limited to: falsification, thesis written by someone else, distortion of research findings or plagiarism. I affirm that my thesis contains honest conclusions based on my own careful research under the guidance of my thesis advisor.
모멘트가 결정하는 오랜 시간 후 확산 방정식의 접근 행동

정재환

위 논문은 한국과학기술원 박사학위논문으로 학위논문심사위원회에서 심사 통과하였음.

2010년 11월 25일

심사위원장 김용정 (인)
심사위원 권순식 (인)
심사위원 김홍오 (인)
심사위원 노준영 (인)
심사위원 Marshall Slemrod (인)
The purpose of this thesis is to investigate relation between moments of initial data and long-time asymptotics of diffusion equations. More precisely, $L^1$-intermediate asymptotics for nonlinear diffusion equations, approximate solutions to the viscous Burgers equation and long-time asymptotics of the zero level set for the heat equation will be discussed.

In Chapter 2 Newtonian potential is introduced in a relative sense for radial functions. This makes us possible to treat the potential theory for a larger class of functions in a unified manner for all dimensions $d \geq 1$. For example, Newton’s theorem can be restated in a simpler form without concerning dimensions. The relative potential is then used to obtain the $L^1$-convergence order $O(t^{-1})$ as $t \to \infty$ for radially symmetric solutions to the porous medium and fast diffusion equations. Similar technique is also applied to radial solutions of the $p$-Laplacian equations to obtain the same convergence order.

In Chapter 3 two kinds of approximate solutions to the heat equation are discussed. They will be used in the following chapters.

In Chapter 4 relation between the moments and the asymptotic behavior of solutions to the viscous Burgers equation is investigated. The Burgers equation is a nonlinear problem having a special property; it can be transformed to a linear problem via the Cole-Hopf transformation. Our asymptotic analysis depends on the transformation. In the chapter an asymptotic approximate solution is constructed, which is given by the inverse Cole-Hopf transformation of a summation of $n$ heat kernels. The $k$-th order moments of exact solution and the approximate solution are contracting with order $O((\sqrt{t})^{k-2n-1+1/p})$ in $L^p$-norm as $t \to \infty$. This asymptotics indicates that the convergence order is increased by a similarity scale whenever the order of controlled moments is increased by one. The theoretical asymptotic convergence orders are tested numerically.

In Chapter 5 we consider the zero set $Z(t) := \{x \in \mathbb{R}^d : u(x, t) = 0\}$ of a solution $u$ to the heat equation in $\mathbb{R}^d$. Under vanishing conditions on moments of the initial data, we will prove the set $Z(t)$ in a ball of radius $C\sqrt{t}$ for some $C > 0$ converges to a specific graph as $t \to \infty$ when the set is divided by $\sqrt{t}$. The graph is zeros of a linear combination of the Hermite polynomials and the coefficients of the linear combination depends on moments of the initial data. Also the graphs to which the zero set $Z(t)$ converges will be classified in some cases.
Contents

Abstract ................................................................. i
Contents ................................................................. ii
List of Tables .......................................................... iv
List of Figures .......................................................... v

Chapter 1. Introduction ................................................... 1

Chapter 2. Relative Newtonian Potentials for Radial Functions and $L^1$-Intermediate Asymptotics for Nonlinear Diffusion 4
  2.1 Introduction ......................................................... 4
  2.2 Relative Potential of Radial Functions ............................ 6
  2.3 The Porous Medium and Fast Diffusion Equation .................. 9
  2.4 The $p$-Laplacian Equation ....................................... 15
  2.5 Open Problems ...................................................... 18

Chapter 3. Approximations for the Heat Equation 19
  3.1 Duoandikoetxea and Zuazua’s Approximation ..................... 19
  3.2 Kim and Ni’s Approximation ....................................... 26
  3.3 Open Problems ...................................................... 27

Chapter 4. Asymptotic Agreement of Moments and Higher Order Contraction in the Viscous Burgers Equation 28
  4.1 Introduction ......................................................... 28
  4.2 Long-time Asymptotics for the Heat Equation .................... 30
    4.2.1 Approximate Solutions to the Heat Equation ............... 30
    4.2.2 Contraction Rates of Moments ............................... 32
  4.3 Long-time Asymptotics for the Burgers Equation ................ 34
  4.4 Fine Asymptotics and the Similarity Scale ....................... 37
  4.5 Numerical Examples ............................................... 38
  4.6 Open Problems ...................................................... 41

Chapter 5. Long-time Asymptotics of the Zero Level Set for the Heat Equation 44
  5.1 Introduction ......................................................... 44
  5.2 Hermite polynomial approximation ................................ 45
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.3</td>
<td>Long-time Asymptotics of the Zero Level Set</td>
<td>46</td>
</tr>
<tr>
<td>5.4</td>
<td>One-dimensional Heat Equation</td>
<td>49</td>
</tr>
<tr>
<td>5.5</td>
<td>Radially Symmetric Initial Data</td>
<td>50</td>
</tr>
<tr>
<td>5.6</td>
<td>Two-dimensional Heat Equation</td>
<td>53</td>
</tr>
<tr>
<td>5.7</td>
<td>Numerical Examples</td>
<td>54</td>
</tr>
<tr>
<td>5.8</td>
<td>Open Problems</td>
<td>55</td>
</tr>
</tbody>
</table>

**Summary (in Korean)** 58

**References** 59
List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>$L^\infty$-errors for Burgers equation</td>
<td>41</td>
</tr>
<tr>
<td>4.2</td>
<td>Approximation error without backward moments</td>
<td>42</td>
</tr>
<tr>
<td>4.3</td>
<td>Approximation error with backward time</td>
<td>42</td>
</tr>
</tbody>
</table>
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Relation among ( \rho, u ) and ( \rho T ).</td>
<td>12</td>
</tr>
<tr>
<td>4.1</td>
<td>Initial data approximation for Burgers equation.</td>
<td>39</td>
</tr>
<tr>
<td>4.2</td>
<td>Approximation to Burgers equation as ( n ) increases.</td>
<td>40</td>
</tr>
<tr>
<td>5.1</td>
<td>The zero level set of the polynomial ( \phi_1(x) ) when ( d = 2 ).</td>
<td>53</td>
</tr>
<tr>
<td>5.2</td>
<td>The zero level set of the polynomial ( \phi_2(x) ) when ( d = 2 ).</td>
<td>54</td>
</tr>
<tr>
<td>5.3</td>
<td>The zero level set is a straight line.</td>
<td>55</td>
</tr>
<tr>
<td>5.4</td>
<td>The zero level set is an ellipse.</td>
<td>55</td>
</tr>
<tr>
<td>5.5</td>
<td>The zero level set is a hyperbola.</td>
<td>56</td>
</tr>
<tr>
<td>5.6</td>
<td>The zero level set is a rectangular hyperbola.</td>
<td>56</td>
</tr>
<tr>
<td>5.7</td>
<td>The zero level set consists of two parallel lines.</td>
<td>57</td>
</tr>
</tbody>
</table>
Chapter 1. Introduction

The heat equation in the whole domain with an integrable initial data \( u_0 \) is
\[
\begin{align*}
    u_t &= \Delta u \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
    u(x,0) &= u_0(x) \in L^1(\mathbb{R}^d)
\end{align*}
\]
and it describes a diffusion process of heat or a chemical. There are many other partial differential equations describing a diffusion process and a general form including them is the diffusion equation
\[
\frac{\partial u}{\partial t}(x,t) = \nabla \cdot [D(u,x) \nabla u(x,t)],
\]
where \( D(u,x) \) is the diffusion coefficient for density \( u \) at location \( x \). (Note that the heat equation comes from the case \( D(u,x) \equiv 1 \).) Density distribution in a diffusion process converges to zero uniformly as time goes to infinity. Nevertheless, scaling by so called self-similar variables, the density distribution looks like a specific solution of the diffusion equation and we call both the phenomenon and the solution self-similar. For example, the heat equation can be rewritten as
\[
U_\tau = \frac{1}{4} \sum_{i=1}^d \frac{\partial}{\partial y_i} \left( e^{-y_i^2} \frac{\partial}{\partial y_i} \left[ e^{y_i^2} U \right] \right)
\]
with change of variables \( u(x,t) = t^{-d/2} U(y, \tau) \), \( y = (y_i) = x/(2\sqrt{t}) \) and \( \tau = \ln t \). Hence a function \( Ce^{-|y|^2} \) is a steady state for any constant \( C \). The function is a heat kernel in the original variables and actually a solution converges to a heat kernel faster than it converges to zero. Accordingly, every solutions looks like a heat kernel in an intermediate scale. In this respect, asymptotic behavior toward self-similar solution is called intermediate asymptotics. A book by Barenblatt \[7\] is a good reference about the intermediate asymptotics using dimensional analysis.

In this thesis, we will deal three topics on long-time asymptotics for diffusion equations in \( \mathbb{R}^d \): \( L^1 \)-intermediate asymptotics, approximations to solutions and the zero level sets. Also it would be clarified that the long-time asymptotics is determined by moments of the initial data.

Consider a nonlinear version of the heat equation
\[
u_t = \Delta(u^m) = \nabla \cdot (mu^{m-1} \nabla u), \quad m > (d-2)_+/d.
\]
If \( m < 1 \), this equation is called the fast diffusion equation (FDE). If \( m > 1 \), it is called the porous medium equation (PME). This equation has a temperature depending conductivity \( mu^{m-1} \). On the other hand, for a fixed \( p > 1 \), the \( p \)-Laplacian equation (PLE) is given by
\[
u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad p > 2d/(d+1).
\]
This equation has gradient depending diffusivity \( |\nabla u|^{p-2} \). For these equations, we prove the following in Chapter 2

**Theorem 1.** Let the initial data \( u_0(x) \geq 0 \) be radially symmetric, compactly supported, and of mass \( M = \int u_0(x) \, dx \). Then the solution \( u(x,t) \) to the PME or FDE satisfies
\[
\|u(\cdot, t) - \rho(\cdot, t)\|_1 = O(t^{-1}) \quad \text{as } t \to \infty,
\]
where $\rho(x,t)$ is the Barenblatt solution given by (2.5). If $u(x,t)$ is the solution to PDE, then the same convergence order holds with a Barenblatt-type solution $\rho(x,t)$ given by (2.2).

The Barenblatt solution is a self-similar solution which becomes a steady state solution when scaled by self-similar variables. Because solutions in the theorem are radially symmetric, they have zero first moment; we are imposing a stronger condition than assuming same zeroth and first moments for $\rho$ and $u$. The contraction rate $O(t^{-1})$ has been shown for $(d - 2)_+ / d < m < d / (d + 2)$ in [13] and a similar rate $O(t^{-1+})$ has been shown for $(d - 1)/d < m < 1$ in [53]. For radial solutions or for one dimension such a contraction rate has been shown for all $m > (d - 2)_+ / d$ using a potential comparison or a mass concentration comparison [17, 13, 47, 60, 68]. Therefore, the contraction rate in the Theorem is not new for FDE and PME cases. However, the point is that we propose a simple proof using the Newtonian potentials in an unified way for all dimensions $d \geq 1$ and all exponents $m > (d - 2)_+ / d$. Chapter 2 is a joint work with Yong Jung Kim.

In Chapter 3, two approximate functions of solutions to the heat equation will be introduced; one by Duoandikoetxea and Zuazua [28] and the other by Kim and Ni [15]. Duoandikoetxea and Zuazua’s approximation is a linear combination of partial derivatives of the heat kernels. Their result is about $L^p$-convergence as $t \to \infty$ with fixed number of terms to be summed, but not about what happens when the number of terms to be summed goes to infinity. In the first section we will improve their result by obtaining an estimate in the $L^\infty$-norm considering number of terms to be summed. When the fixed time $t$ is sufficiently large, our estimate shows that an $L^1$-initial data bounded by a Gaussian function guarantees the $L^\infty$-convergence of approximation as more terms are summed. Also we will prove that as numerically observed in [15], when fixed time $t$ is small the $L^\infty$-convergence is not expected. In the subsequent section Kim and Ni’s approximation is introduced. Their approximate function is a linear combination of the heat kernels (not their derivates) having various mass and center of mass. Assuming a solution and their approximation have same moments, they proved the $L^p$-convergence as $t \to \infty$ with fixed number of terms to be summed. Also they conjectured their approximation would converge as the number of terms to be summed goes to infinity unlike the Duoandikoetxea and Zuazua’s approximation.

Chapter 4 is based on a paper [23], which is a joint work with Eugenia Kim and Yong Jung Kim. In the chapter we consider bounded solutions to the (viscous) Burgers equation in one spatial dimension

$$u_t + uu_x = \mu u_{xx} \quad \text{in } \mathbb{R} \times (0, \infty),$$
$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R},$$

where $\mu > 0$ is the viscosity coefficient. The Burgers equation has been played an important role in many problems such as traffic or fluid flows (see [63]). Also it has been shown that the asymptotic behavior of general systems of hyperbolic conservations laws are given as a solution to the Burgers equation [20, 21, 51]. It is well-known that the Cole-Hopf transformation

$$\Phi(x, t) = e^{-\frac{1}{2\mu} \int_0^t u(y, \tau) \, d\tau} - 1$$

is a relation between a solution $u$ to the Burgers equation and a solution $\Phi$ to the heat equation; $\Phi$ satisfies the heat equation if $u$ satisfies the Burgers equation, and vice versa. Transforming Kim and Ni’s approximation $\Psi$ for the heat equation inversely, we can find an approximation $w$ for the Burgers equation:

$$w(x, t) = -\frac{2\mu \Psi_x(x, t)}{1 + \Psi(x, t)}.$$

In the chapter we prove that moments difference between the solution and our approximation converges to zero as time goes to infinity and the convergence order is increased proportionally to self-similarity.
scale. Note that for nonlinear problems, moments of two solutions initially having same mass does not remain same as time passes by. But even in that case, our result suggests that an approximation could be found by compelling its moments to converge to moments of the solution as time goes to infinity.

In Chapter 5, the zero level set $Z(t) := \{x \in \mathbb{R}^d : u(x, t) = 0\}$ of a solution $u$ to the heat equation \[ (1.1) \] is considered. (Note that non-zero level set $\{x \in \mathbb{R}^d : u(x, t) = c \neq 0\}$ becomes empty in finite time since the solution $u$ goes to zero uniformly as time goes to infinity.) Because a topologically complicated curve can be a zero level set of a high-dimensional function, evolution of the zero level set can be used to track an interface like flame front. The level set method is a numerical technique based on this idea and I refer those interested in the level set method to a book by Osher and Fedkiw [56]. In the chapter we study how the zero level set looks like after a long time. Using the Duoandikoetxea and Zuazua’s approximation, we show that the zero level set in a ball of radius $C\sqrt{t}$ for some $C > 0$ converges to a specific graph as $t \to \infty$ when the set is divided by $\sqrt{t}$. The graph is zeros of a linear combination of the Hermite polynomials and the coefficients of the linear combination depends on moments of the initial data. Hence the long-time asymptotics of the zero level set in a ball of radius $C\sqrt{t}$ is completely determined by moments of the initial data. In low dimensions or for radially symmetric initial data, the graphs to which the zero level sets converge will be classified.
Chapter 2. Relative Newtonian Potentials for Radial Functions and $L^1$-Intermediate Asymptotics for Nonlinear Diffusion

2.1 Introduction

The fundamental solution of Laplace’s equation in $\mathbb{R}^d$ has three different shapes depending on the dimension. They are

$$\phi(x) := \begin{cases} 
-\frac{1}{(d-2)\omega_d} |x|^{2-d} & \text{if } d \geq 3, \\
\frac{1}{\omega_2} \ln |x| & \text{if } d = 2, \\
\frac{1}{\omega_1} |x|^{-1} & \text{if } d = 1.
\end{cases} \quad (2.1)$$

where $\omega_d := 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of a unit sphere in $\mathbb{R}^d$. Due to this difference, the Newtonian potential of a function $v(x)$,

$$V(x) := \int_{\mathbb{R}^d} \phi(x-y)v(y)dy, \quad (2.2)$$

should be considered separately for the three cases. Such a difference is an obstacle to obtain simpler statements that work for all dimensions and makes certain analysis lengthy and complicated.

The purpose of this chapter is to propose a potential theory in a relative sense, which conveys the properties of the Newtonian potential of dimensions $d \geq 3$ to all dimensions, and then demonstrate that the new theory provides a unified approach to all dimensions. In fact, we apply the theory to a study of long-time asymptotics in nonlinear diffusion equations. Remember that it is the potential difference but not the potential itself that makes physics. For example, the electrical current is produced by the potential difference. Hence it is desirable to define a potential in a relative sense from the beginning and develop a theory based on it. In this chapter a relative potential of two radial functions, $v_1$ and $v_2$, is defined by

$$E(r; v_1, v_2) := -\int_{r}^{\infty} \left( x^{1-d} \int_0^x y^{d-1}(v_1(y) - v_2(y)) dy \right) dx.$$ 

One may easily see that this relative potential is well defined for all dimensions if $v_1$ and $v_2$ are compactly supported and share the same mass.

This chapter has three sections after this Introduction. In Section 2.2 this relative potential is compared to the Newtonian potential and a generalized Newton’s theorem, Theorem 3, is proved using relative potentials. For $d \geq 3$, the relative potential is actually the Newtonian potential of $v_1 - v_2$. However, it is not the case for $d \leq 2$. One may see the advantage of using the relative potential from the unified version of Newton’s theorem in Corollary 1 for all dimensions $d \geq 1$. It convinces that the right extension of Newton’s theorem for dimensions $d \geq 3$ to lower dimensions $d \leq 2$ is this new version, but not the original one (see Theorem 9.7 of [50]).

In Sections 3 and 4, the relative potentials are applied to obtain intermediate asymptotics of radial solutions to nonlinear diffusion equations. First, consider

$$u_t = \Delta(u^m) = \nabla \cdot (m u^{m-1} \nabla u), \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d, \quad (2.3)$$
where the initial data \( u_0 \geq 0 \) and the exponent \( m > (d - 2)_+/d \). Here, we denote \( f_+ := \max(f, 0) \). If \( m < 1 \), then the equation is called the fast diffusion equation (FDE). If \( m > 1 \), it is called the porous medium equation (PME). This equation is a nonlinear version of the heat equation with a temperature depending conductivity, \( mu^{m-1} \). This model has been used to describe various diffusion processes such as a gas flow through a porous media, heat radiation in plasmas, groundwater flow, curvature flow, and spreading species (see Chapter 2 in [65]). On the other hand, for a fixed \( p > 1 \), the \( p \)-Laplacian equation (PLE), which is given by

\[
u_t = \nabla \cdot (\|\nabla u\|^{p-2} \nabla u), \quad u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^d, \tag{2.4}\]

models nonlinear diffusion processes with gradient depending diffusivity, which is \( \|\nabla u\|^{p-2} \) in the equation.

These two equations have been played as key prototypes of nonlinear diffusion phenomena.

One of the essential structures of these equations is their radial symmetry. For example, if the initial data \( u_0 \) is radially symmetric, then their solutions keep this symmetry all the time \( t > 0 \). If the initial data is not radial, then the solution asymptotically converge to a fundamental solution of the same mass which is a radial function. In fact, the homogeneity of the problem allows a similarity structure and one may find the fundamental solution explicitly. For the PME or FDE, the fundamental solution is called the Barenblatt solution and given by

\[
\rho(x, t) = t^{-d\alpha}(C_M - k |xt^{-\alpha}|^{2/(m-1)}), \tag{2.5}
\]

where \( \alpha = \frac{1}{d(m-1)+2} \), \( k = \frac{a(m-1)}{2m} \), and the constant \( C_M > 0 \) is decided by the total mass \( \int \rho(x, t)dx = M \). Similarly, there is a Barenblatt-type solution for the PLE given by

\[
\rho(x, t) = t^{-d\alpha}(C_M - k |xt^{-\alpha}|^{p/(p-1)})^{\frac{p-1}{p}}, \tag{2.6}
\]

where \( \alpha = \frac{1}{d(p-2)+p} \), \( k = \frac{d-2}{d} \alpha^{1/(p-1)} \), and \( C_M > 0 \) is similarly decided by the given total mass \( \int \rho(x, t)dx = M \).

In this chapter, the potential of a radial solution \( u(x, t) \) relative to the Barenblatt solution \( \rho(x, t) \) is employed to show the following theorem. We denote the \( L^p \)-norm over the space variable as \( \| \cdot \|_p \).

**Theorem 2.** Let the initial data \( u_0(x) \geq 0 \) be radially symmetric, compactly supported, and of mass \( M = \int u_0(x)dx \). Then the solution \( u(x, t) \) to the PME or FDE \((2.3)\) with \( m > (d - 2)_+/d \) satisfies

\[
\|u(t) - \rho(t)\|_1 = O(t^{-1}) \quad \text{as} \quad t \to \infty, \tag{2.7}
\]

where \( \rho(x, t) \) is the Barenblatt solution given by \((2.5)\). If \( u(x, t) \) is the solution to PLE \((2.4)\) with \( p > \frac{2d}{d+1} \), then the same convergence order holds with a Barenblatt-type solution \( \rho(x, t) \) given by \((2.6)\).

Long-time asymptotic contraction to the Barenblatt solution has been studied intensively for the FDE and PME cases. Vazquez has shown that an \( L^1 \)-contraction order is not generally expected among all \( L^1 \)-initial data even if they share the same mass (see [68]). Hence extra restrictions such as finiteness of moments, entropy or relative entropy has been imposed to obtain certain contraction order throughout the literature (see [14] [10]). There are two kinds of optimal contraction rates. The first one is of the similarity scale of \( O(t^{-\alpha}) \) as \( t \to \infty \), which is the order of a space translation \( \|\rho(t) - \rho_{x_0}(t)\|_1 \) with \( \rho_{x_0}(x, t) = \rho(x + x_0, t) \). This rate has been shown for \( (d - 1)/d < m < 2 \) in [16] [17] [18] [25] [68]. The contraction rates in other regime obtained so far are lower than this optimal rate.

The other optimal contraction rate, which requires tuning the center of mass, is \( O(t^{-1}) \) as \( t \to \infty \), which is the one in our theorem. Note that the center of mass of a radial function is the origin. This
contraction rate is the one of a time translation $\| \rho(t) - \rho(t + T) \|_1$ for a fixed $T > 0$. This rate has been shown for $(d - 2)_+ / d < m \leq d/(d + 2)$ in [43] and a similar rate $O(t^{-(1+\tau)})$ for $(d - 1)/d < m < 1$ in [53]. For radial solutions or for one dimension, such a contraction rate has been shown for all $m > (d - 2)_+/d$ using a potential comparison or a mass concentration comparison [17, 43, 47, 66, 68]. Therefore, the contraction rate in Theorem 2 is not new for FDE and PME cases. However, the point is that the relative potential gives a simple proof in a unified way for all dimensions $d \geq 1$ and all exponents $m > (d - 2)_+/d$. The theorem is proved for the FDE and PME case in Section 2.3 and then for PLE case in Section 2.4

The entropy dissipation method (see [13, 15]) has been used to obtain intermediate long-time asymptotics. Even if the entropy is not defined, the theory is applicable if the relative entropy is defined. In fact relative entropy is commonly used. However, it seems that the relative potential introduced in this chapter is new. Pierre [59] employed Newtonian potentials for dimensions $d \geq 3$ only. Kim and McCann [43] fully employed them for all dimensions $d \geq 1$ case by case.

### 2.2 Relative Potential of Radial Functions

Let $V$ be the Newtonian potential of a function $v \geq 0$. In other words $V$ satisfies $\Delta V = v$ in the sense of distributions. Under the radial symmetry assumption for $V$ and $v$, one may write the relation as

$$\Delta V = r^{1-d}(r^{d-1}V'(r))' = v(r), \quad r = |x|. \tag{2.8}$$

Therefore, it is natural to define a potential function by simply decoupling (2.8), which gives

$$V(r) := -\int_r^\infty (x^{1-d} \int_0^x y^{d-1}v(y)dy)dx = -\frac{1}{\omega_d} \int_r^\infty x^{1-d}V(x)dx, \tag{2.9}$$

where $V(r)$ is the mass concentration of $v$, which gives the mass in $B_r(0)$, a ball of radius $r > 0$ centered at $x = 0$, i.e.,

$$V(r) = \omega_d \int_0^r y^{d-1}v(y)dy. \tag{2.10}$$

(Notice the hierarchy of notations. A small letter $v$ is a given function, a calibrated letter $V$ is for its integration or mass (2.10) and a capital letter $V$ is for its double integration or Newtonian potential (2.9).) The total mass $M$ should be the limit $\lim_{r \to \infty} V(r) = M$. It is clear that $V$ is well-defined if $V(r) = O(r^{d-2-\epsilon})$ for $r > 0$ large. Hence, if $d \geq 3$, the potential $V$ is well-defined for any $L^1$-function $v$. Furthermore, even if it is not integrable, the potential can be well-defined if the mass $\mathcal{V}(r)$ diverges slowly as $r \to \infty$. If $v$ is the Dirac-delta measure in $\mathbb{R}^d$, then $V(r) = 1$ for all $r > 0$ and, for $r = |x|$

$$V(r) = -\frac{1}{\omega_d} \int_r^\infty x^{1-d}dx = -\frac{|x|^{2-d}}{(d-2)\omega_d} = \phi(x),$$

which is the fundamental solution of the Laplace’s equation (2.11) for dimension $d \geq 3$.

If $d \leq 2$, then the potential $V$ in (2.10) is not defined since the only nonnegative function that has the mass concentration of order $V(r) = O(r^{d-2-\epsilon})$ for $r > 0$ large is the trivial function $v = 0$. Then, one may consider another version

$$V_1(r) := \int_0^r \left( x^{1-d} \int_0^x y^{d-1}v(y)dy \right)dx, \tag{2.11}$$

which is well-defined for the dimension $d = 1$ only. One may easily check that, if $v$ is the Dirac-delta measure in $\mathbb{R}$, then the corresponding potential $V_1$ is the fundamental solution in (2.11) for $d = 1$. For
the dimension $d = 2$, one may take

$$V_2(r) := \int_1^r \left( x^{1-d} \int_0^x y^{d-1} v(y) \, dy \right) \, dx,$$  \hspace{1cm} (2.12)$$

which becomes the fundamental solution in (2.1) if $v$ is the Dirac-delta measure in $\mathbb{R}^2$. In fact this potential is well-defined for all dimensions. However, this potential is inconvenient since the outside integration is from $x = 1$, which is not a boundary point of domain. This fact partly explains why difficulties in two spatial dimensions arise.

The previous discussions show that the definition of the Newtonian potential is not consistent with dimensions. However, if one considers the potential in a relative sense as given below, one may obtain a consistent approach for all dimensions.

**Definition 1.** Let $v_1(r)$ and $v_2(r)$ be nonnegative radial function in $\mathbb{R}^d$ with $d \geq 1$. The relative potential $E(r; v_1, v_2)$ between $v_1$ and $v_2$ is defined by

$$E(r; v_1, v_2) := - \int_r^\infty \left( x^{1-d} \int_0^x y^{d-1} (v_1(y) - v_2(y)) \, dy \right) \, dx.$$  \hspace{1cm} (2.13)$$

This relative potential is well-defined if the relative mass

$$\mathcal{E}(r; v_1, v_2) := \omega_d \int_0^r y^{d-1} (v_1(y) - v_2(y)) \, dy$$

in the ball of radius $r > 0$ has order

$$\mathcal{E}(r; v_1, v_2) = O(r^{d-2-\epsilon}) \quad \text{as} \quad r \to \infty.$$  \hspace{1cm} (2.14)$$

On the other hand, if the relative potential is well-defined, then the relative mass should satisfy

$$\mathcal{E}(r; v_1, v_2) = o(r^{d-2}) \quad \text{as} \quad r \to \infty.$$  \hspace{1cm} (2.15)$$

Therefore, when dimension $d \leq 2$, the functions $v_1$ and $v_2$ should have the same total mass, i.e., $\|v_1\|_1 = \|v_2\|_1$, to get their relative potential to be well-defined.

From the definition we have

$$\Delta E(r; v_1, v_2) = v_1 - v_2,$$  \hspace{1cm} (2.16)$$

$$E(r; v_1, v_2) = -E(r; v_2, v_1).$$  \hspace{1cm} (2.17)$$

If the given functions $v_1, v_2$ are clearly given from the context, then we simply denote the relative potential and relative mass by $E(r)$ and $\mathcal{E}(r)$, respectively.

**Newton’s theorem** is a comparison between the fundamental solution of Laplace’s equation and the Newtonian potential of a radial function. One may consider it in terms of relative potentials in a unified way for all dimensions $d \geq 1$.

**Theorem 3.** Let $v_i$, $i = 1, 2$, be non-negative radial functions such that

$$\text{supp}(v_i) = [0, L_i] \text{ with } 0 < L_1 < L_2 < \infty, \quad \omega_d \int_0^{L_i} y^{d-1} v_i(y) \, dy = M,$$

and $E(r)(\equiv E(r; v_1, v_2))$ be the corresponding relative potential, i.e.,

$$E(r) := - \int_r^\infty x^{1-d} k(x) \, dx, \quad k(x) := \int_0^x y^{d-1} (v_1(y) - v_2(y)) \, dy.$$

Then,

$$E(r) \leq 0 \quad \text{if} \quad L_1 < r < L_2, \quad E(r) = 0 \quad \text{if} \quad L_2 < r.$$
Proof. Let 

\[ A(x) := \int_0^x y^{d-1} v_1(y) \, dy, \quad B(x) := \int_0^x y^{d-1} v_2(y) \, dy. \]

Then, \( k(x) = A(x) - B(x) \). If \( x > L_2 \), then \( A(x) = B(x) = M/\omega_d \) and hence \( k(x) = 0 \). Therefore, \( E(r) = 0 \) for all \( r > L_2 \). If \( L_1 < x < L_2 \), then \( A(x) = M/\omega_d \) and \( B(x) < M/\omega_d \). Hence, \( k(x) > 0 \). Therefore, for \( L_1 < r < L_2 \), \( E(r) = -\int_r^{L_2} k(x) \, dx < 0 \). \( \square \)

Newton’s theorem is a special case of Theorem 3 where \( v_1 \) is a Dirac-delta distribution. Of course one should consider an appropriate limiting process with a delta sequence.

**Corollary 1** (Newton’s Theorem in terms of relative potentials). Let \( v \) be a non-negative radial function with \( \text{supp}(v) \subset B_L(0) \) for some \( L > 0 \). Then, with \( M = \omega_d \int_0^L r^{d-1} v(r) \, dr \),

\[ E(r; M\delta, v) = 0 \quad \text{if} \quad r > L, \quad E(r; M\delta, v) \leq 0 \quad \text{if} \quad r < L. \]  

(2.18)

Let \( d \geq 3 \). Then,

\[ \int_{\mathbb{R}^d} \phi(x-y)v(y) \, dy = -\int_{|x|}^{\infty} \left( r^{1-d} \int_0^r y^{d-1} v(y) \, dy \right) dr, \]

(2.19)

where \( \phi(x) \) is the fundamental solution in (2.1).

**Proof.** The first part of Corollary 1 is a special case of Theorem 3 where \( v_1 \) is the Dirac-delta distribution multiplied by \( M > 0 \). We show the second part (2.19). Let

\[ A(x) := \int_{\mathbb{R}^d} \phi(x-y)v(y) \, dy, \quad V(x) := -\int_{|x|}^{\infty} \left( r^{1-d} \int_0^r y^{d-1} v(y) \, dy \right) dr. \]

We have observed that the potential defined by (2.10) is the fundamental solution in (2.1) for dimensions \( d \geq 3 \) if the measure is the delta-distribution. Hence the relation \( E(r; M\delta, v) = 0 \) can be written \( M\phi(x) = V(x) = 0 \). Therefore, the original Newton’s theorem and (2.18) imply that \( A(x) = V(x) = M\phi(x) \) for all \( |x| > L \). Since \( v(x) = \Delta A(x) = \Delta V(x) \), \( A(x) - V(x) \) is a bounded harmonic function such that \( A(x) - V(x) = 0 \) for all \( |x| > L \). The only such harmonic function is the trivial one, i.e., \( A(x) - V(x) = 0 \) for all \( x \in \mathbb{R}^d \). \( \square \)

Corollary 1 implies that the Newtonian potential \( V \) of \( v \) satisfies

\[ M\phi(x) - V(x) = 0 \quad \text{if} \quad |x| > L, \quad M\phi(x) - V(x) \leq 0 \quad \text{if} \quad |x| < L, \]

(2.20)

where \( M = \int v(x) \, dx \) and \( \text{supp}(v) \subset B_L(0) \). Therefore, Corollary 1 is identical to Newton’s theorem for dimensions \( d \geq 3 \). However, for dimensions \( d \leq 2 \), the relative potential is not related to the fundamental solutions in (2.1). Hence Corollary 1 is talking about a slightly different concept for dimensions \( d \leq 2 \). If one wants to write Newton’s theorem in terms of potentials, but not of relative ones, then one should define the potential differently for dimensions \( d \leq 2 \). The inequality in (2.20) becomes (see Theorem 9.7 of [59])

\[ |V(x)| \leq M|\phi(x)| \quad \text{if} \quad |x| < L, \]

(2.21)

where \( \phi(x) \) and \( V(x) \) change their signs for \( d \geq 2 \). It seems that the relative potential gives a simple and a unified approach particularly for dimensions \( d \leq 2 \). For dimensions \( d \geq 3 \), either way seems equally acceptable. The asymptotic analysis for the PME and FDE in the following section employs this relative potential and gives a unified proof of intermediate asymptotics of order \( O(1/t) \) in \( L^1 \)-norm for all dimensions and nonlinearities.
2.3 The Porous Medium and Fast Diffusion Equation

In this section we prove Theorem 2 for the solutions to (2.23). Since the solution \( u(x,t) \) and the initial data \( u_0(x) \) are radially symmetric, one may rewrite the equation as

\[
u_t = r^{1-d}(r^{d-1}(u^m_r))_r, \quad u(r,0) = u_0(r) \geq 0, \quad u_r(0,t) = 0,
\]

where \( r = |x|, m > (d - 2)_+/d \) and \( d \geq 1 \). Notice that we are slightly abusing notation by writing \( u(x,t) = u(r,t), u_0(x) = u_0(r) \). The initial data \( u_0 \) is assumed to be compactly supported and has total mass \( M \), i.e.,

\[
\int u_0(x)dx = \omega_d \int_0^\infty r^{d-1}u_0(r)dr = M, \quad \text{supp}(u_0) \subset B_L(0).
\]

The Barenblatt solution can be written in the radial variable \( r \) which is

\[
\rho(r,t) = t^{-d\alpha}(C_M - k(\tau_r^{-\alpha})^{1/(m-1)}).
\]

**Lemma 1.** Let \( u_1(r,t) \) and \( u_2(r,t) \) be solutions to (2.23) with compactly supported initial data \( u_{10}(r) \) and \( u_{20}(r) \) of the same mass \( M > 0 \). Then the corresponding relative potential,

\[
E(r,t) := -\int_r^\infty \left(x^{1-d}\int_0^x y^{d-1}(u_1(y,t) - u_2(y,t))dy\right)dx,
\]

is well-defined for all \( d \geq 1, m > (d - 2)_+/d \), and \( t > 0 \). Furthermore,

\[
\frac{\partial}{\partial t}E(r,t) = u_1^m(r,t) - u_2^m(r,t).
\]

**Proof.** Since \( u_1 \) and \( u_2 \) are non-negative integrable functions for all \( t \geq 0 \), the relative potential \( E(r,t) \) is well-defined for dimensions \( d \geq 3 \). For the regime of the PME, \( m > 1 \), the solutions are compactly supported. Therefore, the relative mass \( \mathcal{E}(r,t) := \int_0^\infty \omega_d y^{d-1}(u_1(y,t) - u_2(y,t))dy \) becomes identically zero for \( r \rightarrow 0 \) large. Hence, this relative potential is well-defined for all dimensions for the PME regime. For the fast diffusion regime, \( (d - 2)_+/d < m < 1 \), it is well known that the solution \( u_i \) has the same decay rate for \( r \) large as the one of the Barenblatt solution. Since \( \mathcal{E}(r,t) \rightarrow 0 \) as \( r \rightarrow \infty \), we have

\[
|\mathcal{E}(r,t)| = \omega_d \int_r^\infty y^{d-1}(u_1(y,t) - u_2(y,t))dy \leq C\omega_d \int_r^\infty y^{d-1}y^{-\frac{m}{2}}dy = O\left(r^{\frac{2d(m-1)}{m-1}}\right) \quad \text{as} \quad r \rightarrow \infty.
\]

Since \( \frac{2d(m-1)}{m-1} - (d - 2) = \frac{2m}{m-1} < 0 \), the relative potential is well-defined. If \( m = 1 \), then it is the heat equation case. Hence one may conclude the lemma easily using the exponential decay of the solution as \( r \rightarrow \infty \).

A formal proof of (2.26) can be given as

\[
E_t = -\int_r^\infty \left(x^{1-d}\int_0^x y^{d-1}(u_1 - u_2)_ydy\right)dx = -\int_r^\infty \left(x^{1-d}\int_0^x y^{d-1}(y^{1-d}(u_1^m - u_2^m)y)dy\right)dx = u_1^m - u_2^m.
\]

For the PME case, \( m > 1 \), taking the derivative inside the integration is simple since the integration is on a compact set. For the FDE case with dimension \( d \geq 3 \), this relation is the one with the original Newtonian potential, which was given in (33). The FDE case with dimension \( d \leq 2 \) can be similarly shown from the same proof of Proposition 10 in (33). \( \square \)
Note that we always compare solutions with same initial mass $\omega_d \int_0^\infty r^{d-1} u_0(r) \, dr = M$. Then, the conservation of mass implies that
\[
\omega_d \int_0^\infty r^{d-1} \, u(r, t) \, dr = M \quad \text{for all } t > 0.
\] (2.27)

**Proposition 1** (Comparison Principle). Let $u_1(r, t)$ and $u_2(r, t)$ be solutions to (2.22) with compactly supported initial values $u_{10}(r)$ and $u_{20}(r)$ of the same mass $M > 0$ and $E(r, t; u_1, u_2)$ be their relative potential given by (2.25). If there exists $t_0 \geq 0$ such that
\[
E(r, t_0; u_1, u_2) \geq 0 \quad \text{for all } r > 0
\]
then
\[
E(r, t; u_1, u_2) \geq 0 \quad \text{for all } r > 0, \ t \geq t_0.
\]

**Proof.** Using the relations in (2.15) and (2.20) one may write
\[
E_t = a(x, t) \Delta E, \quad a(x, t) := (u_1^n - u_2^n)/(u_1 - u_2) \geq 0.
\] (2.28)

First consider the PME case with $m > 1$. Then the relative potential $E$ is also compactly supported for all $t > 0$. Hence for a fixed time $T > 0$, there exists a constant $C_T > 0$ such that $E(x, t) = 0$ for all $|x| \geq C_T$ and $t_0 \leq t \leq T$. Therefore, the maximum principle and the assumption $E(x, t_0) \geq 0$ for all $|x| > 0$ imply that $E(x, t) \geq 0$ for all $t_0 \leq t \leq T$ and $|x| < C_T$. Since we can take $T$ and $C_T$ arbitrarily large, the proof is done for the PME case. For the FDE case, $(d-2)/d < m < 1$, the solutions $u_1$ and $u_2$ become strictly positive for all $t > 0$ and the equation (2.28) becomes uniformly parabolic. Hence the maximum principle on the unbounded domain $\mathbb{R}^d \times [0, T]$ concludes the proposition. \qed

If $d \geq 3$ and $U_1$ and $U_2$ are the Newtonian potentials of $u_1$ and $u_2$ given by (2.20), then obviously
\[
E(r, t; u_1, u_2) = U_1(r, t) - U_2(r, t).
\]

Hence, the proposition implies that $U_1(r, t) \geq U_2(r, t)$ for all $r > 0$ and $t \geq t_0$ if $U_1(r, t_0) \geq U_2(r, t_0)$ for all $r > 0$. Next step is to sandwich the potential $U(r, t)$ of the solution $u(x, t)$; more precisely, we obtain an estimate
\[
R(r, t) \leq U(r, t) \leq R(r, T + t),
\] (2.29)
where $R$ is the potential of the Barenblatt solution. In the following lemma we show this estimate in terms of relative potentials for all dimensions $d \geq 1$.

**Lemma 2.** Let $u(r, t)$ be a solution to (2.22) with compactly supported initial data with mass $M > 0$. Let $\rho(r, t)$ be the Barenblatt solution of the same mass. Then,

(i) The relative potential satisfies
\[
E(r, t; \rho, u) \leq 0 \quad \text{for all } r, t \geq 0.
\]

(ii) There exists $T > 0$ such that
\[
E(r, t; \rho_T, u) \geq 0 \quad \text{for all } r, t \geq 0,
\]
where $\rho_T(r, t) := \rho(r, t + T)$.
Proof. (i) This estimate, which corresponds to the lower estimate in (2.29), comes from the Newton’s theorem in Corollary 11 and the comparison principle.

(ii) Let supp(u₀) ⊂ [0, L] and

\[ V(r, t) := -\int_{r}^\infty \left( x^{1-d} \int_{0}^{x} y^{d-1} (\rho(y, t) - u_0(y)) dy \right) dx. \]

Then, since supp(u₀) ⊂ [0, L], V(r, t) ≥ 0 for r > L and t > 0. For 0 < r < L,

\[ -V(r, t) = \lim_{l \to \infty} \int_{r}^{l} \left( x^{1-d} \int_{0}^{x} y^{d-1} (\rho(y, t) - u_0(y)) dy \right) dx. \]

First, we have

\[ \int_{r}^{l} \left( x^{1-d} \int_{0}^{x} y^{d-1} u_0(y) dy \right) dx \geq \int_{L}^{l} \frac{x^{1-d}}{\omega_d} dx. \]

Since \( \rho(r, t) \to 0 \) uniformly as \( t \to \infty \), there exists \( T > 0 \) such that \( \rho(r, T) < \epsilon := \frac{2d}{\pi} \int_{L}^{1} \frac{x^{1-d}}{\omega_d} dx \). (Note that \( T \) can be chosen independently of \( l \) large since the right hand side is increasing as \( l \to \infty \).) Then,

\[ \int_{r}^{l} \left( x^{1-d} \int_{0}^{x} y^{d-1} \rho(y, T) dy \right) dx \leq \int_{L}^{l} \frac{x^{1-d}}{\omega_d} dx. \]

Addition of these estimates gives \( -V(r, T) \leq 0 \) for all \( r > 0 \). In other words, we have obtained \( E(r, 0; \rho_T, u) \geq 0 \) for all \( r > 0 \). The potential comparison principle gives \( E(r, t; \rho_T, u) \geq 0 \) for all \( r, t > 0 \). \( \square \)

**Lemma 3.** Let \( \rho(x, t) \) be the Barenblatt solution given in (2.3). Then, for any given \( T > 0 \),

\[ \| \rho(t) - \rho(t + T) \|_1 = O(t^{-1}) \quad \text{as} \quad t \to \infty. \]

**Proof.** Since the Barenblatt solution is explicit, one may explicitly compute the contraction order. However, in the following, we show the lemma in a relatively general way. First, the mean value theorem gives that, for some \( s = s(r) \in (t, t + T) \),

\[ \frac{1}{\omega_d} \| \rho(t) - \rho(t + T) \|_1 = \int_{0}^{\infty} r^{d-1} |\rho(r, t) - \rho(r, t + T)| \, dr \]

\[ = T \int_{0}^{\infty} r^{d-1} |\rho_t(r, s)| \, dr \]

\[ \leq T \int_{0}^{\infty} r^{d-1} (|\rho_t(r, t)| + |\rho_t(r, t + T)|) \, dr. \]

The Barenblatt solution in the radial variable has the form of \( \rho(r, t) = t^{-\alpha} f(rt^{-\alpha}) \) for \( \alpha > 0 \) and

\[ \rho_t(r, t) = -t^{-1} (\frac{d\alpha}{\alpha} t^{-\alpha} f(rt^{-\alpha}) + \alpha t^{-\alpha} f'(rt^{-\alpha}) r^{-\alpha}). \]

Therefore, using the similarity variable \( \zeta = rt^{-\alpha} \), we can see that

\[ \int_{0}^{\infty} r^{d-1} |\rho_t(r, t)| \, dr \leq \frac{\alpha}{T} \int_{0}^{\infty} \zeta^{d-1} (df(\zeta) + \zeta |f'(\zeta)|) \, d\zeta = O(t^{-1}) \]

as \( t \to \infty \). Similarly, \( \int_{0}^{\infty} r^{d-1} |\rho_t(r, t + T)| \, dr = O(t^{-1}) \) as \( t \to \infty \), which completes the proof. \( \square \)

The proof of Theorem 2 employs the well known theory that the number of intersection points between two solutions decreases as time increases in one spatial dimension (see [3] 52 and Corollary 15.10 in [65]). Since the Barenblatt solution is a delta sequence as \( t \to 0 \), there exists exactly one
intersection point between $\rho(r,t)$ and $u(r,t)$ for $r > 0$. In other words, there exists a unique point $r = \beta(t)$ such that

$$(0, \beta(t)) = \{r > 0 : \rho(r,t) > u(r,t)\}.$$ 

We reserve the notation $\beta(t)$ for this unique intersection point between $\rho$ and $u$.

**First proof of Theorem 2 for PME and FDE.** Let $T > 0$ be the one given in Lemma 2. It is well known that there exists a finite time $t_0 > 0$ such that $u^{n-1}(\cdot, t_0)$ becomes concave (see [6, 48, 49]). Hence, by taking larger $T > 0$ if needed, there exists a unique intersection point between $\rho(r, t_0 + T)$ and $u(r, t_0)$. Hence the zero set theory implies that $\rho(r, t_0)$, $u(r, t)$, and $\rho(r, t + T)$ intersects to each other exactly once for all $t > t_0$. Let $\gamma(t)$ be the intersection point between $\rho(\cdot, t)$ and $\rho(\cdot, t + T)$. Then, the lemma implies that

$$\rho(0, t + T) < u(0, t) < \rho(0, t)$$

for all $t > 0$. First suppose that $\beta(t) \leq \gamma(t)$. One may easily see that (c.f., Figure 2.1a)

$$\int_{0}^{\beta(t)} r^{d-1}\left|\rho(r,t) - u(r,t)\right| dr \leq \int_{0}^{\gamma(t)} r^{d-1}\left|\rho(r,t) - \rho(r,t + T)\right| dr.$$ 

Since $u$, $\rho$, and $\rho_T$ are of the same mass, we have

$$\|u(t) - \rho(t)\|_1 = 2\omega_d \int_{0}^{\beta(t)} r^{d-1}\left|u(r,t) - \rho(r,t)\right| dr,$$

$$\|\rho(t) - \rho_T(t)\|_1 = 2\omega_d \int_{0}^{\gamma(t)} r^{d-1}\left|\rho(r,t) - \rho(r,t + T)\right| dr.$$ 

Therefore, we have

$$\|u(t) - \rho(t)\|_1 \leq \|\rho(t) - \rho(t + T)\|_1 = O(t^{-1}) \quad \text{as} \quad t \to \infty. \quad (2.30)$$

Now suppose that $\gamma(t) \leq \beta(t)$. In a similar fashion we have (c.f., Figure 2.1b)

$$\int_{\beta(t)}^{\infty} r^{d-1}\left|\rho(r,t) - u(r,t)\right| dr \leq \int_{\gamma(t)}^{\infty} r^{d-1}\left|\rho(r,t) - \rho(r,t + T)\right| dr.$$
Since
\[
\|u(t) - \rho(t)\|_1 = 2\omega_d \int_{(t)}^{\infty} r^{d-1} |u(r, t) - \rho(r, t)| \, dr,
\]
\[
\|\rho(t) - \rho_T(t)\|_1 = 2\omega_d \int_{(t)}^{\infty} r^{d-1} |\rho(r, t) - \rho(r, t + T)| \, dr,
\]
one obtains (2.30) again.

Proof. In the following we give another proof based on a rescaling method that has been used to find intermediate asymptotics for various cases (see chapters 3 and 4 of [32]). This method has been used for the FDE case in [33]. Hence we consider the PME case in the following. From the explicit formula of the Barenblatt solution in (2.5), one may easily observe the following.

**Lemma 4.** The Barenblatt solution \( \rho \) and the intersection point \( \beta(t) \) between \( \rho \) and a \( L^1 \)-solution \( u \) satisfy that
\[
\|\rho^m(\cdot, t)\|_{\infty} = O\left(t^{-\alpha m}\right) \quad \text{as} \quad t \to \infty,
\]
\[
\beta(t) \leq \zeta(t) = O\left(t^\alpha\right) \quad \text{as} \quad t \to \infty,
\]
where \( \alpha := \frac{1}{d(m-1)+2} \) and \( \zeta(t) > 0 \) denotes the positive interface of the Barenblatt solution, that is, \( \zeta(t) := \sup\{r > 0 : \rho(r, t) > 0\} \).

**Proposition 2** \((L^\infty\text{-distance between potentials})\). Let \( u(r, t) \) be the solution to (2.22) with compactly supported initial data with total mass \( M > 0 \). Let \( \rho(r, t) \) be the Barenblatt solution. Then, the relative potential has the order
\[
\|E(\cdot, t; u, \rho)\|_{\infty} = O\left(t^{-\alpha m}\right) \quad \text{as} \quad t \to \infty.
\]

**Proof.** It is clear that
\[
E(r, t; \rho_T, \rho) = E(r, t; \rho_T, u) + E(r, t; u, \rho) \geq E(r, t; u, \rho) \geq 0
\]
for all \( r, t \geq 0 \). Therefore we have
\[
0 \leq E(r, t; u, \rho) \leq E(r, t; \rho_T, \rho)
\]
\[
\leq \left| \int_{r}^{\infty} \left( x^{1-d} \int_{0}^{x} y^{d-1} (\rho(y, t) - \rho(y, t + T)) \, dy \right) \, dx \right|
\]
\[
= \left| \int_{r}^{\infty} \left( x^{1-d} \int_{0}^{\infty} y^{d-1} T \rho_T(y, c(y)) \, dy \right) \, dx \right|
\]
\[
\leq \sup_{t \in (t, t + T)} \, T |\rho^m(r, \tilde{t})| \leq T |\rho^m(r, t)|,
\]
for some \( c = c(y) \in (t, t + T) \). Therefore by Lemma 4, we conclude that
\[
\|E(\cdot, t; u, \rho)\|_{\infty} \leq T \|\rho^m(\cdot, t)\|_{\infty} = O\left(t^{-\alpha m}\right) \quad \text{as} \quad t \to \infty.
\]

Now we provide the second proof of Theorem 2 for the solutions of PME by translating the above uniform estimate of the relative potential to the required \( L^1 \)-convergence order. The same proof has given for the FDE case in [33]. We now apply the technique to the PME case. To complete the mission we need additional information on the intersection point \( \beta(t) \); which is
\[
\frac{\beta(t)}{t^\alpha} \to \sqrt{2C_M md} \quad \text{as} \quad t \to \infty. \tag{2.31}
\]
This is a conjecture. An related topic for the heat equation is discussed in Chapter 5. Note that the tail analysis was enough for the FDE case in [43] since the solution is positive everywhere.

**Second proof of Theorem 2 for PME.** First a family of rescaled solutions is introduced:

\[ u_\lambda(r, t) := \lambda^{\alpha_\alpha} u(\lambda^\alpha r, \lambda t), \quad \lambda > 0. \]

The Barenblatt solution is unchanged by this scaling, i.e., \( \rho = \rho_\lambda \). Then simple changes of variables yield that

\[ E(r, t; \rho_\lambda, u_\lambda) = \lambda^{(d-2)\alpha} E(\lambda^\alpha r, \lambda t; \rho, u). \]

Hence by Proposition 2

\[ \|E(\cdot, t; \rho_\lambda, u_\lambda)\|_\infty = O\left(\lambda^{-1} t^{-d\alpha}\right). \]

This verifies that \( \lambda E(r, t; \rho_\lambda, u_\lambda) \) is uniformly bounded on \( \lambda \). Therefore by virtue of an a priori estimate (minutely explained in [43]), their Laplacians are uniformly bounded in a region in which the Barenblatt solution \( \rho \) is strictly positive:

\[ |\Delta \lambda E(r, t; \rho_\lambda, u_\lambda)| = |\lambda (u_\lambda - \rho)(r, t)| \leq C_K, \quad \text{for } \lambda > 0, \ |r| \leq K. \]

When \( t \approx 1, K \) is a fixed constant which is strictly smaller than \( \sqrt{C_M/k} \); this condition assures that the Barenblatt solution \( \rho \) is strictly positive in the region \( |r| \leq K \). Fixing \( t = 1 \) and replacing \( \lambda \) by \( \lambda t \) in the above inequality, we obtain

\[ C_K \geq \lambda t |(u_M - \rho)(r, 1)| \]
\[ = \lambda t \cdot t^{\alpha_\alpha} |(u_\lambda - \rho)(t^{\alpha} r, t)| \quad \text{for all } \lambda > 0 \text{ and } |r| \leq K. \]

On the other hand, since

\[ \rho(t^{\alpha} K, t) = t^{-\alpha_\alpha} (C_M - kK^2)^{1/(m-1)}, \]

we have

\[ |(u_\lambda - \rho)(t^{\alpha} r, t)| \leq \frac{C_K}{\lambda t} t^{-\alpha_\alpha} \leq \frac{\tilde{C}_K}{M} \rho(t^{\alpha} K, t) \]
\[ \leq \frac{\tilde{C}_K}{\lambda t} \rho(t^{\alpha} r, t) \quad \text{for all } |r| \leq K, \]

for some constant \( \tilde{C}_K \) which depends only on \( K \). Substituting \( \lambda = 1 \), we can deduce an inequality

\[ |(u - \rho)(r, t)| \leq \frac{\tilde{C}_K}{t} \rho(r, t) \quad \text{for all } \frac{|r|}{t^{\alpha}} \leq K. \]

Choose a constant \( K \) satisfying

\[ \sqrt{2C_M k} < K < \sqrt{C_M/k} = \sqrt{2C_M m \frac{d(m-1)+2}{m-1}}. \]

Assume \( \beta(t)/t^{\alpha} \) is strictly smaller than \( K \) in a finite time so that

\[ |(u - \rho)(r, t)| \leq \frac{\tilde{C}_K}{t} \rho(r, t) \quad \text{for all } |r| \leq \beta(t). \]
Then integration of the above inequality gives us

\[
\|\rho(\cdot, t) - u(\cdot, t)\|_1 \leq 2\omega_d \int_0^{\beta(t)} r^{d-1} (\rho - u)(r, t) \, dr \\
\leq \frac{\tilde{C}_K}{t} \cdot 2\omega_d \int_0^{\beta(t)} r^{d-1} \rho(r, t) \, dr \\
\leq \frac{2\tilde{C}_K M}{t} = O(1/t) \quad \text{as} \quad t \to \infty,
\]

which completes the proof.

\[\square\]

### 2.4 The \(p\)-Laplacian Equation

In this section we show Theorem 2 for the solutions to the \(p\)-Laplacian equation (PLE) given in (2.4). Since the solution \(u(x, t)\) and the initial data \(u_0(x)\) are radially symmetric, we may rewrite the problem as

\[
u_t = r^{1-d} (u^{d-1}|u_x|^{p-2} u_x)_x, \quad u(r, 0) = u_0(r) \geq 0, \quad u_r(0, t) = 0, \quad r, t > 0. \quad (2.32)
\]

The initial data \(u_0\) is assumed to be nonnegative and compactly supported. The proof is based on the potential comparison technique. First observe that the radial PLE (2.32) is easily transformed to the radial PME (2.22) for the one dimensional case \(d = 1\). Let \(\nu := u_r\). Then, after differentiating (2.32) with respect to \(r\) once, one obtains

\[
u_t = (\text{sign}(\nu) |\nu|^{p-1})_{rr}, \quad \nu(x, 0) = \partial_r(u^0(r)), \quad \nu(0, t) = 0, \quad r, t > 0. \quad (2.33)
\]

In other words, the PME and PLE has an equivalence relation for the one spatial dimension, which is given by

\[
u = u_r, \quad m = p - 1. \quad (2.34)
\]

It seems that the equivalence relation is of independent interest.

Note that, for the case of one spatial dimension, the Newtonian potential of \(\nu := u_r\) is simply the antiderivative of the solution \(u\), which gives the mass of the solution. (This antiderivative successfully played the role of a potential for a convection problem in \([40, 41]\).) In fact, for all dimensions \(d \geq 1\), we take the mass concentration \(U\) in the place of the potential:

\[
U(r, t) := \omega_d \int_0^r x^{d-1} u(x, t) \, dx \quad \text{for} \quad r, t \geq 0.
\]

Since we are considering \(L^1\)-solutions, the concept of relative potential is not needed. However, for a situation without integrability, the relative mass in (2.14) can be useful. In any case, one can see that only the mass difference plays a role in the following asymptotic analysis.

The Barenblatt-type solution \(\rho(r, t)\) of the PLE given in (2.6) can be written in the radial variable which is

\[
\rho(r, t) = t^{-\alpha} (C_M - k(r^{-\alpha})^2)^{\frac{p-1}{p-2}}, \quad (2.35)
\]

where \(\alpha = \frac{1}{d(p-2)+p}, \quad k = \frac{d-2}{d} \alpha^{1/(p-1)}\). Let \(\mathcal{R}(r, t)\) be the mass concentration of \(\rho(r, t)\). Then the mass conservation gives

\[
M = \lim_{r \to \infty} U(r, t) = \lim_{r \to \infty} \mathcal{R}(r, t) \quad \text{for all} \quad t > 0. \quad (2.36)
\]
Proposition 3 (Comparison Principle). Let u₁ and u₂ be two bounded solutions of the radial p-Laplacian equation (2.42) and U₁ and U₂ be their mass concentrations, respectively. If t₀ > 0 and

\[ U₁(r, t₀) - U₂(r, t₀) ≥ 0 \quad \text{for all } r > 0, \]

then

\[ U₁(r, t) - U₂(r, t) ≥ 0 \quad \text{for all } r > 0, \ t ≥ t₀. \]

Proof. Let E := U₁ - U₂ be the relative mass. Then, the initial condition gives E(x, t₀) ≥ 0, and we need to show E(x, t) ≥ 0 for all t > t₀. Note that

\[ U₁ = ω₁ d^{d-1} |u₁|^{p-2} u₁, \]
\[ ω₁ d u₁ = \frac{1}{r^{d-1}} U₁ r - \frac{d-1}{r^d} U₁, \]
\[ E_t = a(r, t) E r - \frac{(d-1)a(r, t)}{r} E r, \]

where

\[ a(r, t) := \frac{|u₁|^{p-2} u₁ - |u₂|^{p-2} u₂}{u₁ - u₂} ≥ 0 \quad \text{for } p > 1. \]

If \( \frac{2d}{d+1} < p < 2 \), the solutions are supported on the whole space and the diffusion is non-degenerate. Hence the maximum principle gives E(x, t) ≥ 0 for all \( t > t₀ \). For \( p > 2 \), the solutions are compactly supported and the diffusion is degenerate at the zero points. Then, for a fixed time \( T > 0 \), there exists a large number \( C_T \) such that \( E(r, t) = 0 \) for all \( r ≥ C_T \) and \( t₀ ≤ t ≤ T \). So we can apply the maximum principle in \([1/n, C_T] × [0, T]\) for any natural number \( n \). Assume the minimum value is negative, say \( -ε < 0 \) in \([1/N] × [0, T]\). Then the minimum values in \([1/n] × [0, T]\) for all \( n > N \) must be less than or equal to \( -ε \). However, this contradicts to the facts \( E(0, t) = 0 \) for \( t₀ ≤ t ≤ T \) and \( E \) is continuous. Therefore, \( E(x, t) ≥ 0 \) for all \( p > \frac{2d}{d+1} \).

Now we sandwich the mass concentration of a solution between the mass concentration of the Barenblatt solution and the one of its time delay.

Lemma 5 (Sandwiched). There exist \( t₀, T > 0 \) such that

\[ R(r, t + T) ≤ U(r, t) ≤ R(r, t), \quad r > 0, \ t > t₀. \quad (2.37) \]

Proof. We first check the second inequality. Let \( ζ(t) > 0 \) be the boundary of the support of \( ρ(r, t) \). Let \( r > 0 \) be given. Then, there exists small \( ϵ > 0 \) such that \( ζ(ε) < r \). Then,

\[ U(r, 0) ≤ M = ω₁ d \int_0^{ζ(t)} x^{d-1} ρ(x, t) dx = R(r, ε) ≤ R(r, 0). \]

Therefore, the second inequality holds for all \( t > 0 \) from the comparison principle.

Since \( \|u(t) - ρ(t)\|_∞ → 0 \) as \( t → ∞ \) and the solution \( u \) becomes strictly positive at the origin in a finite time, there exist positive numbers \( t₀, ε \) and \( δ \) such that

\[ u(r, t₀) ≥ ϵ > 0 \quad \text{for all } |r| ≤ δ. \]

Let \( L(t) \) be the outside boundary of the support of the solution \( u \) at time \( t \), that is,

\[ L(t) := \text{sup}(\text{supp}(u(\cdot, t))). \]

Then obviously \( L(t₀) ≥ δ \). Now pick a number \( T > 0 \) such that \( ε(δ/L(t₀))^d ≥ ρ(r, t₀ + T) \) for all \( r ∈ R \). In this setting we consider three different cases according to the intervals.
1. If $|r| \leq \delta$, then $\mathcal{U}(r, t_0) \geq \omega_d \int_0^{|r|} x^{d-1} \epsilon(\delta/L(t_0))^d dx \geq R(r, t_0 + T)$.

2. If $\delta \leq |r| \leq L(t_0)$, then it holds that

$$\mathcal{U}(r, t_0) \geq \mathcal{U}(\delta, t_0) \geq \omega_d \frac{\epsilon^d}{d} \delta^d$$

$$= \omega_d \int_0^{L(t_0)} x^{d-1} \epsilon \left(\frac{\delta}{L(t_0)}\right)^d dx \geq \mathcal{R}(r, t_0 + T)$$

3. If $|r| \geq L(t_0)$, then $\mathcal{U}(r, t_0) = M \geq \mathcal{R}(r, t_0 + T)$.

Therefore, $\mathcal{R}(r, t_0 + T) \leq \mathcal{U}(r, t_0)$. Hence the comparison principle in Proposition 3 gives the rest of the proof.

**Lemma 6.** The fundamental solution $\rho$ of PLE satisfies

$$||r^{d-1}|\rho_r|^p - 1||_{\infty} = O(t^{-1}) \text{ as } t \to \infty.$$ 

**Proof.** The Barenblatt-type solution is

$$\rho(r, t) = t^{-\alpha} (C_M - k(r t^{-\alpha})^{\frac{p-1}{p}}),$$

where $\alpha = \frac{1}{d(p-2)} + 2/k^p$, $k = \frac{d^2}{d} \alpha^{1/(p-1)}$, and $C_M$ is the positive constant that sets the total mass to the solution to be $M > 0$. Note that $|r^{d-1}|\rho_r|^{p-2}\rho_r| = |r|^{d-1}|\rho_r|^{p-1}$ has the maximum at the interface of $\rho$ and the interface is of order $t^\alpha$. Therefore, it suffices to consider

$$\bar{\rho}(r, t) := t^{-\alpha} (rt^{-\alpha})^{\frac{p-1}{p}}$$

to obtain the order of $||r^{d-1}|\rho_r|^p - 2\rho_r||_{\infty}$. Now compute

$$r^{d-1}|\bar{\rho}_r|^p |_{r=t^\alpha} = t^{(d-1)\alpha} \left(\frac{p}{p-2} t^{-d\alpha - \frac{p}{p-2}} + \frac{2\alpha}{p-2}\right)^{p-1}$$

$$= \left(\frac{p}{p-2}\right)^{p-1} t^{(d-1)\alpha} \cdot t^{-(d+1)(p-1)\alpha}$$

$$= O(t^{-1}) \text{ as } t \to \infty.$$ 

**Proof of Theorem 2 for PLE.** From Lemma 6 there exist positive $T > t_0 > 0$ such that

$$\mathcal{R}(r, t + T) \leq \mathcal{U}(r, t) \leq \mathcal{R}(r, t) \text{ for all } r > 0, t \geq t_0.$$ 

Therefore for all $t \geq t_0$, we have

$$|\mathcal{R}(r, t) - \mathcal{U}(r, t)| \leq |\mathcal{R}(r, t) - \mathcal{R}(r, t + T)|$$

$$\leq \sup_{t \in (t, t + T)} T|\mathcal{R}(r, t)|$$

$$\leq T \omega_d ||r^{d-1}|\rho_r|^p - 1||_{\infty}.$$ 

Hence $||\mathcal{R}(r, t) - \mathcal{U}(r, t)||_{\infty} \leq \omega_d T ||r^{d-1}|\rho_r|^p - 1||_{\infty} = O(1/t)$ as $t \to \infty$. Finally by virtue of the fact that the Barenblatt-type solution $\rho$ and a solution $u$ have the only one positive intersection point $\beta(t)$, we
can translate this \(L^\infty\)-distance between potentials to the required \(L^1\)-convergence order:

\[
\frac{1}{\omega_d} \| \rho(t) - u(t) \|_1 = \int_0^\infty r^{d-1} |\rho(r, t) - u(r, t)| \, dr \\
= 2 \int_0^{\beta(t)} r^{d-1} (\rho(r, t) - u(r, t)) \, dr \\
= \frac{2}{\omega_d} \| \mathcal{R}(\cdot, t) - U(\cdot, t) \|_\infty \\
= O(1/t) \quad \text{as} \quad t \to \infty,
\]

which completes the proof. \( \square \)

### 2.5 Open Problems

1. Higher order asymptotics is still open. For FDE, a spectral calculation has been done by Denzler and McCann \[26\] \[27\]. For PME, evolution completeness of eigenfunctions was proved by Galaktionov and Harwin \[34\].

2. Prove the conjecture (2.31): Let \( \beta(t) \) be the unique intersection point between a radial solution to the PME or FDE and the Barenblatt solution having same mass. Then it holds that

\[
\frac{\beta(t)}{t^\alpha} \to \sqrt{2C_M m d} \quad \text{as} \quad t \to \infty,
\]

here \( \alpha \) is the similarity scale and \( C_M \) is a constant comes from the explicit form of the Barenblatt solution (2.5).

3. Find a potential defined in a unified way like equation (2.9) when the given function \( v \) is not radial. It would be

\[
V(x) = \int_0^\infty r^{1-d} \int_{B_r(x)} v(y) \, dy \, dr.
\]

Does this concept make proofs simple?
Chapter 3. Approximations for the Heat Equation

In this Chapter we introduce two kinds of approximate functions of solutions to the heat equation; one by Duoandikoetxea and Zuazua [28] and the other by Kim and Ni [45]. Duoandikoetxea and Zuazua’s approximation will be used in Chapter 5 to study long-time asymptotics of the zero level sets. Kim and Ni’s approximation will be used in Chapter 4 to find an approximation for the viscous Burgers equation.

3.1 Duoandikoetxea and Zuazua’s Approximation

Duoandikoetxea and Zuazua’s approximation is a linear combination of partial derivatives of the heat kernels. Their result is about $L^p$-convergence as $t \to \infty$ with fixed number of terms to be summed, but not about what happens when the number of terms to be summed goes to infinity. In this section we will find an estimate in the $L^\infty$-norm considering number of terms to be summed and show that when the fixed time $t$ is sufficiently large, an $L^1$-initial data bounded by a Gaussian function guarantees the $L^\infty$-convergence of approximation as more terms are summed.

Let $u(x, t)$ be a solution to the heat equation

$$ u_t = \Delta u \quad \text{in} \quad \mathbb{R}^d \times (0, \infty), $$

$$ u(x, 0) = u_0(x) \quad \text{in} \quad \mathbb{R}^d $$

and $G(x, t)$ be the heat kernel

$$ G(x, t) = (4\pi t)^{-\frac{d}{2}} \exp \left( -\frac{|x|^2}{4t} \right). $$

Then Duoandikoetxea and Zuazua’s original result (Theorem 4 in [28]) is the following:

**Theorem 4 (Duoandikoetxea-Zuazua).** If $1 \leq p < \frac{d}{d-1}$ and $p \leq q \leq \infty$, then there exists a constant $C = C(k, p, q, d) > 0$ such that

$$ \|u(\cdot, t) - \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left( \int x^\alpha u_0(x) \, dx \right) D^\alpha G(\cdot, t) \|_q \leq Ct^{k+1} \|x|^{k+1} u_0(x)\|_p $$

for all $u_0 \in L^1(\mathbb{R}^d; 1 + |x|^{k+1})$ satisfying $|x|^{k+1} u_0(x) \in L^p(\mathbb{R}^d)$. Here $k \geq 0$ is an integer.

Because the constant $C$ in the Theorem depends on $k$, which is related to number of terms to be summed, we don’t know what would happen when we add infinite number of terms in the approximation. Our estimate in the $L^\infty$-norm considering number of terms to be summed can be derived by slightly improving the original proof by Duoandikoetxea and Zuazua. The following lemma is crucial for their and our proof alike.

**Lemma 7.** Assume $f \in L^1(\mathbb{R}^d; 1 + |x|^{k+1})$ for some integer $k \geq 0$. Then there exists a family of functions $\{F_\alpha\}$ such that

$$ f = \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left( \int x^\alpha f(x) \, dx \right) D^\alpha \delta + \sum_{|\alpha|=k+1} D^\alpha F_\alpha $$

(3.1)
in the sense of tempered distribution. Moreover, $F_\alpha \in L^1(\mathbb{R}^d)$ and

$$
\|F_\alpha\|_1 \leq \frac{\|x^\alpha f(x)\|_1}{\alpha!}.
\tag{3.2}
$$

**Proof.** Let $\phi$ be a Schwartz function. By the Taylor’s formula,

$$
\phi(x) - (P_k \phi)(x) = (k + 1) \int_0^1 (1 - t)^k \sum_{|\alpha| = k+1} \frac{x^\alpha}{\alpha!} D^\alpha \phi(tx) dt
$$

where $P_k$ denotes the Taylor polynomial at $x = 0$. Hence

$$
\langle f - \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left( \int x^\alpha f(x) \, dx \right) D^\alpha \delta, \phi \rangle
$$

$$
= \int f(x) \phi(x) \, dx - \sum_{|\alpha| \leq k} \frac{(D^\alpha \phi)(0)}{\alpha!} \int x^\alpha f(x) \, dx
$$

$$
= \int f(x) \phi(x) \, dx - \int f(x) \sum_{|\alpha| \leq k} \frac{(D^\alpha \phi)(0)}{\alpha!} x^\alpha \, dx
$$

$$
= \int f(x) \left( \phi(x) - (P_k \phi)(x) \right) \, dx
$$

$$
= (k + 1) \sum_{|\alpha| = k+1} \int_{\mathbb{R}^d} \int_0^1 f(x)(1 - t)^k \frac{x^\alpha}{\alpha!} D^\alpha \phi(tx) \, dt \, dx
$$

$$
= \sum_{|\alpha| = k+1} \int_{\mathbb{R}^d} D^\alpha \phi(x) \int_0^1 (k + 1) \frac{(1 - t)^k}{t^{d+|\alpha|}} \frac{x^\alpha}{\alpha!} f \left( \frac{x}{t} \right) \, dt \, dx
$$

$$
= \sum_{|\alpha| = k+1} \langle D^\alpha F_\alpha, \phi \rangle,
$$

where

$$
F_\alpha(x) := (-1)^{|\alpha|} \int_0^1 (k + 1) \frac{(1 - t)^k}{t^{d+|\alpha|}} \frac{x^\alpha}{\alpha!} f \left( \frac{x}{t} \right) \, dt.
$$

By the Minkowski’s inequality for integrals,

$$
\|F_\alpha\|_1 \leq \int_0^1 \int_{\mathbb{R}^d} \left| (k + 1) \frac{(1 - t)^k}{t^{d+|\alpha|}} \frac{x^\alpha}{\alpha!} f \left( \frac{x}{t} \right) \right| \, dx \, dt
$$

$$
= \frac{\|x^\alpha f(x)\|_1}{\alpha!} \int_0^1 (k + 1)(1 - t)^k \, dt = \frac{\|x^\alpha f(x)\|_1}{\alpha!}.
$$

Combining this lemma, an explicit formula for solutions and an estimate for the Hermite polynomials, we can obtain the following uniform estimate.

**Theorem 5** (Hermite polynomial approximation). Let $u$ be the unique solution the heat equation with initial data $u_0 \in L^1(\mathbb{R}^d, 1 + |x|^{k+1})$ for some integer $k \geq 0$. Then there holds that for all $x \in \mathbb{R}^d$,

$$
|u(x, t) - e^{-\frac{x^2}{4t}} \sum_{|\alpha| \leq k} \frac{x^\alpha u_0(x)}{\alpha!} \frac{1}{(4t)^{\frac{|\alpha|+d}{2}}} \prod_{i=1}^d H_\alpha \left( \frac{x_i}{2\sqrt{t}} \right) | \leq (2\pi)^{-\frac{d}{2}} (2t)^{-\frac{k+1}{2}} \sum_{|\alpha| = k+1} \frac{\|x^\alpha u_0(x)\|_1}{\sqrt{\alpha!}} \left( \prod_{i=1}^d (\alpha_i + 1) \right)^{-\frac{1}{2}}
\tag{3.3}
$$

where the functions $H_\alpha(x)$ are the (physicists’) Hermite polynomials defined by

$$
H_\alpha(x) := (-1)^\alpha e^{x^2} \frac{d^\alpha}{dx^\alpha} e^{-x^2}.
$$
Proof. The solution \( u \) is given by the convolution with the heat kernel

\[
u(x, t) = [G(\cdot, t) \ast u_0](x) = \int G(x - y, t)u_0(y) \, dy.
\]

We put the representation (3.1) of \( u_0 \) here. Then it holds that

\[
u(x, t) = \langle u_0, G(x - \cdot, t) \rangle
= \left\langle \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left( \int x^\alpha u_0(x) \, dx \right) D^\alpha \delta + \sum_{|\alpha| = k+1} D^\alpha F_\alpha, G(x - \cdot, t) \right\rangle
= \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left( \int x^\alpha u_0(x) \, dx \right) D^\alpha G(x, t) + \sum_{|\alpha| = k+1} \langle F_\alpha, D^\alpha G(x - \cdot, t) \rangle.
\]

Hence by Young’s inequality,

\[
\|u(\cdot, t) - \sum_{|\alpha| = k+1} D^\alpha G(\cdot, t) \ast F_\alpha\|_\infty \leq \sum_{|\alpha| = k+1} \|D^\alpha G(\cdot, t)\|_\infty \|F_\alpha\|_1.
\]

(3.4)

Now we estimate \( \|D^\alpha G(\cdot, t)\|_\infty \). Let \( y = (y_i) = x/(2\sqrt{t}) \). Then derivatives of the heat kernel can be written as a product of the Hermite polynomials:

\[
D^\alpha G(x, t) = (4\pi t)^{-\frac{\alpha}{2}} D^\alpha \exp \left( -\frac{|x|^2}{4t} \right)
= (4\pi t)^{-\frac{\alpha}{2}} D^\alpha \exp(-|y|^2/\sqrt{4t})
= \pi^{-\frac{\alpha}{2}} (4t)^{-|\alpha|/2} (-1)^{|\alpha|} \prod_{i=1}^d \left[ H_{\alpha_i}(y_i) e^{-y_i^2} \right]
\]

and we use an estimate for the Hermite polynomials (see Bonan and Clark [10])

\[
\max_{x \in \mathbb{R}} |H_n(x)|e^{-x^2} \leq 2^{\frac{\alpha}{2}} \sqrt{n!} (n + 1)^{-\frac{1}{\alpha}}
\]

for all \( n \geq 0 \)

to obtain

\[
\|D^\alpha G(\cdot, t)\|_\infty \leq \pi^{-\frac{\alpha}{2}} 2^{-\frac{|\alpha|}{2}} \sqrt{\frac{d}{\alpha}} t^{-|\alpha|/2} \left( \prod_{i=1}^d (\alpha_i + 1) \right)^{-\frac{1}{\alpha}}.
\]

Finally we use the \( L^1 \)-estimate of \( F_\alpha \) in equation (3.2) to complete the proof. \( \square \)

From now on \( u_k \) denote the Hermite polynomial approximation of the solution \( u \):

\[
u_k(x, t) := \pi^{-\frac{\alpha}{2}} e^{-\frac{|x|^2}{4t}} \sum_{|\alpha| \leq k} \frac{\int x^\alpha u_0(x) \, dx}{\alpha!} (4t)^{-|\alpha|/2} \prod_{i=1}^d H_{\alpha_i}(x_i/(2\sqrt{t})).
\]

Now we are ready to show that when the fixed time \( t \) is sufficiently large, an \( L^1 \)-initial data bounded by a Gaussian function guarantees the \( L^\infty \)-convergence of approximation as more terms are summed.

Corollary 2 (Convergence of Approximation). Assume the initial data is bounded by a Gaussian function: \( |u_0(x)| \leq C e^{-\frac{|x|^2}{4t_0}} \) a.e. \( x \) for some positive constants \( C \) and \( t_0 \). Then it holds that

\[
\|u(\cdot, t) - u_k(\cdot, t)\|_\infty \leq C \left( \frac{t_0}{t} \right)^{k+d+1} \left( 1 + \frac{k+1}{d} \right)^{-\frac{d}{2}} \frac{(k+d)\ldots}{(k+1)! (d-1)!} \cdot G(k),
\]

- 21 -
Remark 1. The right hand side $G(k)$ in the corollary contains three terms; $(t_0/t)^{k+d+1}$ is an exponential function of $k$, $(1 + (k + 1)/d)^{-1/2}$ is a power function of $k$ and $(k+d)!/(d-1)!$ is a polynomial of $k$. Hence if the fixed time $t$ is strictly bigger than $t_0$, the right hand side $G(k)$ goes to zero as $k$ goes to infinity and the approximation converges to the solution. If $d = 1$ and $t = t_0$, then $G(k) = C(k + 2)^{-1/2}$ so that the approximation still converges but the convergence is very slow.

Proof. Let $F(k)$ be the right hand side of the estimate \((5.3)\): 

$$F(k) := (2\pi)^{-\frac{d}{2}} (2t)^{-\frac{k+d+1}{2}} \sum_{|\alpha|=k+1} \frac{\|x^\alpha u_0(x)\|_1}{\sqrt{\alpha!}} \left( \prod_{i=1}^d (\alpha_i + 1) \right)^{-\frac{1}{2}}$$

We will show that $F(k)$ is bounded by $G(k)$. First we calculate a bound for moments:

$$\|x^\alpha u_0(x)\|_1 \leq C \|x^\alpha e^{-\frac{|x|^2}{2t_0}}\|_1 = C(4t_0)^{-\frac{|\alpha|+2}{2}} \prod_{i=1}^d \Gamma\left(\frac{\alpha_i + 1}{2}\right).$$

From the duplication formula for Gamma functions [1] p256, 

$$\Gamma(z) \Gamma\left(z + \frac{1}{2} \right) = 2^{1-2z} \sqrt{\pi} \Gamma(2z),$$

it holds that 

$$\Gamma\left(\frac{\alpha_i + 1}{2}\right)^2 \leq \Gamma\left(\frac{\alpha_i + 1}{2} \right) \sqrt{\pi} \Gamma\left(\alpha_i + 1 \right) = 2^{-\alpha_i} \pi \Gamma(\alpha_i + 1).$$

By these inequalities and the inequality of arithmetic and geometric means, we can find a bound of $F(k)$:

$$F(k) \leq C(2\pi)^{-\frac{d}{2}} \left( \frac{2t^2}{t} \right)^{\frac{k+d+1}{2}} \sum_{|\alpha|=k+1} \frac{\prod_{i=1}^d \Gamma\left(\frac{(\alpha_i + 1)/2}{\alpha_i + 1}\right)}{\sqrt{\Gamma(\alpha_i + 1)}} \left( \prod_{i=1}^d (\alpha_i + 1) \right)^{-\frac{1}{2}}$$

$$\leq C \left( \frac{t_0}{t} \right)^{\frac{k+d+1}{2}} \left( 1 + \frac{k+1}{d} \right)^{-\frac{d}{2}} \sum_{|\alpha|=k+1} 1$$

$$= C \left( \frac{t_0}{t} \right)^{\frac{k+d+1}{2}} \left( 1 + \frac{k+1}{d} \right)^{-\frac{d}{2}} \frac{(k+d)!}{(k+1)! (d-1)!} = G(k).$$

\[\square\]

Theorem 6 (Divergence of Approximation). Assume the initial data is a Gaussian function: $u_0(x) = Ce^{-\frac{|x|^2}{2}}$ a.e. $x$ for some positive constants $C$ and $t_0$. Then for any fixed $t$ such that $0 < t < t_0$ we have

$$\|u_k(\cdot, t)\|_\infty \to \infty \quad \text{as } k \to \infty.$$ 

More precisely, when $d = 1$ there is a constant $B > 0$ which does not depend on $k$ such that 

$$|u_k(0, t)| \geq B \left( \frac{t_0}{t} \right)^{\frac{1}{2}} \left[ \frac{1}{\lfloor k/2 \rfloor - 1} \right] \quad \text{for every sufficiently large } k.$$ 

Also when $d \geq 2$ it holds that 

$$|u_k(0, t)| \geq \frac{C}{(4t)^{d/2} \Gamma(d/2)} \left( \frac{t_0}{t} \right)^{\frac{1}{2}} \left( \frac{t_0}{t} \right)^{\frac{1}{2}} \quad \text{for every } k.$$ 

Proof. Because initial data is radially symmetric ($u_0(x) = u_0(r)$, $r = |x|$), the approximation $u_k(x)$ can be simplified by observing moments. Assume $\alpha_i$ is odd. Then 

$$\int_{-\infty}^{\infty} x_i^{\alpha_i} u_0(r) \, dx_i = \int_{0}^{\infty} x_i^{\alpha_i} u_0(r) \, dx_i + \int_{-\infty}^{0} x_i^{\alpha_i} u_0(r) \, dx_i$$

$$= \int_{0}^{\infty} x_i^{\alpha_i} u_0(r) \, dx_i - \int_{0}^{\infty} x_i^{\alpha_i} u_0(r) \, dx_i = 0.$$
Hence for a nonzero moment every \( \alpha_i \)'s are even. Consequently a moment is zero when \(|\alpha|\) is odd and we may assume every \( \alpha_i \)'s are even.

First we consider the one-dimensional case \( d = 1 \). Then we have

\[
\sqrt{\pi} u_k(0, t) = \sum_{j \leq k \atop j \text{ even}} \frac{\int x^j u_0(x) \, dx}{j!} (4t)^{-(j+1)/2} H_j(0)
\]

\[
= C \sum_{j \leq k \atop j \text{ even}} \frac{1}{j!} \left( \frac{t_0}{t} \right)^{j+1} \Gamma \left( \frac{j+1}{2} \right) H_j(0)
\]

\[
= C \sqrt{\pi t_0/t} + C \sum_{0 < j \leq k \atop j \text{ even}} \frac{1}{j!} \left( \frac{t_0}{t} \right)^{j+1} \Gamma \left( \frac{j+1}{2} \right) H_j(0)
\]

\[
= C \sqrt{\pi t_0/t} + C \pi^{-1/2} \sum_{0 < j \leq k \atop j \text{ even}} (-1)^{j/2} \left( \frac{t_0}{t} \right)^{j+1} \Gamma \left( \frac{j+1}{2} \right)^2 2^j.
\]

We may assume \( k \) is even. Also assume \( k/2 \) is even for now. Then it holds that

\[
\frac{\sqrt{\pi} (-1)^{k/2}}{C} u_k(0, t) = (-1)^{k/2} \sqrt{\pi t_0/t} + \pi^{-1/2} \sum_{0 < j \leq k \atop j \text{ even}} (-1)^{(j+k)/2} \left( \frac{t_0}{t} \right)^{j+1} \Gamma \left( \frac{(j+1)/2}{2} \right) 2^j
\]

(by duplication formula)

\[
= \sqrt{\pi t_0/t} + \left( \frac{t_0}{t} \right)^{k+1} \Gamma \left( \frac{(k+1)/2}{2} \right) \Gamma \left( \frac{(k-1)/2}{4} \right) - \left( \frac{t_0}{t} \right)^{k+1} \Gamma \left( \frac{(k-1)/2}{2} \right) \Gamma \left( \frac{(k-3)/2}{4} \right)
\]

\[
+ \cdots + \left( \frac{t_0}{t} \right)^{k+1} \Gamma \left( \frac{5/2}{2} \right) \Gamma \left( \frac{3/2}{4} \right) \Gamma \left( \frac{1/2}{4} \right).
\]

By Stirling’s formula \([11, p257]\), there exists a constant \( B > 0 \) such that

\[
|\frac{\Gamma ((k+1)/2)}{\Gamma (k/2+1)}| - \sqrt{\frac{2}{k}} = \frac{B}{k} \quad \text{for every sufficiently large} \quad k.
\]

Therefore for every sufficiently large \( k \),

\[
\frac{\sqrt{\pi} (-1)^{k/2}}{C} u_k(0, t) \geq \left( \frac{t_0}{t} \right)^{k+1} \left( \sqrt{\frac{2}{k}} - \frac{B}{k} \right) - \left( \frac{t_0}{t} \right)^{k+1} \left( \sqrt{\frac{2}{k-2}} + \frac{B}{k-2} \right)
\]

\[
= \left( \frac{t_0}{t} \right)^{k+1} \sqrt{\frac{2}{k-2}} \left( \frac{t_0}{t} \right)^{k+1} \left( \sqrt{\frac{2}{k-2}} \right) - \left( \frac{t_0}{t} \right)^{k+1} \left( \sqrt{\frac{2}{k-2}} \right) \left( \sqrt{\frac{2}{k-2}} \right)
\]

\[
\geq \left( \frac{t_0}{t} \right)^{k+1} \sqrt{\frac{2}{k-2}}.
\]

Similar computation works when \( k/2 \) is odd. We have just proved that for every sufficiently large \( k \),

\[
|u_k(0, t)| \geq \left( \frac{t_0}{t} \right)^{k/2} k^{-1/2}.
\]
Next consider the case $d \geq 2$. Using hyperspherical coordinates

\[
x_1 = r \cos(\psi_1)
\]
\[
x_2 = r \sin(\psi_1) \cos(\psi_2)
\]
\[
\vdots
\]
\[
x_{d-1} = r \sin(\psi_1) \cdots \sin(\psi_{d-2}) \cos(\psi_{d-1})
\]
\[
x_d = r \sin(\psi_1) \cdots \sin(\psi_{d-2}) \sin(\psi_{d-1}),
\]

moments of initial data can be written as

\[
\int x^n u_0(x) \, dx = \int_0^\infty \cdots \int_0^\infty \int_0^{2\pi} \int_0^\infty \rho^{\alpha_1} \cos(\psi_1)^{\alpha_1} \sin(\psi_1) \cos(\psi_2)^{\alpha_2}
\]
\[
\times \cdots \times (\sin(\psi_1) \cdots \sin(\psi_{d-2}) \cos(\psi_{d-1}))^{\alpha_{d-1}} (\sin(\psi_1) \cdots \sin(\psi_{d-2}) \sin(\psi_{d-1}))^{\alpha_d}
\]
\[
\times r^{d-1} \sin^2(\psi_1) \sin^2(\psi_2) \cdots \sin(\psi_{d-2}) u_0(r) \, dr \, dv_1 \cdots dv_{d-1}
\]
\[
= \int_0^\infty r^{\alpha+d-1} u_0(r) \, dr \times \prod_{i=1}^{d-2} \int_0^{\pi} \cos^{\alpha_i}(\psi_i) \sin^{\sum_{j=1}^{i+1} \alpha_j + d - i - 1}(\psi_i) \, d\psi_i
\]
\[
\times \int_0^{2\pi} \cos^{\alpha_{d-1}}(\psi_{d-1}) \sin^{\alpha_d}(\psi_{d-1}) \, d\psi_{d-1}.
\]

Now we use trigonometric integrals

\[
\frac{1}{n!} \int_0^\pi \cos^n(\psi) \sin^m(\psi) \, d\psi = \left\{
\begin{array}{ll}
\int_0^\pi \sin^n(\psi) \, d\psi & \text{if } n \text{ is even} \\
\frac{1}{(m+n)(m+n-2)\cdots(m+2n+1)} \frac{2n!}{(n!)^2} & \text{if } n \text{ is odd,}
\end{array}
\right.
\]

\[
\frac{1}{n!m!} \int_0^{2\pi} \cos^n(\psi) \sin^m(\psi) \, d\psi = \left\{
\begin{array}{ll}
\frac{2\pi}{(n+m)(n+m-2)\cdots (2n+1)} & \text{if } n, m \text{ are even}.
\end{array}
\right.
\]

\[
to \text{conclude that } \frac{1}{\alpha!} \int x^\alpha u_0(x) \, dx = \frac{C(\alpha, d)}{2^{\alpha/2} \prod_{i=1}^{d} (\alpha_i/2)!} \int_0^\infty r^{\alpha+d-1} u_0(r) \, dr
\]

where $C(\alpha, d)$ is a nonzero constant defined by

\[
C(\alpha, d) := \left\{
\begin{array}{ll}
\frac{(2\pi)^{\alpha/2}}{(\alpha+d-2)(\alpha+d-4)\cdots(2)} & \text{if } d \text{ is even,}
\frac{(2\pi)^{\alpha/2}}{2^{\alpha+d-2} \Gamma((\alpha+d)/2)} & \text{if } d \text{ is odd,}
\end{array}
\right.
\]

Also we recall generalized Laguerre polynomials [1] p775]

\[
L_n^{(a)}(x) := \frac{e^{ax} x^n}{n!} \frac{d^n}{dx^n} (e^{-x} x^n + a),
\]

and their properties [1] p785,p779]

\[
L_n^{(a+b+1)}(x+y) = \sum_{i=0}^{n} L_i^{(a)}(x) L_{n-i}^{(b)}(y),
\]
\[
H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{(-1/2)}(x^2).
\]
Then the Hermite polynomial approximation $u_k$ becomes

\[
\begin{align*}
(4\pi t)^{\frac{|n|}{2}} e^{-\frac{x^2}{2t}} u_k(x, t) &= \sum_{j \leq k} \sum_{|\alpha|=j} (4t)^{-\frac{|\alpha|}{2}} \int x^\alpha u_0(x) \frac{dx}{\alpha!} \prod_{i=1}^{d} H_{\alpha_i}(\frac{x_i}{2\sqrt{t}}) \\
&= \sum_{j \leq k}^{j \text{ even}} (4t)^{-\frac{j}{2}} \int_{0}^{\infty} r^{j+d-1} u_0(r) \, dr \sum_{|\alpha|=j}^{\alpha_i/2} \frac{C(j, d)}{2^{j/2} \prod_{i=1}^{d} H_{\alpha_i}(\frac{x_i}{2\sqrt{t}})} \\
&= \sum_{j \leq k}^{j \text{ even}} (4t)^{-\frac{j}{2}} \frac{C}{2} \Gamma(j + d/2)(-2)^j C(j, d) \prod_{i=1}^{d} L_{\alpha_i/2}((\frac{x_i^2}{4t})) \\
&= \frac{C}{2} \sum_{j \leq k}^{j \text{ even}} \left( -\frac{2t_0}{t} \right)^{\frac{j}{2}} \Gamma(j + d/2) C(j, d) L_{j/2}((\frac{x_0^2}{4t})).
\end{align*}
\]

Because we know [1 p777]

\[
L_n^{(a)}(0) = \binom{n + a}{n} = \frac{\Gamma(n + a + 1)}{\Gamma(n + 1)\Gamma(a + 1)} \quad \text{if} \quad a \geq 0,
\]

it holds that

\[
\frac{2}{C} (4\pi t)^{\frac{|n|}{2}} u_k(0, t) = \sum_{j \leq k}^{j \text{ even}} \left( -\frac{2t_0}{t} \right)^{\frac{j}{2}} \Gamma(j + d/2) C(j, d) L_{j/2}((\frac{x_0^2}{4t})).
\]

We may assume $k$ is even. When the dimension $d$ is even and $k/2$ is even, we have

\[
(4t)^{d/2} C u_k(0, t) = \sum_{j \leq k}^{j \text{ even}} (-1)^{\frac{j+k}{2}} \left( \frac{t_0}{t} \right)^{\frac{j}{2}} \Gamma((j + d/2)) C(j, d) L_{j/2}((\frac{x_0^2}{4t})).
\]

Same result holds when $d$ is even and $k/2$ is odd. Now assume the dimension $d$ is odd. Then we have

\[
(4t)^{d/2} C u_k(0, t) = \sum_{j \leq k}^{j \text{ even}} (-1)^{\frac{j+k}{2}} 2^{d/4} \left( \frac{t_0}{t} \right)^{\frac{j}{2}} \Gamma((j + d/2)C(j, d) L_{j/2}((\frac{x_0^2}{4t})).
\]

(by duplication formula)

\[
\geq 2^{d/4} \sqrt{\pi} \sum_{j \leq k}^{j \text{ even}} (-1)^{\frac{j+k}{2}} \left( \frac{t_0}{t} \right)^{\frac{j}{2}} \Gamma((j + d/2)C(j, d) L_{j/2}((\frac{x_0^2}{4t})).
\]

(by same argument as above).
Summing up, we have just proved that when \( d \geq 2 \),

\[
|u_k(0, t)| \geq \frac{C}{(4t)^{d/2} \Gamma(d/2)} \left( \frac{t_0}{t} - 1 \right) \left( \frac{t_0}{t} \right)^{\frac{d}{2} - 1}.
\]

**Remark 2.** Kim and Ni [45] conjectured that if the dimension \( d = 1 \) and the initial data is a Gaussian function \( u_0(x) = \frac{1}{\sqrt{4\pi t_0}} e^{-x^2/4t_0} \), it holds that

\[
\lim_{k \to \infty} \frac{\|u(\cdot, t) - u_{k+2}(\cdot, t)\|_{L^\infty}}{\|u(\cdot, t) - u_{k}(\cdot, t)\|_{L^\infty}} = \frac{t_0}{t},
\]

where \( u_k(x, t) \) is the Duoandikoetxea and Zuazua’s approximation in one dimension. This is an open problem.

### 3.2 Kim and Ni’s Approximation

This section is devoted to a brief summary of a paper by Kim and Ni [45]. To approximate solutions to the one-dimensional heat equation

\[
u_t = \nu_{xx} \quad \text{in} \quad \mathbb{R} \times (0, \infty),
\]

\[
u(x, 0) = u_0(x) \in L^1(\mathbb{R}),
\]

they employed a linear combination of heat kernels

\[
\phi_k(x, t) := \sum_{i=1}^{k} \frac{\rho_i}{\sqrt{4\pi t}} e^{-(x-c_i)^2/4t}.
\]

First they proved a general result on two solutions having same moments:

**Theorem 7.** Let \( u(x, t) \) and \( v(x, t) \) be two solutions of the heat equation with initial data \( u_0(x) \) and \( v_0(x) \), respectively. Suppose that the initial difference \( E_0(x) := u_0(x) - v_0(x) \) satisfies that \( x^m E_0(x) \in L^1(\mathbb{R}) \) for some nonnegative integer \( m \) and

\[
\int x^k E_0(x) \, dx = 0 \quad \text{for all} \quad 0 \leq k < m,
\]

then there exists a function \( E_m \in W^{m,1}(\mathbb{R}) \) such that \( \partial_x^m E_m(x) = E_0(x) \) and

\[
\lim_{t \to \infty} t^{\left( \frac{m+1}{2} - \frac{d}{4} \right)} \|u(\cdot, t) - v(\cdot, t)\|_p = \frac{1}{\sqrt{4\pi}} \left\| \partial_x^m (e^{-x^2/4t}) \right\|_p \int E_m(x) \, dx
\]

for all \( 1 \leq p \leq \infty \).

Hence two solutions having same moments up to order \( m - 1 \) converges to each other in the \( L^p \)-norm with the order of \( t^{\left( \frac{m+1}{2} - \frac{d}{4} \right)} \). Note that the function \( \phi_k(x, t) \) has 2k degree of freedom; \( \rho_i \)'s and \( c_i \)'s for \( i = 1, 2, \cdots, k \). Hence if \( u_0 \) and the function \( \phi_k \) have same finite moments up to order 2k, then they converges to each other in the \( L^p \)-norm with the order of \( t^{\left( \frac{2k+1}{2} - \frac{d}{4} \right)} \). Kim and Ni provides an algorithm to find such \( \rho_i \)'s and \( c_i \)'s. We refer those interested in the existence and uniqueness of \( \rho_i \)'s and \( c_i \)'s to [42] [43].

Lastly we pay attention to what happens when the number of terms to be summed goes to infinity, i.e., when \( k \to \infty \). Kim and Ni conjectured that their approximation has a geometric convergence order as \( k \to \infty \):

\[
\lim_{k \to \infty} \frac{\|u(\cdot, t) - \phi_k(\cdot, t)\|_{L^\infty}}{\|u(\cdot, t) - \phi_{k+1}(\cdot, t)\|_{L^\infty}} = 1 + \frac{4t}{C},
\]

where \( C > 0 \) is a constant depending on the initial data \( u_0(x) \).
3.3 Open Problems

1. Prove that Kim and Ni’s conjecture for the Duoandikoetxea and Zuazua’s approximation in [15] holds in multi-dimensions: If the initial data is a Gaussian function $u_0(x) = Ce^{-|x|^2/4t_0}$ for some positive constants $C$ and $t_0$, then it holds that

$$\lim_{k \to \infty} \frac{\|u(\cdot,t) - u_{k+2}(\cdot,t)\|_\infty}{\|u(\cdot,t) - u_k(\cdot,t)\|_\infty} = \frac{t_0}{t},$$

where $u_k(x,t)$ is the Duoandikoetxea and Zuazua’s approximation

$$u_k(x,t) := \pi^{-d/2} e^{-|x|^2/4t} \sum_{|\alpha| \leq k} \frac{x^{\alpha} f(x) \, dx}{\alpha!} \left(\frac{4t}{|\alpha| + d/2}\right)^{-|\alpha|/2} \prod_{i=1}^d H_{\alpha i}(x_i/(2\sqrt{t})).$$

2. Prove Kim and Ni’s conjecture for their approximation in equation (3.5).
Chapter 4. Asymptotic Agreement of Moments and Higher Order Contraction in the Viscous Burgers Equation

4.1 Introduction

The main motivation of this chapter is to investigate the relation between the agreement of moments and the asymptotic contraction orders of solutions to convection-diffusion equations. Let \( u \) and \( \psi \) be integrable real-valued solutions to

\[
\frac{\partial u}{\partial t} + \nabla_x \cdot (F(u, \nabla u)) = 0,
\]

where \( x \in \mathbb{R}^d \) and \( F : \mathbb{R}^{1+d} \rightarrow \mathbb{R}^d \). Then one may ask what decides the asymptotic contraction order \( \gamma > 0 \), i.e.,

\[
\| u(x, t) - \psi(x, t) \|_r = O(t^{-\gamma}) \quad \text{as} \quad t \to \infty,
\]

where \( \| \cdot \|_r \) is the \( L^r \)-norm, \( r \geq 1 \), over the spatial domain \( x \in \mathbb{R}^d \).

It is well-known that two solutions to the (linear) heat equation share the same moments all the time if they do initially. Using this property it has been shown that, if

\[
\int x^k (u(x, 0) - \psi(x, 0)) \, dx = 0, \quad 0 \leq k \leq m,
\]

the asymptotic contraction order is \( \gamma = \frac{m+2}{2} - \frac{1}{2^r} \) (see [28, 42, 45]). This one dimensional asymptotics is easily extended to multidimensional ones. However, nonlinear problems do not have such a nice property. For the porous medium equation (PME for brevity) case, only the total mass and the center of mass have such a property (i.e., for \( k = 0, 1 \)). For the \( p \)-Laplacian equation case, even the center of mass do not have this property.

In this chapter we consider bounded solutions to the (viscous) Burgers equation in one spatial dimension,

\[
\frac{\partial u}{\partial t} + uu_x = \mu u_{xx} \quad \text{in} \quad \mathbb{R} \times (0, \infty),
\]

\[
u(x, 0) = u_0(x) \quad \text{in} \quad \mathbb{R},
\]

where \( \mu > 0 \) is the viscosity coefficient and the initial value \( u_0 \) is bounded and has finite moments up to order \( 2n \), i.e., \( x^{2n}u_0(x) \in L^1(\mathbb{R}) \). In this case the total mass \( (k = 0) \) is the only moment of which initial agreement guarantees permanent agreement. The reason why the Burgers equation is picked as an exemplary case is because of the Cole-Hopf transformation (see [38]), which makes an rigorous analysis possible. It is given by

\[
\Phi(x, t) = e^{-\frac{1}{2} \int_0^t u(y, t) \, dy} - 1 =: H(u).
\]

For notational convenience, we denote its spatial derivative as

\[
\phi(x, t) := \partial_x \Phi(x, t) = -\frac{1}{2\mu} u(x, t) e^{-\frac{1}{2} \int_0^t u(y, t) \, dy}.
\]
Then \( \Phi \) and \( \phi \) are solutions to the heat equation \((4.10)\) and \( \phi(x, t) \) has finite moments up to order \( 2n \), i.e., \( x^{2n}\phi(x, 0) \in L^1(\mathbb{R}) \). Its inverse transformation is given by

\[
\psi(x, t) = 2\mu \frac{\phi(x, t)}{1 + \Phi(x, t)} =: H^{-1}(\Phi).
\]

If \( \Phi \) is the Cole-Hope transformation of a function \( u(x) \), then

\[
\Phi(x, t) + 1 = e^{-\frac{r^2}{4t}} \int_\infty^{-\infty} u(y)dy > 0,
\]

and hence \( H^{-1}(\Phi) \) is well-defined. However, for general cases, one should show that the denominator \( 1 + \Phi(x, t) \) is strictly positive. The main theorem of this chapter is the following:

**Theorem 8.** Let \( u(x, t) \) be the solution to the Burgers equation \((4.3)\) with a bounded initial value \( u_0(x) \) such that \( x^{2n}u_0(x) \in L^1(\mathbb{R}) \). Then, for any \( t_0 \geq 0 \), there exist \( \rho_i, c_i \in \mathbb{C} \) and \( T \geq 0 \) such that \( w_n := H^{-1}(\Psi_n) \) is well-defined for \( t \geq T \), and, for \( 1 \leq r \leq \infty \) and \( k \geq 0 \),

\[
\|x^k(\psi(x, t) - w_n(x, t))\|_r = O((\sqrt{7})^{1/r-2n-1+k}) \quad \text{as} \quad t \to \infty,
\]

where

\[
\Psi_n(x, t) := \int_{-\infty}^{x} \psi_n(y, t)dy,
\]

\[
\psi_n(x, t) := \text{Re} \left( \sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi\xi(t + t_0)}} e^{-\frac{(x - c_i)^2}{4\xi(t + t_0)}} \right).
\]

It is clear that \( \psi_n(x, t) \) is a solution to the heat equation. The \( \rho_i \)'s and \( c_i \)'s are chosen to satisfy \( 2n \) equations of

\[
\int x^k \phi(x, t)dx = \int x^k \psi_n(x, t)dx \quad \text{for} \quad 0 \leq k < 2n.
\]

The construction of \( \psi_n(x, t) \) has been made in \((15)\) for the positive solutions with \( t_0 = 0 \) using the classical truncated moment problem. For general cases, the classical theory is not enough. However, we could construct \( \psi_n(x, t) \) using a generalized moment problem in \((42)\). Using these techniques higher order convergence have been obtained \((12)\) \((45)\). One may also control the moments using the derivatives of the heat kernel as in \((4.14)\). This technique has been used in \((28)\) and obtained higher order asymptotics.

The inverse transformation \( w_n \) is a solution to the Burgers equation and valid for \( t \geq T \). For the case with \( t_0 = 0 \) and \( u_0 \geq 0 \), it is proved that \( T = 0 \). Otherwise we only have a numerical evidence that \( T = 0 \). Note that, even if \( \phi \) and \( \psi_n \) have same moments up to order \( 2n - 1 \), their inverse transformations do not, i.e., \( \int x^k(\psi(x, t) - w_n(x, t))dx \neq 0 \). However, the asymptotic convergence order in \((4.7)\) shows that they approach to each other asymptotically. In other words the moment setting after the Cole-Hopf transformation actually gives asymptotic moments agreement for the solutions to the Burgers equation and provides fine asymptotics. The higher order contraction indicates that the solution \( w_n \) is an excellent asymptotic approximation of the solution \( u \) (the case for \( k = 0 \)).

The Cole-Hopf transformation has been a main tool to study the long-time behavior of the Burgers equation. It allows one to study the Burgers equation from the behavior of solutions to the heat equation \((4.1)\) \((48)\) \((54)\) \((71)\). For general nonlinear problems there is no such transformation. We only hope to glimpse the long-time behavior of general nonlinear problems from the study of the Burgers equation.

The solution to the Burgers equation has been played a prototype role in many problems such as traffic or fluid flows (see \((69)\)). It has been shown that the asymptotic behavior of general systems of hyperbolic conservations laws are given as a solution to the Burgers equation \((20)\) \((21)\) \((51)\). On the
other hand, asymptotic convergence to similarity solutions has been done for general convection-diffusion equations including the Burgers equation \((15, 29, 37, 73)\). Special attention has been given to the study of asymptotics of the porous medium equation for the last two decades (see \(9\)). One may find optimal convergence to the Barenblatt solution of similarity order \(O(t^{-1/(m+1)})\) (see \(18, 68\)). The convergence orders in \((4.7)\) indicate that the contraction order in \((4.2)\) will be increased by the similarity scale if the order of asymptotically converging moments increases. A brief discussion about this relation is given in Section 4.4. There is a different kind of optimal convergence order \(O(1/t)\) which was obtained for radially symmetric solutions or for fast diffusion case (see \(43, 68\)).

This chapter is organized as the following. In Section 2, we construct an approximate solution \(\psi_n(x, t)\) to the heat equation so that it share the same 2\(n\) moments with \(\phi(x, t)\) in \((4.5)\). In this construction the generalized moment problem for given backward moments is used. The decay order of \(\|x^k(\psi_n(x, t) - \phi(x, t))\|_r\) as \(t \to \infty\) is also derived. In Section 3, we show that this decay rate is transferred to the Burgers equation after the Cole-Hopf transformation and complete the proof of Theorem 8. In Section 4.4, we briefly discuss the relation between the asymptotic convergence order and the control of moments at \(t = \infty\) for general nonlinear problems. Finally, in Section 5 we provide several numerical examples to demonstrate the convergence orders obtained in Section 3 and the role of the backward moments.

### 4.2 Long-time Asymptotics for the Heat Equation

Consider the heat equation with a bounded and integrable initial data:

\[
\begin{align*}
vt &= \mu v_{xx} \quad \text{in } \mathbb{R} \times (0, \infty), \\
v(x, 0) &= v_0(x) \quad \text{in } \mathbb{R}. 
\end{align*}
\tag{4.10}
\]

One usually sets the diffusion constant \(\mu = 1\) after the time rescaling \(t \to \mu t\). However, we leave \(\mu\) in the equation to observe the dependency on the viscosity.

#### 4.2.1 Approximate Solutions to the Heat Equation

In this section, we decide the \(p_i\)'s and \(c_i\)'s in Theorem 8 using a generalized truncated moment problem developed in \(42\). A similar construction for \(\psi_n(x, t)\) is given in \(45\) for positive solutions. We briefly review this asymptotic approximation method based on a generalized moment problem.

Note that the Cole-Hopf transformation \(\Phi(x, t)\) and its spatial derivative \(\phi(x, t)\) are solutions to the heat equation \((4.10)\) and \(\psi_n(x, t)\) is constructed as an asymptotic approximation of \(\phi(x, t)\). Set the moments of the solution \(\phi(x, t)\) as

\[
\alpha_k(t) = \int x^k \phi(x, t) \, dx, \quad k \geq 0.
\]

One may easily check that the moments of a solution to the heat equation \((4.10)\) satisfy the following algebraic relations:

\[
\begin{align*}
\alpha_{2k}(t) &= \sum_{l=0}^{k} \frac{(2k)!}{(k-l)!(2l)!} t^{k-l} \alpha_{2l}(0), \\
\alpha_{2k+1}(t) &= \sum_{l=0}^{k} \frac{(2k+1)!}{(k-l)!(2l+1)!} t^{k-l} \alpha_{2l+1}(0).
\end{align*}
\]

- 30 -
These relations are valid for all \( t \in \mathbb{R} \) as long as its backward solution exists. The first two moments, \( \alpha_0 \) and \( \alpha_1 \), are constant for all \( t \in \mathbb{R} \), which are called the conservation of mass and its center. However, for \( k \geq 2 \), the moment \( \alpha_k(t) \) are not constant anymore.

It is shown that, for any given real sequence \( \alpha_k \), there exists a real sequence \( \beta_k \)'s such that the following \( 2n \) equations

\[
\sum_{i=1}^{n} \rho_i c_i^k = \alpha_k + i\beta_k, \quad 0 \leq k < 2n
\]  

have a solution set \( \rho_i, c_i \in \mathbb{C}, \, i = 1, \ldots, n \), which is unique up to reordering. If one takes \( \alpha_k(-t_0) \) in the place of \( \alpha_k \)'s for a given \( t_0 > 0 \), then

\[
\text{Re} \left( \sum_{i=1}^{n} \rho_i c_i^k \right) = \alpha_k(-t_0),
\]  

where \( \text{Re}(\cdot) \) takes the real part of a complex number. Note that \( \rho_i \)'s and \( c_i \)'s that satisfy (4.11) is not unique since they may depends on the choice of \( \beta_k \)'s. In the numerical tests in Section 4.3 we simply took \( \beta_k = 0 \) as long as (4.11) is numerically solvable.

Finally, we take the approximate solution \( \psi_n(x,t) \) as

\[
\psi_n(x,t) \equiv \text{Re} \left( \sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi t + t_0}} e^{-\frac{(x-c_i)^2}{4(t+t_0)}} \right),
\]

which is a solution to the heat equation (4.10). Then,

\[
\lim_{t \to -t_0} \int x^k \psi_n(x,t)dx = \text{Re} \left( \sum_{i=1}^{n} \rho_i \int x^k \delta(x-c_i)dx \right) = \text{Re} \left( \sum_{i=1}^{n} \rho_i c_i^k \right) = \alpha_k(-t_0).
\]

Therefore, \( \psi_n(x,t) \) and \( \phi(x,t) \) have the same moments up to order \( 2n - 1 \) at time \( t = -t_0 \) and hence at all time \( t \in \mathbb{R} \). Hence, \( \psi_n(x,t) \) in Theorem 4 satisfies for all \( t \in \mathbb{R} \) that

\[
\int x^k \left( \phi(x,t) - \psi_n(x,t) \right) dx = 0, \quad 0 \leq k < 2n - 1.
\]  

**Remark 3.** Note that \( \rho_i \)'s and \( c_i \)'s depend on the backward time \( t_0 \geq 0 \). We do not have a criterion to choose \( t_0 \) and left it as a free variable. If one may find \( t_0 \) that solves one more moment equation, i.e.,

\[
\sum_{i=1}^{n} \rho_i c_i^{2n} = \alpha_{2n}(-t_0) + i\beta_{2n},
\]

then one may obtain an extra asymptotic convergence order. Furthermore, more importantly, it will give a better initial approximation. However, its solvability seems beyond the scope of this thesis. For the simplest case, \( n = 1 \), Miller and Bernoff [54] gave such an approximation for positive solutions. Using the complex heat kernel in this paper and the generalized moment problem one may extend the result for sign-changing solutions easily.

**Remark 4.** One can easily check that

\[
\tilde{\psi}_n(x,t) \equiv \sum_{k=0}^{2n-1} \frac{(-1)^k \alpha_k(0)}{(k!)^{1/2} \sqrt{\pi t}} e^{-\frac{x^2}{4\pi t}}
\]

is a solution to the heat equation (4.10) and satisfies the relation (4.13) (see Duandikoetxea and Zuazua [28]). Yanagisawa [71] applied this kind of approximation to obtain the higher order asymptotics in the
Burgers equation. In the proof of Theorem 4.13, the choice of $\psi_n$ does not matter as far as (4.13) is satisfied. However, the constants and hence the proof of Theorem 4.13 may depend on its choice. Furthermore, even if we obtain the same convergence order as $t \to \infty$, the convergence as $n \to \infty$ may show different behavior. In fact, one may easily construct an example that $\tilde{\psi}_n(x,t)$ diverges as $n \to \infty$ (see [45]). One may improve this approach using the backward moments as we did in this chapter, i.e.,

$$\tilde{\psi}_n(x,t) \equiv \sum_{k=0}^{2n-1} \frac{(-1)^k \alpha_k}{(k!) \sqrt{4\mu \pi(t+\theta)}} \partial_x^k (e^{-\frac{x^2}{4\mu(t+\theta)}}).$$

(4.15)

In this way one may obtain some initial regularity.

4.2.2 Contraction Rates of Moments

The agreement of the moments in (4.13) does not hold after the inverse Cole-Hopf transformation. However, the key observation is that the contraction order (4.17) in $L^r$-norm is preserved after the inverse transformation. Since $\phi(x,t)$ and $\psi_n(x,t)$ satisfies equation (4.13), they contract to each other having order $O(t^{\frac{4}{7} - \frac{2n-1}{r+1}})$ in $L^r$-norm as $t \to \infty$ (see [28, 32, 45]). This contraction property is extended to a contraction of moments in this section.

Lemma 8. Let $g \in L^1(\mathbb{R})$ satisfy $\int g(x) \, dx = 1$ and $g_r(x) := e^{-x^2} g(x/e)$. Suppose that $\|hf\|_p < \infty$ with $1 \leq p < \infty$ and $\|h(\cdot)(f(\cdot) - f(\cdot - \eta))\|_p \to 0$ as $\eta \to 0$. Then,

$$\|hf * g_r - hf\|_p \to 0 \quad \text{as} \quad \epsilon \to 0.$$ 

That is, $\|hf * g_r\|_p \to \|hf\|_p$ as $\epsilon \to 0$.

Proof. The definition of the convolution and the Minkowski’s inequality in an integral form give

$$\|hf * g_r - hf\|_p = \left( \int \left| \int h(x)f(x-y)g_r(y) \, dy - h(x)f(x) \right|^p \, dx \right)^{1/p}$$

$$= \left( \int \left| \int h(x)(f(x-y) - f(x))g_r(y) \, dy \right|^p \, dx \right)^{1/p}$$

$$\leq \left( \int \left| \int h(x)(f(x-y) - f(x))g_r(y) \right|^p \, dx \right)^{1/p} \, dy$$

$$= \int \|h(\cdot)(f(\cdot - y) - f(\cdot))\|_p |g_r(y)| \, dy$$

$$= \int \|h(\cdot)(f(\cdot - y) - f(\cdot))\|_p |g(y)| \, dy.$$ 

The lemma follows from the dominated convergence theorem.

Theorem 9. Let $\phi(x,t)$ and $\psi(x,t)$ be solutions to the heat equation (4.10). Suppose that $\phi(x,0)$ is bounded, $x^q \phi(x,0) \in L^1(\mathbb{R})$ and

$$\int x^k (\phi(x,0) - \psi(x,0)) \, dx = 0 \quad \text{for} \quad k = 0, \cdots, q - 1.$$ 

Then, there exists $c_q \in W^{q,1}(\mathbb{R})$ that satisfies $\partial_t^2 c_q(x) = \phi(x,0) - \psi(x,0)$. Furthermore, for $1 \leq r \leq \infty$ and $k \geq 0$,

$$\lim_{t \to \infty} t^{\frac{q+1-k}{r}} \left\| x^k (\phi(x,t) - \psi(x,t)) \right\|_r = \frac{\|x^k \partial_t^2 \left( e^{-\frac{x^2}{4\mu}} \right) \|_r}{\sqrt{4\mu \pi}} \left| \int c_q(x) \, dx \right|.$$ 

(4.16)

In other words,

$$\left\| x^k (\phi(x,t) - \psi(x,t)) \right\|_r = O((\sqrt{t})^{1/r - q - 1 + k}) \quad \text{as} \quad t \to \infty.$$ 

(4.17)
Proof. The existence of such \(e_q \in W^{q,1}(\mathbb{R})\) is given in [28] [45] and it depends on the relation (4.13). Let \(e_q(x,t)\) be the solution to the heat equation with this initial value \(e_q(x)\). Then, \(\partial^q_t e_q(x,t)\) is a solution to the heat equation with initial value \(\phi(x,0) - \psi(x,0)\) and hence \(\partial^q_t e_q(x,t) = \phi(x,t) - \psi(x,t)\). The solution \(e_q(x,t)\) can be explicitly written as

\[
e_q(x,t) = \frac{1}{\sqrt{4\pi \mu t}} \int e^{-\frac{(x-y)^2}{4t}} e_q(y)dy.
\]

An integrable solution to the heat equation has the similarity scale \(\sqrt{t}\), and \(\sqrt{t} u(\sqrt{t}x, t)\) converges to a nontrivial bounded function as \(t \to \infty\). Using the self-similar variables

\[
\xi = x/\sqrt{t}, \quad \zeta = y/\sqrt{t},
\]

the solution in self-similar variables \(\tilde{e}_q(\xi, t) = \sqrt{t} e_q(\sqrt{t} \xi, t)\) can be written as

\[
\tilde{e}_q(\xi, t) = \frac{1}{\sqrt{4\mu t}} \int e^{-\frac{(\xi-x)^2}{\mu t}} e_q(\sqrt{t} \xi) d\zeta
\]

and its \(q\)-th order derivative is given by

\[
\partial^q_t \tilde{e}_q(\xi, t) = \partial^q_x e_q(x, t) (\partial_x x)^q = \partial^q_x e_q(x, t) (\sqrt{t})^q.
\]

Let \(A_q := |f e_q(z) dz|\) and suppose \(A_q \neq 0\). Then

\[
(\sqrt{t})^{q-k+1} |x^k \partial^q_x e_q(x,t)| = (\sqrt{t})^{k} |\xi^k \partial^q_x \tilde{e}_q(\xi, t)| = \frac{A_q}{\sqrt{4\mu t}} |\xi^k \int f(\zeta) g_t(\xi - \zeta) d\zeta|,
\]

where \(f(\xi) := \partial^q_x (e^{-\xi^2/4\mu})\) is smooth and \(g_t(\xi) := \sqrt{t} e_q(\sqrt{t} \xi)/A_q\) is a delta-sequence as \(t \to \infty\). Since \(f(\xi)\) decays exponentially as \(|\xi| \to \infty\), the assumptions in Lemma 5 are satisfied with \(h(\xi) = \xi^k\) for any \(k > 0\). Taking limit as \(t \to \infty\) to (4.18) gives

\[
\lim_{t \to \infty} (\sqrt{t})^{q-k+1} |x^k \partial^q_x e_q(x,t)| = \frac{A_q}{\sqrt{4\mu t}} |\xi^k f(\xi)|.
\]

For \(r = \infty\), it holds that

\[
\lim_{t \to \infty} (\sqrt{t})^{q-k+1} \|x^k \partial^q_x e_q(x,t)\|_{\infty} = \frac{A_q}{\sqrt{4\mu t}} \|\xi^k f(\xi)\|_{\infty}.
\]

For \(1 \leq r < \infty\),

\[
(\sqrt{t})^{q-k+1-1/r} \|x^k \partial^q_x e_q(x,t)\|_{r} = \left( \int |(\sqrt{t})^{q-k+1} x^k \partial^q_x e_q(x,t)|^r d\left( \frac{x}{\sqrt{t}} \right) \right)^{1/r} = \left( \int \xi^k |\partial^q_x e_q(\xi, t)|^r d\zeta \right)^{1/r} = \frac{A_q}{\sqrt{4\mu t}} \left( \int \xi^k \int f(\zeta) g_t(\xi - \zeta) d\zeta \right)^{1/r} = \frac{A_q}{\sqrt{4\mu t}} \|\xi^k (f * g_t)(\xi)\|_{r}.
\]

Hence Lemma 5 gives

\[
\lim_{t \to \infty} (\sqrt{t})^{q-k+1-1/r} \|x^k \partial^q_x e_q(t)\|_{r} = \frac{A_q}{\sqrt{4\mu t}} \|\xi^k \partial^q_x (e^{-\frac{\xi^2}{4\mu}})\|_{r}.
\]
Now suppose $A_q = 0$. If $e_0$ is nontrivial, then there exists $l > q$ such that $\int_{-\infty}^{\infty} e_l(x) = \lim_{x \to -\infty} e_{l+1}(x) \neq 0$ for some $l > q$ (see [45]). Let $e_l(x, t)$ be the solution with initial value $e_l(x)$. Then, since $\partial_t^l e_l(x) = e_0(x)$, we obtain for $1 \leq r \leq \infty$

$$
\lim_{t \to \infty} t^{\frac{l-1}{2}} \| x^k \partial_x^l e_l(t) \|_r = \frac{A_l}{4 \pi^{\frac{d}{2}}} \| \xi^k \partial_x^l (e^{-\xi^2}) \|_r < \infty.
$$

Therefore, the convergence order in (4.16) still holds. In fact the convergence order is actually higher in this case. ∎

### 4.3 Long-time Asymptotics for the Burgers Equation

Let $u(x, t)$ be the solution to the Burgers equation, i.e.,

\[
\begin{align*}
    u_t + uu_x &= \mu u_{xx} \quad \text{in } \mathbb{R} \times (0, \infty), \\
    u(x, 0) &= u_0(x) \quad \text{in } \mathbb{R}.
\end{align*}
\]

Then, the Cole-Hopf transformation and its partial derivative,

\[
\Phi(x, t) = e^{-\frac{1}{2\mu} \int_{-\infty}^{x} u(y, t) dy - 1} \quad \text{and} \quad \phi(x, t) = \Phi_x(x, t),
\]

are solutions to the heat equation

\[
\begin{align*}
    v_t &= \mu v_{xx} \quad \text{in } \mathbb{R} \times (0, \infty), \\
    v(x, 0) &= v_0(x) \quad \text{in } \mathbb{R}.
\end{align*}
\]

The approximation $\psi_n(x, t)$ in Theorem 8 has been given in Section 4.2.1. Now consider an asymptotic approximate solution to the Burgers equation given by

\[
w_n(x, t) := -2\mu \frac{\psi_n(x, t)}{1 + \Psi_n(x, t)} = H^{-1}(\Psi_n),
\]

where

\[
\Psi_n(x, t) := \int_{-\infty}^{x} \psi_n(y, t) dy.
\]

We need to show that $w_n(x, t)$ is well-defined since the denominator $1 + \Psi_n(x, t)$ can be zero. In the following lemma we will show that there is a time $T \geq 0$ such that this approximate solution $w_n(x, t)$ is well-defined for $t \geq T$.

**Lemma 9.** Let $M := \int u_0(x) dx$ and $a := \min\{1, e^{-\frac{M}{2\mu}}\} > 0$. Then for any $\epsilon > 0$, there exists $T > 0$ such that

$$1 + \Psi_n(x, t) \geq a - \epsilon \quad \text{for all } x \in \mathbb{R}, \ t \geq T.$$

**Proof.** One can easily compare the boundary values at $x = \pm \infty$. First,

\[
\lim_{x \to -\infty} (1 + \Psi_n(x, t)) = \lim_{x \to -\infty} (1 + \Phi_n(x, t)) = 1.
\]

From the definition of the Cole-Hopf transformation (1.43) and the agreement of zeroth moments between $\phi$ and $\psi_n$, we have

\[
e^{-\frac{M}{2\mu}} = \lim_{x \to -\infty} (1 + \Phi(x, t)) = 1 + \int_{-\infty}^{\infty} \phi(x, t) dx
\]

\[
= 1 + \int_{-\infty}^{\infty} \psi_n(x, t) dx = \lim_{x \to -\infty} (1 + \Psi_n(x, t)).
\]

- 34 -
Therefore, for any fixed time $T_0 > 0$, there exists $L > 0$ such that
\[(1 + \Psi_n)(x, T_0) \geq a - \epsilon/2 \quad \text{for } |x| > L\]
and 
\[\frac{1}{\sqrt{4\pi\mu T_0}} \int_{-2L}^{2L} e^{-\frac{x^2}{4\mu T_0}} \, dy \geq 1/4.\]
Let $m := \min_{|x| \leq L}(1 + \Psi_n)(x, T_0)$ and define
\[G(x, t) := \frac{4(|m| + a)}{\sqrt{4\pi \mu T_0}} \int_{-2L}^{2L} e^{-\frac{y^2}{4\mu T_0}} \, dy.\]
Then, $G(x, t)$ satisfies the heat equation and it holds that
\[G(-L, T_0) = G(L, T_0) = \frac{4(|m| + a)}{\sqrt{4\pi \mu T_0}} \int_{-2L}^{2L} e^{-\frac{y^2}{4\mu T_0}} \, dy \geq |m| + a,
\]
hence,
\[(1 + \Psi_n + G)(x, T_0) \geq m + |m| + a \geq a \quad \text{for } |x| \leq L.
\]
Therefore, the maximum principle (see [60]) gives
\[(1 + \Psi_n + G)(x, t) \geq a - \epsilon/2 \quad \text{for } x \in \mathbb{R}, \ t \geq T_0.
\]
On the other hand, since $G(x, t)$ is a bounded $L^1$ solution to the heat equation, there exists a large time $T \geq T_0$ such that
\[G(x, t) \leq \epsilon/2 \quad \text{for } x \in \mathbb{R}, \ t \geq T.
\]
Finally we obtain the conclusion
\[(1 + \Psi_n)(x, t) \geq a - \epsilon/2 - G(x, t) \geq a - \epsilon \quad \text{for } x \in \mathbb{R}, \ t \geq T.
\]
\[\Box
\]
\textbf{Remark 5.} Suppose that the initial value $u_0$ is negative, $u_0(x) \leq 0$. Then, $\phi(x, 0)$, which is given by (4.5), is positive. The truncated moment problem for a positive measure, without using backward moments ($t_0 = 0$), implies that $\rho_i > 0$ for all $i$. Therefore, the denominator
\[1 + \Psi_n(x, 0) \equiv 1 + \int_{-\infty}^{x} \sum_{i=1}^{n} \rho_i \delta(x - c_i) \, dx\]
is monotone and hence $1 + \Psi(x, 0) > 0$ for all $x \in \mathbb{R}$. Therefore, we may take $T = 0$ in Lemma 3. Same conclusion holds if $u_0$ is positive. If the backward time is positive or if the initial value is not signed, then $1 + \Psi_n(x, 0)$ is not monotone in general. However, our numerical examples always give $1 + \Psi_n(x, 0) > 0$, i.e., $T = 0$.

Now we are ready to prove Theorem 3.

\textbf{Proof of Theorem 3.} The moment differences between $u(x, t)$ and $w_n(x, t)$ are estimated using (4.16) and (4.19):
\[|x^k(u(x, t) - w_n(x, t))| = 2\mu \left| \frac{x^k \phi(x, t)}{1 + \Phi(x, t)} - \frac{x^k \psi_n(x, t)}{1 + \Psi_n(x, t)} \right|\]
\[= 2\mu \left| \frac{x^k \phi(x, t) + x^k \phi(x, t) \Psi_n(x, t) - x^k \psi_n(x, t)(1 + \Phi(x, t))}{(1 + \Phi(x, t))(1 + \Psi_n(x, t))} \right|\]
\[\leq 2\mu \left| \phi(x, t) \right| \frac{|x^k(\Phi(x, t) - \Psi_n(x, t))|}{|1 + \Phi(x, t)| |1 + \Psi_n(x, t)|} \]
\[+ 2\mu \left| \phi(x, t) \right| \frac{|x^k(\phi(x, t) - \psi_n(x, t))|}{|1 + \Phi(x, t)| |1 + \Psi_n(x, t)|}.
\]
Let \( U_0(x) = \int_{-\infty}^{x} u_0(y)dy \). Then, \( A = -\inf_x U_0(x) \) and \( B = \sup_x U_0(x) \) are non-negative. Since 

\[
0 < e^{\frac{-t}{C_1}} \leq 1 + \Phi(x,t) \leq e^{\frac{t}{C_2}} < \infty, \quad t \geq 0.
\]  

(4.21)

Let \( T > 0 \) be the one in Lemma 4.2 and take \( \epsilon = a/2 \), then we obtain a uniform lower bound 

\[
\frac{1}{2} \min\{1, e^{-M/2}\} \leq 1 + \Phi_n(x,t), \quad t \geq T.
\]  

(4.22)

Therefore, the denominators are uniformly bounded below away from zero. Now we show the convergence order of nominators to obtain (4.7). First, since \( \phi \) is an \( L^1 \)-solution to the heat equation, we have 

\[
\|\phi(\cdot,t)\|_\infty = O(t^{-\frac{1}{2}}) \quad \text{as} \quad t \to \infty.
\]  

(4.23)

The \( L' \)-norm estimates of \( x^k(\psi_n(x,t) - \phi(x,t)) \) and \( x^k(\Psi_n(x,t) - \Phi(x,t)) \) are obtained similarly using Theorem 4.9. Recall that 

\[
\Psi_n(x,t) = \int_{-\infty}^{x} \psi_n(y,t)dy \quad \text{and} \quad \Phi(x,t) = \int_{-\infty}^{x} \phi(y,t)dy.
\]

The approximation \( \psi_n \) was constructed to satisfy 

\[
\int_{-\infty}^{x} x^k(\phi(x,0) - \psi_n(x,0)) dx = 0, \quad \text{for} \quad 0 \leq k \leq 2n - 1.
\]

Then, for \( 0 \leq k \leq 2n - 2 \), 

\[
\int_{-\infty}^{x} x^k(\Phi(x,0) - \Psi_n(x,0)) dx \\
= \int_{-\infty}^{x} \int_{-\infty}^{x} x^k(\phi(y,0) - \psi_n(y,0)) dy dx \\
= \int_{-\infty}^{x} \int_{y}^{x} x^k(\phi(y,0) - \psi_n(y,0)) dx dy \\
= -\frac{1}{k+1} \int_{-\infty}^{x} y^{k+1}(\phi(y,0) - \psi_n(y,0)) dy = 0.
\]

Therefore, by Theorem 4.9 we obtain 

\[
\|x^k(\phi(x,t) - \psi_n(x,t))\|_r = O(t^{\frac{3}{2n+1-k}}) \quad \text{as} \quad t \to \infty, \quad \text{(4.24)}
\]

\[
\|x^k(\Phi(x,t) - \Psi_n(x,t))\|_r = O(t^{\frac{3}{2n-1-k}}) \quad \text{as} \quad t \to \infty. \quad \text{(4.25)}
\]

Then, for \( 1 \leq r \leq \infty \) and \( t \geq T \), taking the \( L' \)-norm on (4.20) gives 

\[
\|x^k(u(x,t) - w_n(x,t))\|_r \leq C_1 \|\phi(x,t)\|_\infty \|x^k(\Psi_n(x,t) - \Phi(x,t))\|_r \\
+ C_2 \|x^k(\phi(x,t) - \psi_n(x,t))\|_r,
\]

where constants \( C_1, C_2 > 0 \) are from the uniform estimates (4.21) and (4.22). Combining the asymptotic convergence orders in (4.24), (4.24) and (4.25) gives 

\[
\|x^k(u(t) - w_n(t))\|_r = O(t^{\frac{1}{2n+1-k}}) \quad \text{as} \quad t \to \infty,
\]

which completes the proof of Theorem 4.8.
4.4 Fine Asymptotics and the Similarity Scale

There are many studies on the asymptotic analysis for various problems. The porous medium equation is one of the examples that such a study has been done intensively. We start our discussion with a brief review on it. Let \( m > 0 \) and \( u \) be a \( L^1 \)-solution in one spatial dimension

\[
   u_t = (u^m)_{xx}, \quad u(x, 0) = u_0(x),
\]

where the initial data \( u_0 \) is integrable. Let \( u(x, t) = av(ax, a^{m+1}t) \). Then \( v \) satisfies the equation and preserves the \( L^1 \)-norm of \( u \). The invariance property under this specific dilation is called the \( L^1 \)-similarity structure of the problem. Variables and solutions that are also invariant under the dilation is called self-similar variables and solutions, respectively. The Barenblatt solution, say \( \rho(x, t) \), and variable \( \xi = xt^{-\alpha} \) are the ones, where \( \alpha = 1/(m+1) \),

\[
   \rho(x, t) = A - \frac{m-1}{2m(m+1)}(xt^{-\alpha})^{\frac{1}{m}} \quad \text{and} \quad \xi = xt^{-\alpha}.
\]

Note that the constant \( A \) is a free parameter that decides the total mass and that the self-similar variable \( \xi = xt^{-\alpha} \) show how the support of the solution expands asymptotically. We say that \( t^{\alpha} \) is the asymptotic scale for spatial distribution of the solution.

If an equation contains more terms, say

\[
   u_t + (u^q)_x = (u^m)_{xx} + (|u_x|^{p-2}u_x)_x,
\]

then the problems has no similarity structure anymore. However, in general, there may exist an asymptotic scale \( t^{\alpha} \) that gives the propagating speed of the solution distribution. In the last case the asymptotic scale is given by \( \alpha := \max\{1/q, 1/p, 1/(m+1)\} \). Then, \( t^{\alpha}u(t^\alpha x, t) \) converges to a \( L^1 \)-function as \( t \to \infty \).

It seems that the asymptotic scale exists for more general kind of problems.

In the literature, two kinds of optimal asymptotic convergence rates appear. They actually correspond to the temporal and spatial shifts. In the \( L^1 \)-norm they can be written as, for \( t > 0 \) large,

\[
   \|\rho(x, t) - \rho(x - c, t)\|_1 = O(t^{-\alpha}), \quad \|\rho(x, t) - \rho(x, t + T)\|_1 = O(t^{-1}).
\]

Mechanism of these two asymptotics are different. The first one in (4.26) is actually related to the convergence order in Theorem 8. This convergence order corresponds to the one with zeroth moment agreement. The other one (4.27) is not actually related. In the following we formally investigate the relation between the asymptotic convergence orders and the control of moments at \( t = \infty \). Even though we do not have a rigorous proof, it seems reasonable to put this formal arguments in this chapter since the convergence order of moments in Theorem 8 motivates them.

Let \( v \) be another solution with initial data \( v_0 \). Set

\[
   e(x, t) := u(x, t) - v(x, t).
\]

Suppose that

\[
   \||x|^N e(x, t)\|_1 = \int |x|^N |e(x, t)| \, dx = O(1) \quad \text{as} \quad t \to \infty.
\]

We want to derive the decay rate of \( ||e(x, t)||_1 \) as \( t \to \infty \). Change the space variable using

\[
   x = t^\beta y, \quad dx = t^\beta dy.
\]
Then,
\[ \int |x|^N |e(x,t)| \, dx = t^{N\alpha} \int |y|^N |e(t^\beta y,t)| t^\beta \, dy. \]
If one obtains two positive constants \( c \) and \( C \) that are independent of the time \( t > 0 \) and satisfy
\[ c \int |e(t^\beta y,t)| t^\beta \, dy \leq \int |y|^N |e(t^\beta y,t)| t^\beta \, dy \leq C \int |e(t^\beta y,t)| t^\beta \, dy, \]
then the correct convergence order for \( \|e(x,t)\|_1 \) is obtained. Suppose that \( t^{\alpha} \) is the similarity scale or the asymptotic scale that gives the propagating speed of the support of the solution. Then, if \( \beta > \alpha \), then \( t^\beta |e(t^\beta y,t)| \) behaves like a delta sequence and hence lower bound in \( \ref{4.20} \) should fail. Similarly, if \( \beta < \alpha \), then the support of \( t^\beta |e(t^\beta y,t)| \) expands as \( t \to \infty \) and hence the upper bound of \( \ref{4.20} \) is not expected. Hence, \( \beta = \alpha \) is the only case that one may obtain the correct convergence order for \( \|e(x,t)\|_1 \).

Then, there exists \( c^* = c^*(t) > 0 \) such that \( c \leq c^* \leq C \) and
\[ \int |e(t^\beta y,t)| t^\beta \, dy = c^* \int |y|^N |e(t^\beta y,t)| t^\beta \, dy. \]
Using these relations, one obtains
\[ t^{N\alpha} \|e(x,t)\|_1 = t^{N\alpha} \int |e(t^\alpha y,t)| t^\alpha \, dy = c^* t^{N\alpha} \int |y|^N |e(t^\alpha y,t)| t^\alpha \, dy = O(\|x|^N e(x,t)\|_1) = O(1) \]
as \( t \to \infty \). Hence, the decay of the \( N \)-th order moment at \( t = \infty \) in \( \ref{4.28} \) gives
\[ \|e(x,t)\|_1 = \int |e(x,t)| \, dx = O(t^{-N\alpha}) \quad \text{as} \quad t \to \infty. \]

In summary, if one may show \( \ref{4.20} \), then the following claim is obtained.

**Fine asymptotics and the similarity scale:** Let \( u(x,t) \) and \( v(x,t) \) be integrable solutions to a nonlinear problem given in \( \ref{4.7} \) in one spatial dimension. Suppose that \( c := u - v \) satisfies \( \ref{4.28} \) and \( t^\alpha, \alpha > 0, \) is the asymptotic scale of the problem. Then, for \( 0 \leq k \leq N, \)
\[ \| |x|^k e(x,t)\|_v = O(t^{(k-N-1+1/r)\alpha}) \quad \text{as} \quad t \to \infty. \]
This convergence order is the one corresponding to \( \ref{4.7} \). One may say that the convergence order \( \ref{4.21} \) is not of this kind. However, the one in \( \ref{4.20} \) is this kind with \( N = 1. \)

**Remark 6.** For the fast diffusion case, \( 0 < m < 1, \) the similarity scale \( t^\alpha \) is with \( \alpha = 1/(m+1) > 1/2. \) Hence, if \( \| |x|^2 (\rho(x,t) - \rho(x,t+T))\|_1 = O(1) \) as \( t \to \infty, \) then above discussion implies that \( \|\rho(t) - \rho(t+T)\|_1 = O(t^{-2\alpha}) \). However, the order \( \ref{4.21} \) is an optimal one and hence one should expect that \( \| |x|^2 (\rho(x,t) - \rho(x,t+T))\|_1 \to \infty \) as \( t \to \infty. \) In fact, this is true and one may check that using the explicit formula of the Barenblatt solution.

### 4.5 Numerical Examples

In this section, we demonstrate the asymptotic convergence orders obtained in Theorem 8 numerically. The effect of backward moments is also tested. Note that the convergence order in \( \ref{4.7} \) is for \( t > 0 \) large and hence we should wait certain amount of time to observe such a convergence order. However,
Figure 4.1: The initial data $u_0$ (solid line) and its approximation (dashed line) are figured. The three figures in the first row show the convergence as $n$ increases. The backward time is fixed with $t_0 = 0.3$. The three figures in the second row show the role of the backward time $t_0$. One may see that a better backward time gives better results for a given $n$. In the example with the given initial value and $n = 8$, the backward time $t_0 = 1.1$ seems a limit. After this limit of backward time the error of the approximation increases suddenly (see Table 4.1).

at that stage, the error $\|u(t) - w_n(t)\|_r$ can be very small. Hence it is important to compute the exact solution,

$$u(x, t) = H^{-1}\left(\int_{-\infty}^{x} \phi(y, z) e^{-z^2/4t} dz \right) dy,$$

with an error smaller than this asymptotic approximation error. However, it is unrealistic to do the required integrations with such small tolerance. Hence one should test the convergence order with a case that an explicit solution exists. One easy way to do that is to set $\phi(x, t)$ first (not $u(x, t)$). Let

$$\phi(x, t) := \frac{5}{\sqrt{4\pi \mu(t + 2)}} e^{-\frac{(x+1)^2}{4\mu(t+2)}} + \frac{20}{\sqrt{4\pi \mu(t + 1)}} e^{-\frac{(x+0.5)^2}{4\mu(t+1)}}$$

$$- \frac{16}{\sqrt{4\pi \mu(t + 0.5)}} e^{-\frac{(x+0.5)^2}{4\mu(t+0.5)}} - \frac{9}{\sqrt{4\pi \mu(t + 2)}} e^{-\frac{(x-1)^2}{4\mu(t+2)}},$$

and $u(x, t)$ be the inverse Cole-Hopf transformation of

$$\Phi(x, t) = \int_{-\infty}^{x} \phi(y, t) dy.$$

Remember that from the definition of the Cole-Hopf transformation,

$$\int_{-\infty}^{x} \phi(y, 0) dy = \Phi(x, 0) = e^{-\frac{x^2}{4\mu}} \int_{-\infty}^{x} u_0(y) dy - 1 > -1.$$  (4.31)

Hence one should choose $\phi(x, 0)$ that satisfies (4.31) for all $x \in \mathbb{R}$. The one given in (4.30) satisfies it.

The numerical test in this section has two purposes. The first one is to observe approximation properties of the method suggested in this chapter. In Figures 4.1 and 4.2 we have compared the
approximations to the exact one varying the backward time and the number of heat kernels. The graph of the initial data $u(x,0)$, which is the inverse Cole-Hopf transformation of $\Phi(x,0)$, is given in Figure 4.1 in solid lines. For a given backward time $t_0 \geq 0$, the approximate solution $\psi_n(x,t)$ to the heat equation is given by (4.9). The approximation $w_n(x,t)$ in Theorem 5 is the inverse Cole-Hopf transformation of $\Psi_n(x,t) := \int_{-\infty}^x \psi_n(y,t) dy$. The initial approximation $w_n(x,0)$ are given in Figure 4.1 in dashed lines. The three figures in the first row show the convergence as $n$ increases with a fixed backward time $t_0 = 0$.3. If the backward moments are not used, i.e., $t_0 = 0$, then the approximation is just a collection of delta distributions. Hence, initial convergence as $n \to \infty$ is not expected without using backward moments.

The three figures in the second row of Figure 4.1 show the role of the backward time $t_0$. One may see that a better backward time gives better results for a fixed $n$. In this example, the backward time $t_0 = 1.1$ seems the best. One should not be misled by assuming that the approximation converges as $t_0 \to \infty$. In fact, the approximation error increases suddenly for $t_0 > 1.1$. This behavior is related to the initial data $u(x,0)$ given by (4.30) and the number of heat kernels $n = 8$. To verify this property, an error comparison is given in Table 4.1 for $n = 2, 4, 8$ and 16 as increasing $t_0$. One may observe that the backward time improves the approximation only up to certain limit and, after that, the performance becomes poor suddenly. For a bigger $n$, the best backward time becomes smaller. This property seems related to the age of the initial heat distribution $\phi(x,0)$ in (4.30). In this example, the age is $t = 0.5$, and the best backward time $t_0$ seems to approach to this age as $n \to \infty$. However, we only have numerical evidence.

In Figure 4.2, the solution to the Burgers equation at time $t = 1$ is given in solid lines. Approximate solutions are given in dashed lines. The three figures in the first row are without using backward moments, i.e., $t_0 = 0$. The others are with $t_0 = 0.3$. In both cases one may observe convergence as $n$ increases. One may also see that the effects of the backward time $t_0 > 0$ becomes smaller as $t$ increases (compare
Tables 4.2 and 4.3).

The second purpose of this section is to demonstrate the asymptotic convergence order in (4.7). In the followings we only test the zeroth moment in the uniform norm, i.e., the $L^\infty$-contraction order between $u$ and $w_n$. The convergence rate $\gamma_n(t)$ is computed by the formula

$$\gamma_n(t) := \ln \left( \frac{\| u(x,t) - w_n(x,t) \|_\infty}{\| u(x,t/2) - w_n(x,t/2) \|_\infty} \right) / \ln \left( \frac{1}{2} \right).$$  (4.32)

In Table 4.2 the error and the convergence rates are given in the uniform norm. The approximate solutions $w_n$ are constructed for $n = 2, 4, 8$ and zero backward time $t_0 = 0$. One may roughly observe that the convergence order increases to $n + 0.5$ which is given by Theorem 8. In Table 4.3 the same comparisons are given for the approximate solutions using backward moment $t_0 = 0.3$. For a small time $t > 0$, the result is considerably better if a backward time is used. However, as $t \to \infty$, both of them become similar. The asymptotic convergence order in (4.7) is for $t > 0$ large. In conclusion, the approximate solution $w_n$ constructed in this chapter well behaves for a small time, too. This is partly due to the use of backward time. The approach using the derivatives of heat kernels as in (4.14) can be also improved by considering backward time as in (4.15).

4.6 Open Problems

1. Prove that $T = 0$ is always possible in Theorem 8

2. Find an algorithm to choose efficient backward time $t_0$; see Remark 3
Table 4.2: Approximation error without backward moments, i.e., $t_0 = 0$. The numerical order $\gamma_n(t)$ computed by (4.32) converges to the theoretical one as $t \to \infty$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$e_2(t)$</th>
<th>$\gamma_2(t)$</th>
<th>$e_4(t)$</th>
<th>$\gamma_4(t)$</th>
<th>$e_8(t)$</th>
<th>$\gamma_8(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.125</td>
<td>1.11e-00</td>
<td>7.51e-01</td>
<td></td>
<td></td>
<td>3.89e-01</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>7.23e-01</td>
<td>0.62</td>
<td>4.47e-01</td>
<td>0.75</td>
<td>2.00e-01</td>
<td>0.96</td>
</tr>
<tr>
<td>0.5</td>
<td>4.48e-01</td>
<td>0.69</td>
<td>2.36e-01</td>
<td>0.92</td>
<td>8.95e-02</td>
<td>1.16</td>
</tr>
<tr>
<td>1</td>
<td>2.53e-01</td>
<td>0.82</td>
<td>1.05e-01</td>
<td>1.17</td>
<td>2.19e-02</td>
<td>2.03</td>
</tr>
<tr>
<td>2</td>
<td>1.23e-01</td>
<td>1.04</td>
<td>3.14e-02</td>
<td>1.74</td>
<td>2.12e-03</td>
<td>3.37</td>
</tr>
<tr>
<td>4</td>
<td>4.77e-02</td>
<td>1.37</td>
<td>5.71e-03</td>
<td>2.46</td>
<td>7.36e-05</td>
<td>4.85</td>
</tr>
<tr>
<td>8</td>
<td>1.41e-02</td>
<td>1.76</td>
<td>6.35e-04</td>
<td>3.17</td>
<td>1.04e-06</td>
<td>6.15</td>
</tr>
<tr>
<td>16</td>
<td>3.33e-03</td>
<td>2.08</td>
<td>4.78e-05</td>
<td>3.73</td>
<td>7.97e-09</td>
<td>7.03</td>
</tr>
<tr>
<td>32</td>
<td>6.81e-04</td>
<td>2.29</td>
<td>3.03e-06</td>
<td>3.98</td>
<td>4.07e-11</td>
<td>7.61</td>
</tr>
<tr>
<td>64</td>
<td>1.31e-04</td>
<td>2.38</td>
<td>1.72e-07</td>
<td>4.14</td>
<td>1.66e-13</td>
<td>7.94</td>
</tr>
<tr>
<td>128</td>
<td>2.46e-05</td>
<td>2.41</td>
<td>9.34e-09</td>
<td>4.20</td>
<td>6.18e-16</td>
<td>8.07</td>
</tr>
</tbody>
</table>

Table 4.3: Approximation error and contraction order with backward time $t_0 = 0.3$. The error in this case is smaller than the case with $t_0 = 0$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$e_2(t)$</th>
<th>$\gamma_2(t)$</th>
<th>$e_4(t)$</th>
<th>$\gamma_4(t)$</th>
<th>$e_8(t)$</th>
<th>$\gamma_8(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.125</td>
<td>5.22e-01</td>
<td></td>
<td>2.88e-01</td>
<td></td>
<td>1.02e-01</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>4.31e-01</td>
<td>0.28</td>
<td>2.15e-01</td>
<td>0.42</td>
<td>6.94e-02</td>
<td>0.56</td>
</tr>
<tr>
<td>0.5</td>
<td>3.19e-01</td>
<td>0.43</td>
<td>1.38e-01</td>
<td>0.64</td>
<td>3.26e-02</td>
<td>1.09</td>
</tr>
<tr>
<td>1</td>
<td>2.03e-01</td>
<td>0.65</td>
<td>6.69e-02</td>
<td>1.04</td>
<td>8.16e-03</td>
<td>2.00</td>
</tr>
<tr>
<td>2</td>
<td>1.06e-01</td>
<td>0.94</td>
<td>2.09e-02</td>
<td>1.68</td>
<td>8.42e-04</td>
<td>3.28</td>
</tr>
<tr>
<td>4</td>
<td>4.24e-02</td>
<td>1.32</td>
<td>4.00e-03</td>
<td>2.39</td>
<td>3.19e-05</td>
<td>4.72</td>
</tr>
<tr>
<td>8</td>
<td>1.25e-02</td>
<td>1.76</td>
<td>4.66e-04</td>
<td>3.10</td>
<td>4.95e-07</td>
<td>6.01</td>
</tr>
<tr>
<td>16</td>
<td>3.07e-03</td>
<td>2.03</td>
<td>3.58e-05</td>
<td>3.70</td>
<td>4.01e-09</td>
<td>6.95</td>
</tr>
<tr>
<td>32</td>
<td>6.39e-04</td>
<td>2.26</td>
<td>2.30e-06</td>
<td>3.96</td>
<td>2.14e-11</td>
<td>7.55</td>
</tr>
<tr>
<td>64</td>
<td>1.24e-04</td>
<td>2.37</td>
<td>1.33e-07</td>
<td>4.11</td>
<td>8.95e-14</td>
<td>7.90</td>
</tr>
<tr>
<td>128</td>
<td>2.38e-05</td>
<td>2.38</td>
<td>7.32e-09</td>
<td>4.18</td>
<td>3.54e-16</td>
<td>7.98</td>
</tr>
</tbody>
</table>
3. Does asymptotic agreement of moments at $t = \infty$ guarantee higher contraction order of solutions?
Chapter 5. Long-time Asymptotics of the Zero Level Set for the Heat Equation

5.1 Introduction

Study of zero level set is important in both theoretical and applied aspects. In theoretical aspects, for solutions \( u \) of a one-dimensional semilinear parabolic equation, there are important facts about the zero level set \( Z(t) := \{ x \in \mathbb{R} : u(x, t) = 0 \} \). Angenent [3] proved that the set is discrete if the initial data is nontrivial. Also, Sturm (see [33]), Matano [52] and Sakaguchi [62] showed that number of zeros does not increase as time passes. Local behavior and decrease of number of zeros near a multiple zero has been studied by Angenent [3] and Chen [19]. Brunovský and Fiedler [12] showed that a solution \( u(\cdot, t) \) to one-dimensional reaction-diffusion equation has only simple zeros for \( t \) in an open dense subset of \( \mathbb{R}^+ \). Nonincreasing property of number of zeros have many applications including asymptotic stability of nonlinear parabolic equations. I refer those interested to a book by Galaktionov [33] and a survey by Galaktionov and Harwin [35]. In multi-dimensions, Chen [19] also classified local behavior of solutions to a system of second-order parabolic inequalities and obtained upper bounds for dimensions of zero level sets. However, counting number of zeros is not applicable to multi-dimensions where the zero level set consists of curves, not points.

In applied aspects, a topologically complicated curve can be a zero level set of a high-dimensional function. Hence evolution of zero level set can represent a moving interface like flame front. A numerical technique based on this idea is the level set method, which was developed by Osher and Sethian [57]. Those interested in the method are referred to a book by Osher and Fedkiw [56]. In this aspect, long-time asymptotics of zero level set shows us the ultimate shape of a moving interface.

The purpose of this chapter is to investigate long-time asymptotics of zero level set in multi-dimensions. As an initial work in this direction, the heat equation in the whole domain \( \mathbb{R}^d \), \( d \geq 1 \) will be considered:

\[
\begin{align*}
  u_t &= \Delta u & \text{in } \mathbb{R}^d \times (0, \infty), \\
  u(x, 0) &= u_0(x) \in L^1(\mathbb{R}^d).
\end{align*}
\]

(5.1)

Note that nonzero level set \( \{ x \in \mathbb{R}^d : u(x, t) = c \neq 0 \} \) becomes empty in finite time since the solution \( u \) goes to zero uniformly as time passes. So we consider zero level set only. We start with a simple example. Consider the heat kernel of the whole space \( \mathbb{R}^d \)

\[ G(x, t) := (4\pi t)^{-\frac{d}{2}} \exp \left( -\frac{|x|^2}{4t} \right) \]

(5.2)

and define

\[ u(x, t) := G(x, t) - G(x, t + T) \quad \text{for some fixed } T > 0. \]

Then the function \( u \) is a solution to the heat equation. Let \( z(t) \) be a zero of the function \( u \), i.e., \( u(z(t), t) = 0 \) for each time \( t > 0 \). Then we can show that

\[ \frac{|z(t)|}{\sqrt{t}} \to \sqrt{2d} \quad \text{as } t \to \infty. \]
This example suggests that the order of $\sqrt{t}$ describes long-time asymptotics of zeros. It is natural because $\sqrt{t}$ is a self-similar scale which represents speed of spatial propagation in diffusion process. In this chapter we will show that this is true in general. More precisely, under vanishing conditions on moments of the initial data, we will prove that the zero level set $Z(t) := \{ x \in \mathbb{R}^d : u(x, t) = 0 \}$ in a ball of radius $C \sqrt{t}$ for any $C > 0$ converges to a specific graph as $t \to \infty$ when the set is divided by $\sqrt{t}$. Solving a linear combination of the Hermite polynomials gives the graph and coefficients of the linear combination depend on moments of the initial data. Also the graphs to which the zero level set $Z(t)$ converges are classified in some cases.

**Remark 7.** Consider a parabolic equation

$$v_t = \Delta v + b(x, t) \cdot \nabla v + c(x, t)v, \quad b(x, t) = (b_i(x, t)).$$

Then using a transform given by Angenent [3],

$$w(x, t) := \exp \left\{ \frac{1}{2} \sum_{i=1}^{d} \int_{0}^{x_i} b_i(s, t) \, ds \right\} v(x, t),$$

we have a reaction-diffusion equation

$$w_t = \Delta w + w \left( \frac{1}{2} \sum_{i=1}^{d} \int_{0}^{x_i} \frac{\partial b_i}{\partial t}(s, t) \, ds + c - \frac{1}{2} (\nabla \cdot b) - \frac{1}{4} |b|^2 \right).$$

Also two functions $v$ and $w$ have exactly same zeros. Hence when the reaction coefficient of the above equation vanishes, our study on zeros of the heat equation can be applicable to more general parabolic equations without any change.

### 5.2 Hermite polynomial approximation

Our result is based on an approximation of solutions to the heat equation given by Duoandikoetxea and Zuazua [28]. As a special case of their estimates, we can yield the following result:

**Theorem 10** (Duoandikoetxea-Zuazua). Let $u$ be the solution to the heat equation (5.1) with initial data $u_0 \in L^1(\mathbb{R}^d; 1 + |x|^{k+1})$ for some nonnegative integer $k$. Then there exists a constant $C(k, d) > 0$ such that

$$\| u(\cdot, t) - \sum_{|\alpha|\leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left( \int x^\alpha u_0(x) \, dx \right) D^\alpha G(\cdot, t) \|_\infty \leq C t^{-(k+d+1)/2} \| x^{k+1} u_0(x) \|_1.$$ 

Here function $G(x, t)$ is the heat kernel (5.2) and $D^\alpha G(x, t)$ is its partial derivative with respect to multi-index $\alpha$.

The theorem states that a linear combination of partial derivatives of the heat kernel can describe long-time asymptotics of a solution to the heat equation. Partial derivatives of the heat kernel can be written as a product of the Hermite polynomials $H_n(x)$, which are given by

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$
The first few polynomials are

\begin{align*}
H_0(x) &= 1, \\
H_1(x) &= 2x, \\
H_2(x) &= 4x^2 - 2, \\
H_3(x) &= 8x^3 - 12x, \\
H_4(x) &= 16x^4 - 48x^2 + 12, \\
H_5(x) &= 32x^5 - 160x^3 + 120x.
\end{align*}

By the change of variables \( y = x/(2\sqrt{t}) \), we can yield that

\[
D^2_y G(x, t) = (4\pi t)^{-d/2} D^2_x \exp \left( -\frac{|x|^2}{4t} \right)
= (4\pi t)^{-d/2} D^2_x \exp(-|y|^2) \frac{1}{(2\sqrt{t})^d} \\
= \pi^{-d/2} (4t)^{-(|\alpha| + d)/2} (-1)^{|\alpha|} \prod_{i=1}^d \left[ H_{\alpha_i}(y_i) e^{-y_i^2} \right] \\
= \pi^{-d/2} (4t)^{-(|\alpha| + d)/2} (-1)^{|\alpha|} e^{-\frac{|x|^2}{2t}} \prod_{i=1}^d H_{\alpha_i} \left( \frac{x_i}{2\sqrt{t}} \right).
\]

Using this relation we can obtain the following estimate via the Hermite polynomials:

**Theorem 11** (Hermite polynomial approximation). Let \( u \) be the solution to the heat equation \( \Box u - \partial_t u = 0 \) with initial data \( u_0 \in L^1(\mathbb{R}^d; 1 + |x|^{k+1}) \) for some nonnegative integer \( k \). Then there exists a constant \( C(k, d) > 0 \) such that

\[
\left| u(x, t) - \pi^{-d/2} e^{-\frac{|x|^2}{2t}} \sum_{|\alpha| \leq k} \frac{\int x^\alpha u_0(x) \, dx}{\alpha!} (4t)^{-(|\alpha| + d)/2} \prod_{i=1}^d H_{\alpha_i} \left( \frac{x_i}{2\sqrt{t}} \right) \right| \\
\leq C t^{-(d+1)/2} \left\| |x|^{k+1} u_0(x) \right\|_1 \quad \text{for all } x = (x_i) \in \mathbb{R}^d.
\]  

**Remark 8.** Because the constant \( C \) depends on \( k \), the inequality (5.3) does not tell us what happens when \( k \to \infty \), i.e., when the number of terms in the sum goes to infinity. Actually Kim and Ni [23] provided a numerical example in which the Hermite polynomial approximation diverges. Also they suggested a linear combination of the heat kernels (not their derivatives) as an approximation to solutions of the heat equation and conjectured it would converge even if the number of terms in the linear combination goes to infinity. Their idea can be applied to find an approximation of solutions to the viscous Burgers equation [23].

**Remark 9.** The Hermite polynomial approximation is also can be thought as a formal eigenfunction expansion. See [70].

### 5.3 Long-time Asymptotics of the Zero Level Set

The Hermite polynomial approximation (5.3) will play a leading role in studying asymptotics of zeros. Let \( z(t) \) be zeros of a solution \( u(x, t) \) defined for each time \( t \in (T, \infty) \), \( T > 0 \), i.e., \( u(z(t), t) = 0 \). If \( u_0 \in L^1(\mathbb{R}^d; 1 + |x|) \) and the mass of initial data \( \int u_0(x) \, dx \) is nonzero, then the inequality (5.3) with \( k = 0 \) and \( x = z(t) \) implies that

\[
\left| (4\pi t)^{-d/2} e^{-\frac{|x|^2}{2t}} \int u_0(x) \, dx \right| \leq C t^{-(d+1)/2} \left\| |x| u_0(x) \right\|_1.
\]
Assume $|z(t)|/\sqrt{t}$ is bounded. Then for some constant $\tilde{C}$, it holds that
\[
\left| \int u_0(x) \, dx \right| \leq \tilde{C} t^{-1/2} \left\| |x| u_0(x) \right\|_1
\]
and $\int u_0(x) \, dx \to 0$ as $t \to \infty$. But this contradicts to the fact that the mass is nonzero. Hence $|z(t)|/\sqrt{t}$ cannot be bounded. This result is stated in the following:

**Theorem 12 (Zeros with nonzero mass).** Let $u$ be the solution to the heat equation (5.1) with initial data $u_0 \in L^1(\mathbb{R}^d; 1 + |x|)$. If the mass of initial data $\int u_0(x) \, dx$ is nonzero, then for any zeros $z(t)$ of the solution $u(\cdot, t)$, defined for $t \in (T, \infty)$, $T > 0$, it holds that
\[
\limsup_{t \to \infty} \frac{|z(t)|}{\sqrt{t}} = \infty.
\]

**Remark 10.** For one-dimensional heat equation, Mizoguchi [55] proved that any zero level set $Z(t)$ is contained in $[-Ct, Ct]$ for large $t > 0$ with some $C > 0$ if the initial data changes sign finite time. Because $u(x, t) := e G(x, t) - G(x - 2, t)$ has a unique zero $z(t) = t + 1$, Mizoguchi’s upper bound $t$ is optimal. Theorem 12 suggests a (strictly) lower bound $\sqrt{t}$. But we don’t know whether this lower bound is optimal or not.

Another easy consequence we can obtain from the Hermite polynomial approximation (5.3) is about asymptotic behavior of bounded zeros. Assume $|z(t)|/\sqrt{t}$ is bounded and $u_0 \in L^1(\mathbb{R}^d; 1 + |x|^{k+1})$. Then multiplying both sides of the inequality (5.3) with $x = z(t)$ by $e^{\|\|u_0\|\|_{L^1}}$, we obtain
\[
\left| \pi^{-d/2} \sum_{|\alpha| \leq k} \int \frac{x^\alpha u_0(x) \, dx}{\alpha!} \right| (4t)^{-|\alpha|/2} \prod_{i=1}^d H_{\alpha_i} \left( \frac{z_i(t)}{2\sqrt{t}} \right) \leq C e^{\|u_0\|_{L^1}} t^{-(k+d+1)/2} \left\| |x|^{k+1} u_0(x) \right\|_1.
\]

Because $|z(t)|/\sqrt{t}$ is bounded, the right hand side is just a constant times $t^{-(k+d+1)/2}$ and so we proved the following theorem:

**Theorem 13 (Bounded zeros).** Let $u$ be the solution to the heat equation (5.1) with initial data $u_0 \in L^1(\mathbb{R}^d; 1 + |x|^{k+1})$ and $z(t) = (z_i(t))$ be zeros of the solution $u(\cdot, t)$ defined for $t \in (T, \infty)$, $T > 0$. If $|z(t)|/\sqrt{t}$ is bounded, then it holds that, as $t \to \infty$,
\[
\left| \sum_{|\alpha| \leq k} \int \frac{x^\alpha u_0(x) \, dx}{\alpha!} (4t)^{-|\alpha|/2} \prod_{i=1}^d H_{\alpha_i} \left( \frac{z_i(t)}{2\sqrt{t}} \right) \right| = O(t^{-(k+1)/2}).
\]

If zeroth moment (mass) is nonzero, Theorem 12 tells us that any zeros cannot be inside of a ball of radius $\sqrt{t}$. If zeroth moment is zero, then we may assume first $k - 1$ moments ($k \geq 1$) are zero and $k$-th moment is nonzero. Then the above theorem can be simplified to the following theorem:

**Theorem 14 (Bounded zeros with vanishing moments).** Let $u$ be the solution to the heat equation (5.1) with initial data $u_0 \in L^1(\mathbb{R}^d; 1 + |x|^{k+1})$ and $z(t) = (z_i(t))$ be zeros of the solution $u(\cdot, t)$ defined for $t \in (T, \infty)$, $T > 0$. If $|z(t)|/\sqrt{t}$ is bounded and the moments of the solution $u$ satisfies
\[
\int x^\alpha u_0(x) \, dx = 0 \quad \text{for all } |\alpha| < k,
\]
then it holds that, as $t \to \infty$,
\[
\left| \sum_{|\alpha| = k} \int \frac{x^\alpha u_0(x) \, dx}{\alpha!} \prod_{i=1}^d H_{\alpha_i} \left( \frac{z_i(t)}{2\sqrt{t}} \right) \right| = O(t^{-1/2}). \tag{5.4}
\]
The theorem tells us that when the propagation speed of zeros is bounded by $\sqrt{t}$, the zeros divided by $\sqrt{t}$ eventually become close to zeros of a linear combination of the Hermite polynomials. And we can go further; near simple zeros of the polynomial, there is at least one zero of solution under the same moments condition. To put it concretely, the theorem will prove is the following:

**Theorem 15 (Existence of zeros).** Let $u$ be the solution to the heat equation with initial data $u_0 \in L^1(\mathbb{R}^d; 1 + |x|^{k+1})$, $k \geq 1$. Assume the initial data $u_0$ has zero moments up to order $k - 1$ and nonzero $k$-th moment. Then for each simple zero $x_0$ of a polynomial $\phi_k(x)$, there exist a constant $T > 0$ and zeros $z(t) = (z_i(t))$ of the solution $u(\cdot, t)$ defined for $t \in (T, \infty)$ such that

$$\frac{z(t)}{2\sqrt{t}} \to x_0 \quad \text{as } t \to \infty.$$  

Here the polynomial $\phi_k(x)$ is defined by

$$\phi_k(x) := \sum_{|\alpha| = k} \frac{\int x^\alpha u_0(x) \, dx}{\alpha!} \prod_{i=1}^d H_{\alpha_i}(x_i), \quad x = (x_i).$$

The zeros $z(t)$ in Theorem 15 satisfies the boundedness condition of Theorem 14 and so the convergence order (5.3) is obtained.

**Proof.** First notice that from the Hermite polynomial approximation (5.3) there exist a constant $C$ such that

$$|u^{d/2}(4t)^{(k+d)/2}e^{\frac{|x|^2}{4t}}u(x, t) - \phi_k \left( \frac{x}{2\sqrt{t}} \right)| \leq Ce^{\frac{|x|^2}{4t}} \leq \frac{1}{(2\sqrt{t})} \leq \frac{1}{2\sqrt{t}}$$

for any $x \in \mathbb{R}^d$. Now let $x_0$ be a zero of the polynomial $\phi_k(x)$. Then there are two points $x_1, x_2 \in \mathbb{R}^d$ such that

$$\phi_k(x_1) > 0, \quad \phi_k(x_2) < 0$$

and on the line segment $I := \{\lambda x_1 + (1 - \lambda)x_2 : 0 \leq \lambda \leq 1\}$ joining $x_1$ and $x_2$, the polynomial $\phi_k(x)$ have a unique zero $x_0$. Put $x = 2\sqrt{t}x_1$ and $x = 2\sqrt{t}x_2$ to the equation (5.5) and observe that as $t \to \infty$,

$$u^{d/2}(4t)^{(k+d)/2}e^{\frac{|x|^2}{4t}}u(2\sqrt{t}x_j, t) \to \phi_k(x_j), \quad j = 1, 2.$$  

Hence there exist a constant $T > 0$ such that

$$u(2\sqrt{t}x_1, t) > 0 \quad \text{and} \quad u(2\sqrt{t}x_2, t) < 0 \quad \text{for every } t \geq T.$$  

Because the solution $u(\cdot, t)$ is continuous, there is at least one zero $z(t)$ on the line segment joining $2\sqrt{t}x_1$ and $2\sqrt{t}x_2$, i.e., $z(t)/(2\sqrt{t}) \in I$. (Of course $z(t)$ is defined for $t \geq T$.) Therefore $|z(t)|/\sqrt{t}$ is bounded and we can apply the Theorem 14. By the equation (5.4) it holds that

$$\frac{\phi_k \left( \frac{z(t)}{2\sqrt{t}} \right)}{2\sqrt{t}} \to 0 \quad \text{as } t \to \infty.$$  

But $z(t)/(2\sqrt{t}) \in I$ and the only zero of the polynomial $\phi_k(x)$ on the line segment $I$ is $x_0$. Consequently we can conclude that $z(t)/(2\sqrt{t}) \to x_0$ as $t \to \infty$.  

In summary, the zero level set $Z(t)$ in a ball of radius $C\sqrt{t}$ for some $C > 0$ converges to zeros of a linear combination of the Hermite polynomial when the set is divided by $\sqrt{t}$. Hence asymptotics of the zero level set is closely related to zeros of the Hermite polynomials. In the following sections we will present more detailed results on the asymptotics for some specific cases.
5.4 One-dimensional Heat Equation

In this section we assume the dimension \( d = 1 \). Hence the zeros can be counted [3] and the number of zeros is nonincreasing \( \frac{d}{2} \) [62]. Furthermore we can count initial number of zeros under vanishing moments condition as the following lemma gives us:

**Lemma 10.** Suppose a nonzero continuous function \( f : \mathbb{R} \rightarrow \mathbb{R} \) has zero moments up to order \( m \). Then the function \( f \) changes sign at least \( m + 1 \) times.

**Proof.** If \( m = 0 \), the non-zero continuous function \( f \) should change sign at least one time because otherwise the mass (zeroth moment) cannot be zero. Now assume \( m \geq 1 \) and the function \( f \) changes sign \( k \) times where \( 1 \leq k < m + 1 \). Then there are \( k \) points \( c_1, c_2, \cdots, c_k \) such that

1. \( f(x) \) does not change sign in the interval \([c_i, c_{i+1}]\)
2. \( f(\xi) \cdot f(\xi+1) < 0 \) for some \( \xi \in (c_i, c_{i+1}) \), \( \xi+1 \in (c_{i+1}, c_{i+2}) \)

where \( c_0 = -\infty, c_{k+1} = \infty \) and \( 0 \leq i \leq k - 1 \). Then \( \int (x - c_1)(x - c_2) \cdots (x - c_k)f(x) \, dx \) is a linear combination of moments of order less than or equal to \( m \) and should be zero. But it cannot be because a continuous function \((x - c_1)(x - c_2) \cdots (x - c_k)f(x)\) does not change sign and nonzero.

On the other hand, Theorem 15 can be simplified because the polynomial \( \phi_k(x) \) is just

\[
\phi_k(x) = \frac{\int x^k u_0(x) \, dx}{k!} H_k(x).
\]

Thus if \( k \)-th moment is nonzero, then zeros of \( \phi_k(x) \) and zeros of \( H_k(x) \) are exactly same. Also we know that every zero of the Hermite polynomials is simple. (Actually all the zeros of the orthogonal polynomials are simple. See [11, p787]) Hence a simplified version of Theorem 15 can be found:

**Theorem 16** (Existence of zeros in 1-d). Let \( u \) be the solution to the one-dimensional heat equation \( \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \) with initial data \( u_0 \in L^1(\mathbb{R}; 1 + |x|^{k+1}) \), \( k \geq 1 \). If the initial data \( u_0 \) has zero moments up to order \( k - 1 \) and nonzero \( k \)-th moment, then for each zero \( x_0 \) of the Hermite polynomial \( H_k(x) \), there exist a constant \( T > 0 \) and zeros \( z(t) \) of the solution \( u(\cdot, t) \) defined for \( t \in (T, \infty) \) such that

\[
\frac{z(t)}{2\sqrt{t}} \rightarrow x_0 \quad \text{as} \quad t \rightarrow \infty.
\]

We may assume the initial data \( u_0 \) is continuous because the solution \( u \) becomes continuous immediately and the vanishing moments condition at initial time holds for all time \( t > 0 \) (See [45]). If the initial data \( u_0 \) has zero moments up to order \( k - 1 \), then by the Lemma 10 the solution \( u \) has at least \( k \) zeros for all time \( t > 0 \). Furthermore if the initial data \( u_0 \) has \( k \) zeros initially, then by the nonincreasing property of the zero level set, the solution \( u \) has at most \( k \) zeros for all time \( t > 0 \). Therefore there are exactly \( k \) zeros for all time \( t > 0 \) and by Theorem 15 we can describe asymptotics of all those \( k \) zeros:

**Theorem 17** (Asymptotics of zeros in 1-d). Let \( u \) be the solution to the one-dimensional heat equation \( \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \) with initial data \( u_0 \in L^1(\mathbb{R}; 1 + |x|^{k+1}) \), \( k \geq 1 \). Assume the initial data \( u_0 \) has zero moments up to order \( k - 1 \) and nonzero \( k \)-th moment. If the initial data \( u_0 \) has \( k \) zeros, then there are exactly \( k \) continuous curves of zeros \( z_i(t), i = 1, \cdots, k \) defined for all time \( t \in (0, \infty) \) and each zeros satisfies that

\[
\frac{z_i(t)}{2\sqrt{t}} \rightarrow x_i \quad \text{as} \quad t \rightarrow \infty.
\]

where \( x_i, i = 1, \cdots, k \) are different zeros of the Hermite polynomial \( H_k(x) \).
The zeros of the first few Hermite polynomials $H_k(x)$ are given below:

<table>
<thead>
<tr>
<th>$k$</th>
<th>zeros of $H_k(x)$</th>
<th>approximate values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$\pm \sqrt{2}/2$</td>
<td>$\pm 0.707$</td>
</tr>
<tr>
<td>3</td>
<td>$0, \pm \sqrt{6}/2$</td>
<td>$0, \pm 1.225$</td>
</tr>
<tr>
<td>4</td>
<td>$\pm \sqrt{6 - 2\sqrt{6}}/2, \pm \sqrt{6 + 2\sqrt{6}}/2$</td>
<td>$0.525, \pm 1.651$</td>
</tr>
<tr>
<td>5</td>
<td>$0, \pm \sqrt{10 - 2\sqrt{10}}/2, \pm \sqrt{10 + 2\sqrt{10}}/2$</td>
<td>$0, \pm 0.959, \pm 2.020$</td>
</tr>
</tbody>
</table>

Now we are ready to explain the example given in the introduction for the one spatial dimension.

**Example 1.** Let the initial data $u_0$ be the difference of two time-delayed heat kernels:

$$u_0(x) := G(x, t) - G(x, t + T), \quad \text{for some fixed } T > 0.$$  

Then due to the symmetry, zeroth and first moments of $u_0$ are zero and second moment is nonzero. Also $u_0$ has only two zeros. Therefore by Theorem 17, there are exactly two continuous curves of zeros $z_{\pm}(t)$ and they satisfy

$$\frac{z_{\pm}(t)}{\sqrt{t}} \to \pm \sqrt{2} \quad \text{as } t \to \infty.$$

We can verify this fact via computation because solution $u$ is explicitly given by

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} - \frac{1}{\sqrt{4\pi(t+T)}} e^{-\frac{x^2}{4(t+T)}}.$$  

Assuming $u(z(t), t) = 0$ implies that

$$\sqrt{1 + \frac{T}{t}} = \exp \left( -\frac{z^2(t)}{4(t + T)} + \frac{z^2(t)}{4t} \right) = \exp \left( \frac{T}{4(t + T)} z^2(t) \right).$$

Hence $z^2(t)/t = 2(1 + t/T) \ln(1 + T/t) \to 2$ as $t \to \infty$.

### 5.5 Radially Symmetric Initial Data

In this section we assume the initial data $u_0$ is radially symmetric, i.e., $u_0(x) = u_0(r)$ where $r = |x|$. Then the solution is radially symmetric for all time $t > 0$ and the heat equation (5.1) can be rewritten as

$$u_t = u_{rr} + (d - 1) \frac{u_r}{r} \quad \text{in } \mathbb{R} \times (0, \infty),$$  

$$u(r, 0) = u_0(r) \quad \text{in } \mathbb{R}.$$  

Therefore the situation is similar to the one-dimensional case; the circles of zeros can be counted and their number is nonincreasing. Also the polynomial $\phi_k(x)$ in Theorem 15 can be simplified by observing moments. Assume $\alpha_i$ is odd. Then

$$\int_{-\infty}^{\infty} x_i^{\alpha_i} u_0(r) \, dx_i = \int_0^{\infty} x_i^{\alpha_i} u_0(r) \, dx_i + \int_{-\infty}^0 x_i^{\alpha_i} u_0(r) \, dx_i$$

$$= \int_0^{\infty} x_i^{\alpha_i} u_0(r) \, dx_i - \int_0^{\infty} x_i^{\alpha_i} u_0(r) \, dx_i = 0.$$

Hence for a nonzero moment every $\alpha_i$’s are even. Consequently a moment is zero when $|\alpha|$ is odd.
Now we assume every $\alpha_i$'s are even and $d \geq 2$. Using hyperspherical coordinates

\[
x_1 = r \cos(\psi_1)
\]
\[
x_2 = r \sin(\psi_1) \cos(\psi_2)
\]
\[
\vdots
\]
\[
x_d = r \sin(\psi_1) \cdots \sin(\psi_{d-2}) \sin(\psi_{d-1}),
\]

moments of initial data can be written as

\[
\int x^\alpha u_0(x) \, dx = \int_{r=0}^{\infty} \int_{\psi_1=0}^{\pi} \cdots \int_{\psi_{d-2}=0}^{\pi} \int_{\psi_{d-1}=0}^{\pi} r^{|\alpha|} (\cos(\psi_1))^{\alpha_1} (\sin(\psi_1) \cos(\psi_2))^{\alpha_2} \\
\times \cdots \times (\sin(\psi_1) \cdots \sin(\psi_{d-2}) \cos(\psi_{d-1}))^{\alpha_{d-1}} (\sin(\psi_1) \cdots \sin(\psi_{d-2}) \sin(\psi_{d-1}))^{\alpha_d} \\
\times r^{d-1} \sin^{d-2}(\psi_1) \sin^{d-3}(\psi_2) \cdots \sin(\psi_{d-2}) u_0(r) \, dr \, d\psi_1 \cdots d\psi_{d-1}
\]
\[
= \int_0^{\infty} r^{|\alpha|+d-1} u_0(r) \, dr \times \prod_{i=1}^{d-2} \int_{\alpha_i = 0}^{\pi} \cos^{\alpha_i}(\psi_i) \sin^{\sum_{j=1}^{i} \alpha_j + d - i - 1}(\psi_i) \, d\psi_i
\]
\[
\times \int_0^{2\pi} \cos^{\alpha_{d-1}}(\psi_{d-1}) \sin^{\alpha_d}(\psi_{d-1}) \, d\psi_{d-1}.
\]

Now we use trigonometric integrals

\[
\frac{1}{n!} \int_{0}^{\pi} \cos^n(\psi) \sin^m(\psi) \, d\psi = \int_{0}^{\pi} \sin^n(\psi) \, d\psi = \begin{cases} 
\frac{(m+n)(m+n-2) \cdots (m+2)}{(m+n+1)(m+n+2) \cdots (m+n+2)} \frac{2^{m+n}}{2^{m+n+2}!} & \text{if } n \text{ is even} \\
\frac{2^{m+n}}{m+n+1} \frac{2^{m+n+2}!}{(m+n+1)(m+n+2) \cdots (m+n+2)} & \text{if } m \text{ is odd,} \\
\frac{2^{m+n}}{m+n+1} \frac{2^{m+n+2}!}{(m+n+1)(m+n+2) \cdots (m+n+2)} & \text{if } m \text{ is even,}
\end{cases}
\]

\[
\frac{1}{n! m!} \int_{0}^{2\pi} \cos^n(\psi) \sin^m(\psi) \, d\psi = \begin{cases} 
\frac{2\pi}{(n+m)(n+m-2) \cdots (n+2)} \frac{2^{n+m+2}!}{2^{n+m}/2!(n/2)!} & \text{if } n, m \text{ are even.}
\end{cases}
\]

to conclude that

\[
\frac{1}{\alpha!} \int x^\alpha u_0(x) \, dx = C(|\alpha|, d) \frac{2^{\alpha/2}}{\prod_{i=1}^{d} (\alpha_i/2)!} \int_{0}^{\infty} r^{|\alpha|+d-1} u_0(r) \, dr
\]

where $C(|\alpha|, d)$ is a nonzero constant defined by

\[
C(|\alpha|, d) := \begin{cases} 
\frac{(2\pi)^{d/2}}{|\alpha|+d-2} & \text{if } d \text{ is even,} \\
\frac{(2\pi)^{d-1/2}}{|\alpha|+d-2} (|\alpha|+d-4) & \text{if } d \text{ is odd.}
\end{cases}
\]

Also we recall generalized Laguerre polynomials [Π p775]

\[
L_n^{(a)}(x) := \frac{x^{-a} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+a}),
\]

and their properties [Π p785, p779]

\[
L_n^{(a+b+1)}(x + y) = \sum_{i=0}^{n} L_i^{(a)}(x) L_{n-i}^{(b)}(y),
\]
\[
H_{2n}(x) = (-1)^n \frac{2^{2n}}{n!} L_n^{(-1/2)}(x^2).
\]
Then the polynomial \( \phi_k(x) \) in Theorem 15 becomes

\[
\phi_k(x) \bigg| \int_0^\infty r^{[\alpha]+d-1} u_0(r) \, dr = \sum_{|\alpha|=k} \frac{\int x^\alpha u_0(x) \, dx}{\alpha!} \prod_{i=1}^d H_{\alpha_i}(x_i) \bigg| \int_0^\infty r^{[\alpha]+d-1} u_0(r) \, dr
\]

\[
= \sum_{|\alpha|=k} \frac{C(|\alpha|, d)}{2^{\alpha/2} \prod_{i=1}^d (\alpha_i/2)!} \prod_{i=1}^d H_{\alpha_i}(x_i)
\]

\[
= (-2)^{k/2} C(k, d) \sum_{|\alpha|=k} \prod_{i=1}^d L_{\alpha_i/2}(-1)^2(x_i^2)
\]

\[
= (-2)^{k/2} C(k, d) L_{k/2}^{(d-2)/2}(r^2).
\]

Therefore \( \phi_k(x) \) and \( L_{k/2}^{(d-2)/2}(r^2) \) have exactly same zeros. Hence we can conclude the following theorem:

**Theorem 18** (Radial symmetric initial data). Let \( u \) be the solution to the heat equation \([5.1]\) with radially symmetric initial data \( u_0 \in L^1(\mathbb{R}^d; 1 + |x|^{k+1}) \). Assume the initial data \( u_0 \) has zero moments up to order \( k-1 \) and a nonzero \( k \)-th moment. Then \( k \) should be an even number and for each zero \( r_0 \) of a generalized Laguerre polynomial \( L_{k/2}^{(d-2)/2}(r^2) \) there exist a constant \( T > 0 \) and zeros \( z(t) = (z_i(t)) \) of the solution \( u(\cdot, t) \) defined for \( t \in (T, \infty) \) such that

\[
\frac{|z(t)|}{2 \sqrt{t}} \to r_0 \quad \text{as} \quad t \to \infty.
\]

Hence if the initial data \( u_0 \) has exactly \( k \) circles of zeros, then there are always exactly \( k \) circles of zeros and their radii divided by \( 2 \sqrt{T} \) converges to separate zeros of the generalized Laguerre polynomial \( L_{k/2}^{(d-2)/2}(r^2) \) as \( t \to \infty \). From this observation we can generalize Example 11 to multi-dimensions.

**Example 2.** Let the initial data \( u_0 \) be the difference of two time-delayed heat kernels:

\[
u_0(x) := G(x, t) - G(x, t + T) \quad \text{for some fixed} \quad T > 0.
\]

Then due to the radial symmetry, zeroth and first moments of \( u_0 \) are zero and \( \int x^\alpha u_0(x) \, dx \) is nonzero. Also \( u_0 \) has only one circle of zeros. Because

\[
L_1^{(d-2)/2}(x) = x^{-(d-2)/2} e^{-x/2} \frac{d}{dx} (e^{-x} x^{1+(d-2)/2}) = -x + d/2,
\]

the unique positive zero of \( L_1^{(d-2)/2}(r^2) \) is \( r = \sqrt{d/2} \) and therefore by Theorem 18 and non-increasing property of the zeros, there are only one circle of zeros with radius \( r(t) \) and they satisfy

\[
\frac{r(t)}{2 \sqrt{t}} \to \sqrt{d/2} \quad \text{as} \quad t \to \infty.
\]

We can verify this fact via computation because the solution \( u \) is explicitly given by

\[
u(x, t) = (4\pi t)^{-d/2} e^{-|x|^2/4t} - (4\pi (t + T))^{-d/2} e^{-|x|^2/4(t + T)}.
\]

Assuming \( u(z(t), t) = 0 \) implies that

\[
\left(1 + \frac{T}{t}\right)^{d/2} = \exp \left( -\frac{|z(t)|^2}{4(t + T)} + \frac{|z(t)|^2}{4t} \right)
\]

\[
= \exp \left( \frac{T}{4t(t + T)} |z(t)|^2 \right).
\]

(5.6)

Hence \( |z(t)|^2/t = 2d(1 + t/T) \ln(1 + T/t) \to 2d \) as \( t \to \infty \).
It is worth noting that asymptotics of radially symmetric solution is independent of exact values of moments; “nonzero” moment is enough to describe asymptotics as in the one-dimensional case. But for general cases, this is not true as we will see in the following section.

5.6 Two-dimensional Heat Equation

This section is provided to demonstrate variety of asymptotic behavior of zeros for multi-dimensions. We assume \( d = 2 \) and \( k \leq 2 \) in Theorem 15. When \( k = 1 \), the polynomial \( \phi_1(x) \) is

\[
\phi_1(x) = \sum_{|\alpha|=1} \frac{\int x^\alpha u_0(x) \, dx}{\alpha!} H_{\alpha_1}(x_1)H_{\alpha_2}(x_2)
\]

\[
= \int x_1 u_0(x) \, dx \, H_1(x_1)H_0(x_2) + \int x_2 u_0(x) \, dx \, H_0(x_1)H_1(x_2)
\]

\[
= 2x_1 \int x_1 u_0(x) \, dx + 2x_2 \int x_2 u_0(x) \, dx.
\]

Hence the zero level set of \( \phi_1(x) \) is a straight line passing through the origin.

![Figure 5.1: The zero level set of the polynomial \( \phi_1(x) \) when \( d = 2 \). If \( \int x_1 u_0(x) \, dx = 0 \), the zero level set is a straight line \( x_2 = 0 \) (left). If not, the zero level set is a straight line \( x_1 = -R \frac{x_2}{u_0(x)} \) (right).](image)

When \( k = 2 \), the polynomial \( \phi_2(x) \) is

\[
\phi_2(x) = \int x_1^2 u_0(x) \, dx \, H_2(x_1)H_0(x_2) + \int x_2^2 u_0(x) \, dx \, H_0(x_1)H_2(x_2)
\]

\[
+ \int x_1 x_2 u_0(x) \, dx \, H_1(x_1)H_1(x_2)
\]

\[
= (4x_1^2 - 2)A + 4x_1 x_2 B + (4x_2^2 - 2)C
\]

\[
= 4(Ax_1^2 + Bx_1 x_2 + Cx_2^2) - 2(A + C),
\]

where

\[
A := \int x_1^2 u_0(x) \, dx, \quad B := \int x_1 x_2 u_0(x) \, dx \quad \text{and} \quad C := \int x_2^2 u_0(x) \, dx.
\]

Hence we are looking for the graph of a quadratic equation

\[
Ax_1^2 + Bx_1 x_2 + Cx_2^2 = \frac{A + C}{2}, \tag{5.7}
\]

which is a conic section. (Note that second order moments are not all zero, i.e., \( A \neq 0 \) or \( B \neq 0 \) or \( C \neq 0 \).) The graph can be two lines, one line, one point and the empty set; they are called degenerate curves. If the graph is non-degenerate, by the discriminant classification, it holds that

- the graph is an ellipse if \( B^2 - 4AC < 0 \). (The graph is a circle if \( A = C \) and \( B = 0 \).)
• the graph is a **parabola** if $B^2 - 4AC = 0$. This shape is **impossible** because the equation (5.7) does not have a first degree term.

• the graph is a **hyperbola** if $B^2 - 4AC > 0$. (The graph is a **rectangular hyperbola** if we also have $A + C = 0$.)

Even if the graph is degenerate, certain cases can be excluded; we can verify that the conic section (5.7) cannot be one line, one point nor the empty set. Now assume the graph represents two lines $ax_1 + bx_2 + c = 0$ and $\tilde{a}x_1 + \tilde{b}x_2 + \tilde{c} = 0$. Then we have

$$
\phi_2(x) = 4(Ax_1^2 + Bx_1x_2 + Cx_2^2) - 2(A + C)
= (ax_1 + bx_2 + c)(\tilde{a}x_1 + \tilde{b}x_2 + \tilde{c})
= a\tilde{a}x_1^2 + (a\tilde{b} + \tilde{a}b)x_1x_2 + b\tilde{b}x_2^2 + (a\tilde{c} + \tilde{a}c)x_1 + (\tilde{b}c + bc)x_2 + \tilde{c}c.
$$

Comparing zeroth and first degree terms we have

$$
a\tilde{c} + \tilde{a}c = b\tilde{b} + \tilde{b}c = 0 \quad \text{and} \quad c\tilde{c} = -\frac{1}{2}(a\tilde{a} + b\tilde{b}).
$$

If $c = 0$ and $\tilde{c} \neq 0$, then $a = b = 0$ and the graph is not two lines. If $c = \tilde{c} = 0$, then $A + C = 0$ and the graph is a rectangular hyperbola, or more specifically, two lines intersecting at the origin. Now assume $c \neq 0$ and $\tilde{c} \neq 0$. Then from $a\tilde{c} = \tilde{a}c = a\tilde{b} + \tilde{b}c = 0$ we have $a\tilde{c} = \tilde{a}c$ or $\tilde{a}b = a\tilde{b}$. Therefore two lines are parallel. Let $\tilde{a}/a = \tilde{b}/b = \kappa$. Then we have

$$
\phi_2(x) = \kappa\left\{(ax_1 + bx_2)^2 - (a^2 + b^2)/2\right\}.
$$

Thus the zero level set of $\phi_2(x)$ is not changed by reflection through the origin.

We completed the classification on the graph of the zero level set and the result is summarized in the Figure 5.2.

![Figure 5.2](image)

Figure 5.2: The zero level set of the polynomial $\phi_2(x)$ when $d = 2$.

### 5.7 Numerical Examples

Finally we give numerical examples which show that our classification on the graph of the zero level set really works. There are eight figures; four figures in the left are initial data and four figures in the right are their solution at time $t = 1$. We can see the zero level set of the solution from the contour at the bottom of the graph.
5.8 Open Problems

1. Mizoguchi [55] proved that when the number of initial sign changes is finite, every zero of solutions to the one-dimensional heat equation is bounded by $t$. On the other hand, by Theorem [12] every zero cannot be bounded by $\sqrt{t}$ if the initial mass is not zero. We know Mizoguchi’s upper bound is optimal. Also we obtained a (strictly) lower bound $\sqrt{T}$. Is there any zero of speed between $\sqrt{t}$ and $t$ for one-dimensional heat equation with finite sign changes? Moreover, can we extend the result to multi-dimensional heat equation case? It seems that multi-dimensional extension is related to a multi-dimensional concept for sign changes.

2. Study the long-time asymptotics of zero set and intersection points for linear, uniformly parabolic partial differential equations. Estimates on derivatives of the fundamental solutions may be helpful. Refer to a book by Friedman [31].

3. Study the long-time asymptotics of intersection points for nonlinear diffusion equations including the porous medium equation and the fast diffusion equation.

4. Angenent [4] proved that solutions of the one-dimensional porous medium equation are determined by their free boundary. More specifically local behaviours of free boundary after a waiting time
Figure 5.5: The zero level set is a hyperbola.

Figure 5.6: The zero level set is a rectangular hyperbola intersecting at the origin.

determine solutions. What can we say about the heat equation?
Figure 5.7: The zero level set consists of \textit{two parallel lines}. The set is invariant under the reflection through the origin.
요 약 문

모멘트가 결정하는 오랜 시간 후 확산 방정식의 접근 행동

본 논문에서는 초기치의 모멘트와 확산 방정식에서의 오랜 시간 후 접근 행동 간의 상관 관계를 다룬다. 구체적으로는 비선형 방정식에서의 $L^1$-중간 접근 행동, 점성 버커스 방정식의 근사해, 열 방정식에서의 영(zero) 등이 접근의 오랜 시간 후 접근 행동을 논의한다.

2장에서는 상대적 관점에서 방사형(radial) 해의 뉴턴 퍼텐셜을 소개한다. 이를 통해 차원에 관계 없이 동일한 방법으로 더 많은 함수들에 퍼텐셜 이론을 적용할 수 있다. 이것의 한 예로 차원에 대한 고려 없이 뉴턴 정리를 간단하게 다시 쓸 수 있음을 드린다. 그 후 상대적 퍼텐셜을 이용해 다양한 해질 방정식과 빠른 확산 방정식에서 방사형 해의 $L^1$-수렴차수가 시간 $t$가 무한대로 갈 때 $O(t^{-1})$임을 보인다. 비슷한 방법으로 $p$-라플라스 방정식 해에 대해서도 같은 수렴차수를 얻는다.

3장에서는 이어지는 장들에서 사용할 열 방정식의 근사해 두 종류를 소개한다. 비선형 방정식인 버커스 방정식은 콜-호프 변환을 통해 선형 방정식으로 변환 가능한 특별한 성질을 가지고 있다. $n$개의 열핵(heat kernel)의 합을 역 콜-호프 변환하여 접근적 근사해를 만들고 원래 해와 근사해의 $L^2$-수렴을 시간 $t$가 무한대로 갈 때 $O((\sqrt{t})^{n-1/2})$의 수렴차수로 가까워짐을 보인다.

4장에서는 간적 $R^d$에서 열 방정식의 해 $u$의 영 위 답합 $Z(t) := \{x \in R^d : u(x, t) = 0\}$를 고려한다. 초기치의 모멘트가 정의하는 가정하에 $C^1$의 공간 $C^1$는 임의의 양수 $\alpha$에서 집합 $Z(t)$가 특정한 그레프로 수렴함을 보인다. 그레프는 에르미트 다항식의 일차결합의 해집합과 같으며 일차결합의 계수는 초기치의 모멘트에 의존한다. 또한 몇몇 경우에 영 집합 $Z(t)$가 수렴하는 그레프를 분류한다.
References


[41] Y.-J. Kim, Potential comparison and asymptotics in scalar conservation laws without convexity, J. Differential Equations, 244 (2008), 40–51.


[59] M. Pierre, Uniqueness of the solutions of $u_t - \Delta \varphi(u) = 0$ with initial datum a measure, Nonlinear Anal., 6 (1982), 175–187.


감 사 의 글

6년 간의 대학원 생활 내내 부족한 저를 너그러이 이해해 주시고 귀중한 조언들로 연구자의 자세를 일깨워 주신 김용중 교수님께 진심으로 감사드립니다. 앞으로도 교수님의 가르침을 잃지 않고 노력하는 자세로 즐겁게 수학 연구를 하겠습니다.

또한 연구를 진행하는 동안 많은 지원을 해주신 한국과학기술원, (구)한국학술진흥재단, (구)한국과학재단 그리고 한국연구재단에 감사드립니다.

열정적인 강의로 많은 가르침을 주신 수리과학과 교수님들께도 감사드립니다. 특히 김홍오 교수님께서는 실변수 함수론과 복소변수 함수론 강의로 해석학에 깊은 흥미를 느끼게 해주셨습니다. 또한 권길현 교수님의 함수해석학과 초등수론 강의는 편미분 방정식을 공부하는데 큰 도움이 되었습니다.

바쁘신 와중에도 논문 심사를 해주신 권순식 교수님, 김홍오 교수님, 문화기술대학원의 노준용 교수님 그리고 위스콘신 대학의 Marshall Slemrod 교수님께도 감사드립니다. 또한 예비 심사를 해주신 권길현 교수님께도 감사드립니다.

많은 가르침과 도움을 받은 박사후 연구원님들과 연구실 후배님들, 학부생 연구원들에게도 감사드립니다. 수치해석을 연구하시는 하영수 박사님, 생물수학을 연구하시는 조은주 박사님, 슈프딩거 방정식을 연구하시는 이영란 박사님, 함수방정식을 연구하시는 이영수 박사님, 과동 방정식을 연구하시는 김정에 박사님, 불변 방정식을 연구하시는 윤석현 박사님 그리고 타원형 편미분방정식을 연구하시는 권요상 박사님, 모든 분들의 가르침에 감사드립니다. 오랫동안 연구실 생활을 함께 한 박희호씨와 성격 좋은 민기 형, UC 버클리에서 열심히 공부 중인 경림이, 취미가 비슷했던 원상이, 매의 눈을 가진 장옥 씨, 배려심 깊고 든든한 민수 씨, 뉴욕대에 있을 때 큰 씨, 멘탈리스트 지은이, 뉴욕에 있을 연수, 차가운 도시 남자 정준이, 훈훈한 남자 성철이 그리고 공부도 잘하고 농구도 잘하는 응대, 모두 고맙습니다.

오랫동안 함께 생활한 만큼 수리과학과 대학원생 여러분들에게도 많은 도움을 받았습니다. 선급 프로젝트를 함께 한 성연아를 비롯해 병선이 형, 수미 누나, 기완이 형, 재만이 형, 찬용이 형, 계선이 형, 신욱이 형, 의진이 형, 동선이 형, 홍일이, 남길이, 은희 누나, 은미 누나, 은경 누나, 전진 누나, 감사드립니다.

오랜 끼메이트 병기를 비롯한 고교 동창들 우성아, 영화, 상영이, 두형이, 스كان이, 세휘, 수영이, 대홍이, 세환 그리고 대학 친구들이인 진형이, 상민이에게도 고마운 말을 전합니다.

마지막으로 해야될 수 없는 관심과 사랑으로 저를 길러주신 아버지와 어머니께 감사드립니다. 앞으로는 제가 두 분을 행복하게 해드리고 살겠습니다.
이 력 서

이 름 : 정재환
생 년 월 일 : 1982년 10월 27일
출 생 지 : 강원도 속초시
본 적 지 : 강원도 원주시 무실동
주 소 : 강원도 원주시 무실동
E-mail 주 소 : jaywan.chung@gmail.com

학 력

1998. 3. – 2000. 2.  : 강원과학고등학교 (2년 수료)
2000. 3. – 2005. 2.  : 한국과학기술원 수학과 수학전공 (B.S.)
2005. 3. – 2007. 2.  : 한국과학기술원 수학과 응용수학전공 (M.S.)
2007. 2. – 2011. 1.  : 한국과학기술원 수리과학과 (Ph.D.)

경 력

2010. 7. – 2011. 2.  : 국방과학연구소 프로젝트 “Homing Guidance Loop Design and Analysis Based on Confluent Hypergeometric Kummer Differential Equation” 연구조원
2009. 5. – 2011. 2.  : 한국과학재단 프로젝트 “Virtual Resistive Network & a Development of an Anisotropic Conductivity Reconstruction Method for a Medical Imaging” 연구조원
2006. 7. – 2007. 6.  : 한국산업기술청 프로젝트 “Development of Potential Comparison Technique and Convergence Study in Several Nonlinear Dynamics” 연구조원
2005. 3. – 2011. 1.  : 한국과학기술원 수리과학과 일반조교

학회 활 동


**연구 업적**

