The Stokes equations

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January 7th, 2013

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§ 1. Fundamental tensor and layer potentials for the Stokes equations.

The Stokes equations

$$\begin{cases} -\nu\Delta u + \nabla p = 0, & \text{in} \quad D \\ \nabla \cdot u = 0 \quad \text{in} \quad D, \end{cases} \tag{1.1}$$

where ν is the viscosity, D is an open domain in $\mathbf{R^n}$, $u : D \to \mathbf{R^n}$ is the velocity and $p : D \to \mathbf{R}$ is the pressure.

Nonstationary problem

$$\begin{cases} u_t - \nu\Delta u + \nabla p = 0, & \text{in} \quad D \times (0,\infty) \\ \nabla \cdot u = 0 \quad \text{in} \quad D \times (0,\infty) \end{cases} \tag{1.2}$$

where ν is the viscosity, $u : D \times (0,\infty) \to \mathbf{R^n}$ is the velocity and $p : D \times (0,\infty) \to \mathbf{R}$ is the pressure.

For simplicity we will assume $\nu = 1$ from now on. The fundamental tensors to (1.1) are

$$\begin{cases} E^{ij}(x) = \frac{1}{2(n-2)\omega_n}\left(\frac{\delta_{ij}}{|x|^{n-2}} + \frac{(n-2)x_i x_j}{|x|^n}\right) \\ F^i(x) = \frac{1}{\omega_n}\frac{x_i}{|x|^n} \end{cases}$$

which satisfy

$$\begin{cases} -\Delta E^{ij} + \frac{\partial}{\partial x_j} F^i = \delta_{ij}\delta(x) \quad \text{in} \quad \mathbf{R}^n \\ \sum_{j=1}^n \frac{\partial}{\partial x_j} E^{ij} = 0 \end{cases}$$

in the sense of distributions. To introduce the nonstationary Stokes tensor, we first define the fundamental solution to the equation:

$$\Gamma(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4t}\right) & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

which satisfy $\frac{\partial \Gamma}{\partial t} - \Delta\Gamma = \delta(t)\delta(x)$ in $\mathbf{R}^n \times (\mathbf{R}^+ \cup \{0\})$ in the sense of distributions.

The fundamental tensor for the nonstationary Stokes equations (1.2) are

$$\begin{cases} ET^{ij}(x,t) = \delta_{ij}\Gamma(x,t) + \frac{1}{(n-2)\omega_n}\frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathbf{R^n}} \frac{\Gamma(y,t)}{|x-y|^{n-2}}dy \\ \qquad\qquad\quad = \delta_{ij}\Gamma(x,t) + R_i R_j \Gamma(x,t) \\ FT^i(x,t) = \frac{\delta(t)}{\omega_n}\frac{x_i}{|x|^n} \end{cases}$$

which satisfy

$$\begin{cases} (ET^{ij})_t - \Delta(ET^{ij}) + \frac{\partial}{\partial x_j}(FT^i) = \delta_{ij}\delta(x)\delta(t) \\ \sum_{j=1}^n \frac{\partial}{\partial x_j}(ET^{ij}) = 0 \quad \text{in} \quad \mathbf{R}^n \times (\mathbf{R}^+ \cup \{0\}) \end{cases}$$

in the sense of distributions.

By the easy calculus, we can obtain the following result.

Lemma 3.1

$$\begin{aligned} ET^{ij}(x,t) &= \delta_{ij}\Gamma(x,t) - \frac{x_i x_j}{|x|^2}\Gamma(x,t) \\ &\quad -\left(\frac{\delta_{ij}}{|x|^n} - n\frac{x_i x_j}{|x|^{n+2}}\right)\frac{1}{\pi^{\frac{n}{2}}}\int_0^{\frac{|x|}{\sqrt{4t}}} \alpha^{n-1}e^{-\alpha^2}d\alpha. \end{aligned}$$

Proof.

Let $w(x) = \frac{1}{(n-2)\omega_n}\int_{\mathbf{R^n}} \frac{\Gamma(y,t)}{|x-y|^{n-2}}dy$ which satisfying

$$\begin{cases} v''(r) - \frac{n-1}{r}v'(r) = -\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-\frac{r^2}{4t}\right) \\ v(0) = \frac{2t}{(n-2)(4t\pi)^{\frac{n}{2}}}, \quad v'(0) = 0. \end{cases}$$

By uniqueness of the solution of O.D.E., w is the unique solution of the above equation. Solving this equation, we obtain

$$w(r) = -\frac{1}{(4\pi t)^{\frac{n}{2}}} \int_0^r \frac{1}{\alpha^{n-1}} \int_0^\alpha s^{n-1} e^{-\frac{s^2}{4t}} ds d\alpha + w(0) \quad , r = |x|.$$

It says that

$$\begin{aligned} \frac{\partial^2 w}{\partial x_i \partial x_j} &= -\frac{x_i x_j}{r^2}\Gamma - (\frac{\delta_{ij}}{r^n} - n\frac{x_i x_j}{r^{n+2}})\frac{1}{(4t\pi)^{\frac{n}{2}}} \int_0^r s^{n-1} e^{-\frac{s^2}{4t}} \\ &= -\frac{x_i x_j}{r^2}\Gamma - (\frac{\delta_{ij}}{r^n} - n\frac{x_i x_j}{r^{n+2}})\frac{1}{\pi^{\frac{n}{2}}} \int_0^{\frac{r}{\sqrt{4t}}} \alpha^{n-1} e^{-\alpha^2} d\alpha, \quad r = |x|. \quad \square \end{aligned}$$

Therefore we prove the lemma.

Definition

We call $D \subset \mathbf{R}^n$, $n \geq 2$ is Lipschitz if for every $Q \in S(= \partial D)$, there is a ball $B(r, Q) = \{P \in \mathbf{R}^n \,|\, |P - Q| < r\}$ and a coordinate system such that

$$B(r, Q) \cap D = B(r, Q) \cap \{(x', x_n) \mid x_n > \phi(x'),\ ||\nabla\phi||_{L^\infty} \leq M\},$$

By a cone, we mean an open circular truncated cone. For each $Q \in S$, we assign a cone $\gamma(Q)$ such that there are three cones $\gamma_1, \gamma_2, \gamma_3$ with vertices at origin and axis along x_n satisfying

$$\gamma_1 \subset \overline{\gamma_2} \setminus \{0\} \subset \gamma_3, \quad \gamma_1 + Q \subset \overline{\gamma} \setminus \{Q\} \subset \gamma_2 + Q \subset \gamma_3 + Q$$

We define the single layer potential of f for the stationary Stokes equations :

$$\begin{cases} u^i(X) = (E^{ij} * f^j)(X) = \int_S E^{ij}(X-Q) f^j(Q) dQ \\ p(X) = (F^j * f^j)(X) = \int_S F^j(X-Q) f^j(Q) dQ. \end{cases} \tag{1.3}$$

where dQ is the surface area measure. Similarly, we define the single layer potential for the nonstationary Stokes equations:

$$\begin{cases} u^i(X,t) = (ET^{ij} * f^j)(X,t) = \int_0^t \int_S ET^{ij}(X-Q, t-s) f^j(Q,s) dQ ds \\ p(X,t) = (FT^j * f^j)(X,t) = \int_S FT^j(X-Q,t) f^j(Q,t) dQ. \end{cases}$$

We remark that the expression of the pressure in the single layer potential involves only space integral on S. This implies that the nonstationary Stokes Theory is lack of time control and this cause some significant difficulties to develop higher regularity theory for the Naver-Stokes equations and Stokes equations. In fact, without a measurable sigularity assumption of boundary data in time variable, we can not achieve the regularity in the interior in time. In principle, the time regularity in the interior is most the same as the regularity of the boundary data. We need to find the trace formula for ∇u. When $x \notin S$, from direct computations for stationary case we have

$$\begin{aligned} D_k u^i(x) &= \frac{1}{2\omega_n}\int_S \Bigg(-\delta_{ij}\frac{X_k - Q_k}{|X-Q|^n} + \delta_{ik}\frac{X_j - Q_j}{|X-Q|^n} + \delta_{jk}\frac{X_i - Q_i}{|X-Q|^n} \\ &\quad -n\frac{(X_i - Q_i)(X_j - Q_j)(X_k - Q_k)}{|X-Q|^{n+2}}\Bigg) f^j(Q)dQ. \end{aligned}$$

On the contrary, the nonstationary case is much more complicated. Considering the integral expressions,

$$D_k u^i(x) = A_0 + A_1 + A_2 + A_3$$

where

$$\begin{aligned}
A_0 &= \int_0^t \int_S \delta_{ij} \frac{\partial}{\partial x_k} \Gamma(x-Q, t-s) f^i(Q,s)\, dQ\, ds, \\
A_1 &= -\int_0^t \int_S \left(\delta_{ij} \frac{x_k - Q_k}{|x-Q|^2} + \delta_{ik} \frac{x_i - Q_i}{|x-Q|^2} + \delta_{jk} \frac{x_i - Q_i}{|x-Q|^2} \right. \\
&\quad \left. - (n+2) \frac{(x_i - Q_i)(x_j - Q_j)(x_k - Q_k)}{|x-Q|^4} \right) \Gamma(x-Q, t-s) f^j(Q,s)\, dQ\, ds, \\
A_2 &= -\int_0^t \int_S \frac{(x_i - Q_i)(x_j - Q_j)}{|x-Q|^2} \frac{\partial}{\partial x_k} \Gamma(x-Q, t-s) f^j(Q,s)\, dQ\, ds \\
A_3 &= \frac{1}{\pi^{\frac{n}{2}}} \int_0^t \int_S \left(n\delta_{ij} \frac{x_k - Q_k}{|x-Q|^{n+2}} + n\delta_{ik} \frac{x_j - Q_j}{|x-Q|^{n+2}} + n\delta_{jk} \frac{x_i - Q_i}{|x-Q|^{n+2}} \right. \\
&\quad \left. - n(n+2) \frac{(x_i - Q_i)(x_j - Q_j)(x_k - Q_k)}{|x-Q|^{n+4}} \right) \int_0^{\frac{|x-Q|}{\sqrt{4(t-s)}}} \alpha^{n-1} e^{-\alpha^2}\, d\alpha\, f^j(Q,s)
\end{aligned}$$

We let $\Omega = D$. The following facts are going to be useful. We assume the boundary of the domain $\Omega \subset \mathbf{R}^n$ is flat and $N(P)$ is the exterior normal vector at $P \in \partial\Omega$. Let ω_n be the surface area of the unit sphere in $\mathbf{R}^n$.

Lemma 3.3

We have for all $\epsilon > 0$

$$\lim_{X \in \Omega \to P} \frac{1}{\omega_n} \int_{\Omega \cap \partial B(P,\epsilon)} \frac{(X_i - Q_i) N^j(Q)}{|X - Q|^n} dQ = \frac{-1}{2n} \delta_{ij},$$

and

$$\lim_{X \in \Omega \to P} \frac{1}{\omega_n} \int_{\Omega \cap \partial B(P,\epsilon)} \frac{(X_i - Q_i)(X_j - Q_j)(X_k - Q_k) N^j(Q)}{|X - Q|^{n+2}} dQ$$
$$= \begin{cases} -\frac{3}{2n(n+2)} & \text{if} \quad i = j = k \\ -\frac{\delta_{ik}}{2n(n+2)} & \text{otherwise} \end{cases}$$

where $N(Q)$ is the exterior normal vector at $Q \in \Omega \cap \partial B(P, \epsilon)$.

Lemma 3.4

We have

$$\lim_{X\in\Omega\to P}\frac{1}{\omega_n}\int_{\partial\Omega\cap B(P,\epsilon)}\frac{X_i-Q_i}{|X-Q|^n}dQ=\frac{-1}{2}N^i(P)+p.v.\frac{1}{\omega_n}\int_{\partial\Omega\cap B(P,\epsilon)}\frac{P_i-Q_i}{|P-Q|^n}dQ$$

and

$$\begin{aligned}\lim_{X\in\Omega\to P}\quad \tfrac{1}{\omega_n} &\quad \int_{\partial\Omega}\frac{(X_i-Q_i)(X_j-Q_j)(X_k-Q_k)}{|X-Q|^{n+2}}dQ\\ = &\quad \frac{-1}{2n}\left(N^i(P)\delta_{jk}+N^j(P)\delta_{ik}+N^k(P)\delta_{ij}\right)+\frac{1}{n}N^i(P)N^j(P)N^k(P)\\ &\quad +p.v.\frac{1}{\omega_n}\int_{\partial\Omega}\frac{(P_i-Q_i)(P_j-Q_j)(P_k-Q_k)}{|P-Q|^{n+2}}dQ\end{aligned}$$

where $N^i(P)$ is the i-th component of the unit normal vector $N(P)$ at $P\in\partial\Omega$.

Using the Lemma 3.3 and Lemma 3.4, we can show the trace formula of stationary equation for $f \in L^2(\partial\Omega)$.

$$\begin{aligned}
(D_k u^j)_\pm &= \lim_{X \to P} D_k u^i(X) \\
&= \pm\frac{1}{2}\left(N^k(P) f^i(P) - N^k(P) N^i(P) < N(P), f(P)\right) \\
&\quad + p.v. \int_{\partial\Omega} D_k E^{ij}(P-Q) f^j(Q)\, dQ, \\
p_\pm(P) = \lim_{X \to P} p(X,t) &= \mp\frac{1}{2} N(P) \cdot f + p.v. \int_{\partial\Omega} F_k(P-Q) f^k(Q,t)\, dQ
\end{aligned}$$

where we denote " + " for interior and " − " for exterior. Now we choose $f(Q) = e_j$, where e_j is the j-th unit vector, and obtain

$$\begin{aligned}
\lim_{X \in \Omega \to P} \quad \frac{1}{2\omega_n} & \int_{\partial\Omega} -\frac{X_k - Q_k}{|X-Q|^n}\delta_{ij} + \frac{X_i - Q_i}{|X-Q|^n}\delta_{jk} + \frac{X_j - Q_j}{|X-Q|^n}\delta_{ik}\, dQ \\
& - \frac{n}{2\omega_n} \int_{\partial\Omega} \frac{(X_i - Q_i)(X_j - Q_j)(X_k - Q_k)}{|X-Q|^{n+2}}\, dQ \\
= \quad & \frac{1}{2}\left(N^k(P)\delta_{ij} - N^i(P) N^j(P) N^k(P)\right) + p.v. \int_{\partial\Omega} D_k E^{ij}(P-Q)\, dQ.
\end{aligned}$$

Considering the relation

$$\lim_{X\in\Omega\to P}\frac{1}{\omega_n}\int_{\partial\Omega}\frac{X_i-Q_i}{|X-Q|^n}dQ=\frac{-1}{2}N^i(P)+p.v.\frac{1}{\omega_n}\int_{\partial\Omega}\frac{P_i-Q_i}{|P-Q|^n}dQ.$$

we have

$$\begin{aligned}\lim_{X\in\Omega\to P}\quad \tfrac{1}{2\omega_n}\quad & \int_{\partial\Omega}-\frac{X_i-Q_i}{|X-Q|^n}\delta_{jk}+\frac{X_k-Q_k}{|X-Q|^n}\delta_{ij}+\frac{X_j-Q_j}{|X-Q|^n}\delta_{ik}dQ\\ = \quad & \frac{1}{4}(N^k(P)\delta_{ij}-N^j(P)\delta_{ik}-N^i(P)\delta_{jk})\\ & +p.v.\frac{1}{2\omega_n}\int_{\partial\Omega}-\frac{P_i-Q_i}{|P-Q|^n}\delta_{jk}+\frac{P_k-Q_k}{|P-Q|^n}\delta_{ij}+\frac{P_j-Q_j}{|P-Q|^n}\delta_{ik}dQ.\end{aligned}$$

Hence comparing with the trace formula, we conclude that

$$\begin{aligned}\lim_{X\in\Omega\to P}\quad \tfrac{1}{\omega_n}\quad & \int_{\partial\Omega}\frac{(X_i-Q_i)(X_j-Q_j)(X_k-Q_k)}{|X-Q|^{n+2}}dQ\\ = \quad & \frac{-1}{2n}\left(N^i(P)\delta_{jk}+N^j(P)\delta_{ik}+N^k(P)\delta_{ij}\right)+\frac{1}{n}N^j(P)N^j(P)N^k(P)\\ & +p.v.\frac{1}{\omega_n}\int_{\partial\Omega}\frac{(P_i-Q_i)(P_j-Q_j)(P_k-Q_k)}{|P-Q|^{n+2}}dQ\end{aligned}$$

Now we are in position to compute the trace of time dependent double layer potentials for Stokes equations. To this purpose we assume our domain is flat near $P \in \partial D$ and take a small cylinder $C(P, t, \epsilon, \delta)$ centered at (P, t) such that $C(P, t, \epsilon, \delta) = B(P, \epsilon) \times (t - \delta, t)$. ∂C is only the lateral boundary of the cylinder C. We are going assume

$$\epsilon = o(\sqrt{\delta}) \quad \text{as} \quad \delta \to 0.$$

We are interested in computing

$$\lim_{(X,\tau)\to(P,t)} \int_{C(P,t,\epsilon,\delta)\cap\partial D_+} \frac{\partial}{\partial N(Q)} ET^{ij}(X - Q, \tau - s)dQds.$$

where $D_+ = D \times (0, \infty)$. Let

$$H_{ij}(X, \tau) = ET^{ij}(X, \tau) - \delta_{ij}\Gamma(X, \tau).$$

From the Stokes equations and integration by parts, we have that

$$\int_{\partial(C(P,t,\epsilon,\delta)\cap D_+)} \frac{\partial}{\partial N(Q)} H_{ij}(X-Q,\tau-s)dQds = B_1 + B_2 + B_3$$

where

$$\begin{aligned}
B_1 &= -\int_{B(P,\epsilon)\cap D} \left(H_{ij}(X-Q,\tau-t+\delta) - H_{ij}(X-Q,\tau-t)\right) dQds \\
B_2 &= \int_{\partial(C(P,t,\epsilon,\delta)\cap D_+)} FT^j(X-Q,\tau-s)N^iQ)dQds + O(\delta) \\
B_3 &= \int_{B(P,\epsilon)\cap D} \left(\frac{\partial}{\partial N(Q)} H_{ij}(X-Q,\tau-t) - \frac{\partial}{\partial N(Q)} H_{ij}(X-Q,\tau-t+\delta)\right) d
\end{aligned}$$

From the definition $FT^j(X-Q,\tau-s)=\frac{\delta(\tau-s)}{\omega_n}\frac{X_j-Q_j}{|X-Q|^n}$, Lemma 3.3 and 3.4, we have

$$\begin{aligned}\lim_{\delta\to 0} B_2 &= \frac{1}{\omega_n}\int_{\partial(B(P,\epsilon)\cap D)} \frac{X_j-Q_j}{|X-Q|^n} N^i(Q)dQ \\ &= -\frac{1}{2n}\delta_{ij}-\frac{1}{2}N^i(P)N^j(P).\end{aligned}$$

It is not difficult to see that

$$\int_{\partial B^+(0,1)\cap D}\frac{Q_iQ_j}{|Q|^2}dQ=\frac{\delta_{ij}}{2n}$$

and

$$\int_0^\infty \alpha^{n-1}\exp(-\alpha^2)d\alpha=\frac{1}{2}\Gamma(\frac{n}{2}).$$

We are assuming that (X,τ) converges nontangentially to (P,t). Since $\epsilon=o(\sqrt{\delta})$, we see that

$$\begin{aligned}\lim_{\delta\to 0} B_{11} &= \lim_{\delta\to 0}\int_{B(P,\epsilon)\cap D}\frac{(X_i-Q_i)(X_j-Q_j)}{|X-Q|^2}\Gamma(X-Q,\tau-t+\delta)dQ \\ &\quad+\lim_{\delta\to 0}\int_{B(P,\epsilon)\cap D}\left[\left(\frac{\delta_{ij}}{|X-Q|^n}-n\frac{(X_i-Q_i)(X_j-Q_j)}{|X-Q|^{n+2}}\right)\right.\end{aligned}$$

Similarly, we can show that $\lim_{\delta\to 0} B_{12} = 0$. It is easy to show that $\lim_{\delta\to 0} B_3 = 0$.Therefore

$$\lim_{\delta\to 0}\int_{\partial(C(P,t,\epsilon,\delta)\cap D_+)}\frac{\partial}{\partial N(Q)}H_{ij}(X-Q,\tau-s)dQds = -\frac{1}{2n}\delta_{ij} - \frac{1}{2}N^i(P)N^j(P).$$

It remains to compute

$$\int_{\partial C(P,t,\epsilon,\delta)\cap D_+}\frac{\partial}{\partial N(Q)}H_{ij}(X-Q,\tau-s)dQds = A_1 + A_2 + A_3$$

$$\begin{aligned}
A_1 &= -\sum_{k=1}^{k=n}\int_{t-\delta}^{t}\int_{\partial B(P,\epsilon)\cap D}\Bigg[\delta_{ij}\frac{X_k-Q_k}{|X-Q|^2} + \delta_{ik}\frac{X_j-Q_j}{|X-Q|^2} + \delta_{jk}\frac{X_i-Q_i}{|X-Q|^2} \\
&\quad - (n+2)\frac{(X_i-Q_i)(X_j-Q_j)(X_k-Q_k)}{|X-Q|^4}\Bigg]\Gamma(X-Q,t-s)N^k(Q)dQds \\
A_2 &= -\sum_{k=1}^{k=n}\int_{t-\delta}^{t}\int_{\partial(P,\epsilon)\cap D}\frac{(X_i-Q_i)(X_j-Q_j)}{|X-Q|^2}\frac{\partial}{\partial X_k}\Gamma(X-Q,t-s)N^k(Q)dQds \\
A_3 &= \sum_{k=1}^{k=n}\frac{1}{\pi^{\frac{n}{2}}}\int_{t-\delta}^{t}\int_{\partial B(P,\epsilon)\cap D}\Bigg[n\delta_{ij}\frac{X_k-Q_k}{|X-Q|^{n+2}} + n\delta_{ij}\frac{X_j-Q_j}{|X-Q|^{n+2}} + n\delta_{jk}\frac{X_i-Q_i}{|X-Q|^{n+2}} \\
&\quad - n(n+2)\frac{(X_i-Q_i)(X_j-Q_j)(X_k-Q_k)}{|X-Q|^{n+4}}\Bigg]\left[\int^{\frac{|X-Q|}{\sqrt{4(t-s)}}}\alpha^{n-1}\exp(-\alpha^2)d\alpha\right]N^k(Q)dQ
\end{aligned}$$

From Lemma 3.3 we have

$$\lim_{X\to P}\int_{\partial B(p,\epsilon)\cap D}\left[\delta_{ij}\frac{X_k-Q_k}{|X-Q|}+\delta_{ij}\frac{X_j-Q_j}{|X-Q|}+\delta_{jk}\frac{X_i-Q_i}{|X-Q|}-(n+2)\frac{Q_iQ_jQ_k}{|Q|^3}\right] N^k(Q)d(\frac{X-Q}{|X-Q|})=0$$

and hence considering the symmetry and polar coordinate we have

$$\begin{aligned}\lim_{\delta\to 0}A_1 &= \lim_{\delta\to 0}\int_{\partial B(P,\epsilon)\cap D}\left[\delta_{ij}\frac{X_k-Q_k}{|X-Q|^2}+\delta_{ik}\frac{X_j-Q_j}{|X-Q|^2}+\delta_{jk}\frac{X_i-Q_i}{|X-Q|^2}\right.\\ &\quad\left.-(n+2)\frac{(X_i-Q_i)(X_j-Q_j)(X_k-Q_k)}{|X-Q|^4}\right]N^k(Q)d(\frac{X-Q}{|X-Q|})\int_{t-\delta}^{t}\Gamma(\epsilon,t-s)ds\\ &= 0\end{aligned}$$

Under the same token we also get

$$\lim_{\delta\to 0}A_3=0.$$

Finally we have

$$
\begin{aligned}
\lim_{\delta\to 0} A_2 &= \lim_{\delta\to 0}\int_{\partial B(P,\epsilon)\cap D\cap D}\frac{(X_i-Q_i)(X_j-Q_j)(X_k-Q_k)}{|X-Q|^2}N^k(Q)d(\frac{X-Q}{|X-Q|}) \\
&\quad \cdot\int_{t-\delta}^{t}\frac{-1}{2(t-s)}\frac{1}{(4\pi(t-s))^{\frac{n}{2}}}\exp{(-\frac{\epsilon^2}{4(t-s)})}ds \\
&= -\frac{1}{2n}\delta_{ij}
\end{aligned}
$$

and obtain

$$
\lim_{\delta\to 0}\int_{\partial C(P,t,\epsilon,\delta)\cap D_+}\frac{\partial}{\partial N(Q)}H_{ij}(X-Q,\tau-s)dQds=-\frac{1}{2n}\delta_{ij}.
$$

Therefore combining all the previous computations, we have

$$\begin{aligned}
\lim_{\delta\to 0}\int_{C(P,t,\epsilon,\delta)\cap\partial D_+} & \frac{\partial}{\partial N(Q)}H_{ij}(X-Q,\tau-s)dQds \\
= & \lim_{\delta\to 0}\left[\int_{\partial(C(P,t,\epsilon,\delta)\cap\partial D_+)}\frac{\partial}{\partial N(Q)}H_{ij}(X_Q,\tau-s)dQds\right. \\
& \left.-\int_{\partial C(P,t,\epsilon,\delta)\cap D_+}\frac{\partial}{\partial N(Q)}H_{ij}(X_Q,\tau-s)dQds\right] \\
= & -\frac{1}{2n}\delta_{ij}-\frac{1}{2}N^i(P)N^j(P) \\
& -\lim_{\delta\to 0}\int_{\partial C,t,\epsilon,\delta)\cap D_+}\frac{\partial}{\partial N(Q)}H_{ij}(X-Q,\tau-s)dQds \\
= & -\frac{1}{2n}\delta_{ij}-\frac{1}{2}N^i(P)N^j(P)+\frac{1}{2n}\delta_{ij} \\
= & -\frac{1}{2}N^i(P)N^j(P)
\end{aligned}$$

From standard computation for heat equations we have

$$\lim_{(X,\tau)\to(P,t)} \int_{\partial(C(P,t,\epsilon,\delta)\cap D_+)} \frac{\partial}{\partial N(Q)} \Gamma(X-Q,\tau-s)dQds = 1.$$

Thus

$$\begin{aligned}
&\lim_{\delta\to 0} \int_{C(P,t,\epsilon,\delta)\cap\partial D_+} \frac{\partial}{\partial N(Q)} E_{ij}(X-Q,\tau-s)dQds \\
=&\lim_{\delta\to 0} \int_{C(P,t,\epsilon,\delta)\cap\partial D_+} \delta_{ij}\frac{\partial}{\partial N(Q)} \Gamma(X-Q,\tau-s)dQds \\
+&\lim_{\delta\to 0} \int_{C(P,t,\epsilon,\delta)\cap\partial D_+} \frac{\partial}{\partial N(Q)} H_{ij}(X-Q,\tau-s)dQds \\
&= \frac{1}{2n}\delta_{ij} - \frac{1}{2}N^i(P)N^j(P).
\end{aligned}$$

Although there is a bump along the normal direction, the tangential derivatives of single layer potential are continuous. We let $e^i =< e^i, N(P) > N(P)+ < e^i, T(P) > T(P)$, where $T(P)$ is tangent vector at $P \in \partial D$. Hence we have

$$\begin{aligned}
& D_i u^j(P,t) \\
= \ & \lim_{(X,\tau)\in D_+ \to (P,t)} \int_{\partial D_+} \frac{\partial}{\partial N(Q)} ET^{jk}(X-Q,\tau-s) f^k(Q,s) N^i(P) dQds \\
= \ & \frac{1}{2} N^i(P) f^j(P) - \frac{1}{2} N^i(P) N^j(P) < N(P), f(P) > \\
& + p.v. \int_{\partial D_+} D_i ET^{jk}(P-Q,t-s) f^k(Q,s) dQds
\end{aligned}$$

With the same argument for the velocity, we have

$$\lim_{(X,s)\in D_+\to(P,t)} p(X,t) = -\frac{1}{2} < N(P), f(P) > +p.v. \int_{\partial D} F_k(P-Q) f^k(Q,t) dQ$$

We define the conormal derivative

$$\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial N} - Np.$$

Theorem

$$\frac{\partial u_\pm}{\partial \nu}(P) = \pm\frac{1}{2} f(P) + K f(P)$$

where

$$K f(p) = p.v. \int_{\partial D_t} \frac{E_{ij}}{\partial N}(P-Q, t-s) f^k(Q,s) dQ ds$$
$$-p.v. \int_{\partial D} F_k(P-Q) f^k(Q,t) dQ.$$

Contents

§ 2. Caccioppoli inequality.

Let (u, p) be the solution of the Stokes equations in $(0, \infty) \times \Omega$

$$u_t - \Delta u + \nabla p = 0, \tag{2.1}$$

$$\mathrm{div} u = 0. \tag{2.2}$$

Fix $(x_0, t_0) \in (0, T) \times \mathbb{R}^n_+$, where $x_0 = (x_{0,1}, \cdots, x_{0,n})$. Take $r > 0$ so that $0 < r < \frac{1}{2} \min\{\sqrt{t_0}, \mathrm{dist}(x_0, \partial\Omega)\}$. Set $B_r = B_r(x_0)$ and $Q_r = B_r \times (-r^2 + t_0, t_0]$.

Lemma 27

$$\|\nabla u\|^2_{Q_r} \le C \frac{1}{r^2} \|u\|^2_{Q_{2r}}. \tag{2.3}$$

Our aim is to show the proof of the above lemma.

Choose $\phi_r \in C_0^\infty(\mathbb{R}^n)$, $\psi_r \in C_0^\infty(\mathbb{R})$ so that $\phi_r(x) = 1$ on $B_r = B_r(x_0)$ and $\phi_r(x) = 0$ on B^c_{2r}, and $\psi_r(t) = 1$ on $(-r^2 + t_0, t_0]$ and $\psi_r(t) = 0$ on $(-\infty, t_0 - 4r^2]$.

Let N be the fundamental solution of the Laplace equation

$$N(x) = \frac{1}{n(2-n)\omega_n |x|^{n-2}}, \ n \geq 3.$$

Define a function v by $v(x,t) = \int_{\mathbb{R}^n} N(x-y)[\phi_r(y)\nabla_y \times u(y,t)]dy$. Later we will see that $\nabla \times v$ is comparable to $\phi_r u$.
Take inner product by $\nabla \times (\phi_r v)$ to the equation (2.1) and integrate over Ω. By the orthogonality of ∇ and $\nabla \times$, we have the identity

$$\begin{aligned} 0 &= \int_\Omega (u_t - \Delta u + \nabla p) \cdot \nabla \times (\phi_r v) dx \\ &= \int_\Omega u_t \cdot \nabla \times (\phi_r v) dx - \int_\Omega u \cdot \Delta[\nabla \times (\phi_r v)] dx. \end{aligned} \tag{2.4}$$

Now we investigate the properties of v to derive Caciapolli inequality from (2.4).

Since $\nabla \times (\phi_r v) = \nabla \phi_r \times v + \phi_r \nabla \times v$ and $\Delta v = -\phi_r \nabla \times u$,

$$\Delta[\nabla \times (\phi_r v)] = \nabla \times [(\Delta \phi_r) v + 2(\nabla \phi_r) \cdot \nabla v - \phi_r^2 \nabla \times u]$$

Hence (2.4) can be rewritten by

$$\begin{aligned} 0 = & \int_\Omega u_t \cdot \nabla \times (\phi_r v) dx + \int_\Omega u \cdot \nabla \times [\phi_r^2 \nabla \times u] dx \\ & - \int_\Omega u \cdot \nabla \times [(\Delta \phi_r) v + 2(\nabla \phi_r) \cdot \nabla v] dx. \end{aligned} \tag{2.5}$$

Note that $\int_\Omega u_t \cdot \nabla \times (\phi_r v) dx = \frac{1}{2} \frac{d}{dt} \int_\Omega u \cdot \nabla \times (\phi_r v) dx$, since

$$\frac{d}{dt} \int_\Omega u \cdot \nabla \times (\phi_r v) dx = \int_\Omega u_t \cdot \nabla \times (\phi_r v) dx + \int_\Omega u \cdot \nabla \times (\phi_r v_t) dx$$

and

$$\begin{aligned} \int_\Omega u \cdot \nabla \times (\phi_r v_t) dx &= \int_{\mathbb{R}^n} v_t \cdot (\phi_r \nabla \times u) dx \\ &= \int_{\mathbb{R}^n} (\phi_r \nabla \times u_t) \cdot N * (\phi_r \nabla \times u) dx = \int_{\mathbb{R}^n} (\phi_r \nabla \times u_t) \cdot v dx \\ &= \int_\Omega u_t \cdot \nabla \times (\phi_r v) dx. \end{aligned}$$

Note also that

$$\int_\Omega u \cdot \nabla \times [\phi_r^2 \nabla \times u] dx = \int_\Omega \phi_r^2 |\nabla \times u|^2 dx$$

and

$$\int_\Omega u \cdot \nabla \times [(\Delta \phi_r) v + 2(\nabla \phi_r) \cdot \nabla v] dx = \int_\Omega \nabla \times u \cdot [(\Delta \phi_r) v + 2(\nabla \phi_r) \cdot \nabla v] dx$$

Hence (2.5) reduces to the identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u \cdot \nabla \times (\phi_r v) dx + \int_\Omega \phi_r^2 |\nabla \times u|^2 dx = \int_\Omega \nabla \times u \cdot [(\Delta \phi_r) v + 2(\nabla \phi_r) \cdot \nabla v] dx \tag{2.6}$$

Now multiply ψ_r to (2.6) and integrate over $(0, t_0)$, then we have the identity

$$\begin{aligned}
&\int_\Omega u(t_0) \cdot \nabla \times (\phi_r v)(t_0) dx + \int_0^{t_0} \psi(t) \int_\Omega \phi_r^2 |\nabla \times u|^2 dx dt \\
&= \int_0^{t_0} \psi_r'(t) \int_\Omega u \cdot \nabla \times (\phi_r v) dx dt \quad + \int_0^{t_0} \psi_r(t) \int_\Omega \nabla \times u \cdot [(\Delta \phi_r) v + 2(\nabla \phi_r) \cdot \nabla v] dx dt \\
&= I + II.
\end{aligned} \tag{2.7}$$

By the definition of v we have that

$$\int_\Omega u(t_0)\cdot\nabla\times(\phi_r v)(t_0)dx = \int_\Omega \phi_r \nabla\times u(t_0)\cdot v(t_0)dx$$
$$= -\int_{\mathbb{R}^n}\Delta v(t_0)\cdot v(t_0)dx = \int_{\mathbb{R}^n}|\nabla v(t_0)|^2 dx.$$

By the properties of the ψ_r and ϕ_r we have

$$\begin{aligned} I &\leq \frac{C}{r^2}\int_{r_0-4r^2}^{t_0-r^2}\int_{B_{2r}}|u|(\frac{1}{r}|v|+|\nabla v|)dxdt \\ &\leq \frac{C}{r^2}\int_{r_0-4r^2}^{t_0-r^2}(\|u\|_{L^2(B_{2r})}\|v\|_{L^{2n/(n-2)}}+\|u\|_{L^2(B_{2r})}\|\nabla v\|_{L^2})dt \\ &\leq \frac{C}{r^2}\int_{r_0-4r^2}^{t_0-r^2}\|u\|_{L^2(B_{2r})}\|\nabla v\|_{L^2}dt \end{aligned}$$

$$\begin{aligned} II &\leq \int_0^{t_0}\psi_r\int_\Omega |u|(|\nabla^2\phi_r||v|+|\nabla\phi_r||\nabla v|)dxdt \\ &\leq \int_0^{t_0}\psi_r\|u\|_{L^2(B_{2r})}(\frac{1}{r}\|v\|_{L^{\frac{2n}{n-2}}}+\frac{1}{r}\|\nabla v\|_{L^2})dt \leq \frac{1}{r}\int_0^{t_0}\psi_r\|u\|_{L^2(B_{2r})}\|\nabla v\|_{L^2}dt. \end{aligned}$$

Note that $v = \nabla_x \times \int_{\mathbb{R}^n} N(x-y) \times [\phi_r(y)u(y,t)]dy - \int_{\mathbb{R}^n} N(x-y)[(\nabla_y \phi_r)(y) \times u(y,t)]dy$. Hence

$$\begin{aligned}\nabla_x \times v = \nabla_x \times \nabla \times \int_{\mathbb{R}^n} N(x-y)[\phi_r(y)u(y,t)]dy \\ - \nabla_x \times \int_{\mathbb{R}^n} N(x-y)[(\nabla_y \phi_r)(y) \times u(y,t)]dy.\end{aligned}$$

Recall Biot-Sawart law that

$$\nabla \times \nabla \times f = -\Delta f + \nabla \mathrm{div} f$$

and

$$\begin{aligned}\mathrm{div}_x \int_{\mathbb{R}^n} N(x-y) \times [\phi_r(y)u(y,t)]dy &= \int_{\mathbb{R}^n} N(x-y)\mathrm{div}[\phi_r(y)u(y,t)]dy \\ &= \int_{\mathbb{R}^n} N(x-y)[(\nabla_y \phi_r)(y) \cdot u(y,t)]dy,\end{aligned}$$

since $\mathrm{div} u = 0$. Hence we have the identity

$$\nabla \times v = \phi_r u + \nabla N * [(u \cdot \nabla)\phi_r] - \nabla \times N * [(\nabla \phi_r) \times u]. \qquad (2.8)$$

Recall Sobolev inequality that

$$\|f\|_{L^{np/(n-p)}(\mathbb{R}^n)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)}, 1 \leq p < n$$

and Calderon-Zygmund inequality that

$$\|\nabla^2 N * f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}, 1 < p < \infty.$$

By Sobolev inequality and Calderon-Zygmund inequality,

$$\begin{aligned} \|v\|_{L^{2n/(n-2)}} &\leq C\|\nabla v\|_{L^2} \leq C\|\phi_r u\|_{L^2} + c\|\nabla N * ((\nabla\phi_r) \times u)\|_{L^2} \\ &\leq C\|\phi_r u\|_{L^2} + c\|\nabla^2 N * ((\nabla\phi_r) \times u)\|_{L^{2n/(n+2)}} \\ &\leq C\|\phi_r u\|_{L^2} + c\||\nabla\phi_r|u\|_{L^{2n/(n+2)}} \leq C\|u\|_{L^2(B_{2r})}. \end{aligned} \quad (2.9)$$

Hence (2.7) reduces to the following inequality

$$\frac{1}{2}\int_{\mathbb{R}^n} |\nabla v(t_0)|^2 dx + \int_0^{t_0} \psi(t) \int_\Omega \phi_r^2 |\nabla \times u|^2 dx dt \leq \frac{C}{r^2} \int_{t_0-4r^2}^{t_0} \|u\|^2_{L^2(B_{2r})} dt \leq \frac{C}{r^2} \|u\|^2_{L^2(Q_{2r})}. \tag{2.10}$$

This implies that

$$\|\nabla v(t_0)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u\|^2_{L^2(Q_r)} \leq \frac{C}{r^2} \|u\|^2_{L^2(Q_{2r})}.$$

Now we would like to derive the Caciapolli inequality for the higher derivatives.

Lemma

$$\|\nabla u(t_0)\|_{L^2(B_r)}, \|\Delta u\|_{L^2(Q_r)} \leq C\frac{1}{r^2}\|u\|_{Q_{2r}}. \tag{2.11}$$

proof. Take inner product by $\nabla \times (\phi_r \nabla \times u)$ to the equation (2.1) and integrate over Ω. By the orthogonality of ∇ and $\nabla\times$, we have the identity

$$\begin{aligned} 0 &= \int_\Omega (u_t - \Delta u + \nabla p) \cdot \nabla \times (\phi_r \nabla \times u) dx \\ &= \int_\Omega u_t \cdot \nabla \times (\phi_r \nabla \times u) dx - \int_\Omega \Delta u \cdot [\nabla \times (\phi_r \nabla \times u)] dx. \end{aligned} \tag{2.12}$$

Since $\nabla \times (\phi_r^2 \nabla \times u) = \nabla \phi_r \times \nabla \times u + \phi_r \nabla \times \nabla \times u$ and $\nabla \times \nabla \times u = -\Delta u$, we have

$$\int_\Omega \Delta u \cdot [\nabla \times (\phi_r^2 \nabla \times u)] dx = -\int_\Omega \phi_r^2 |\Delta u|^2 dx + \int_\Omega \Delta u \cdot [\nabla \phi_r^2 \times \nabla \times u] dx.$$

Note that

$$\int_\Omega u_t \cdot \nabla \times (\phi_r^2 \nabla \times u) dx = \frac{1}{2} \frac{d}{dt} \int_\Omega \phi_r^2 |\nabla \times u|^2 dx.$$

Hence, (2.20) reduces to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \phi_r^2|\nabla\times u|^2dx+\int_\Omega \phi_r^2|\Delta u|^2dx=\int_\Omega \Delta u\cdot[\nabla\phi_r^2\times\nabla\times u]dx. \tag{2.13}$$

Now multiply ψ_r to (2.13) and integrate over $(0,t_0)$, then we have the identity

$$\frac{1}{2}\int_\Omega \phi_r^2|\nabla\times u(t_0)|^2dx+\int_0^{t_0}\psi(t)\int_\Omega \phi_r^2|\Delta u|^2dxdt \tag{2.14}$$

$$=\int_0^{t_0}\psi(t)\int_\Omega \Delta u\cdot[\nabla\phi_r^2\times\nabla\times u]dxdt+\frac{1}{2}\int_0^{t_0}\psi'(t)\int_\Omega \phi_r^2|\nabla\times u|^2dxdt \tag{2.15}$$

$$=I+II. \tag{2.16}$$

Apply Hölder inequality to the right hand side of (2.14), we have

$$
\begin{aligned}
I &\le \frac{C}{r}\int_0^{t_0}\psi(t)\int_\Omega \phi_r|\Delta u||\nabla u|dxdt \\
&\le \epsilon\int_0^{t_0}\psi(t)\int_\Omega \phi_r^2|\Delta u|^2dxdt + \frac{C}{\epsilon r^2}\int_0^{t_0}\psi(t)\int_\Omega |\nabla u|^2dxdt \\
&\le \epsilon\int_0^{t_0}\psi(t)\int_\Omega \phi_r^2|\Delta u|^2dxdt + \frac{C}{\epsilon r^2}\|\nabla u\|^2_{L^2(Q_{2r})} \\
&\le \epsilon\int_0^{t_0}\psi(t)\int_\Omega \phi_r^2|\Delta u|^2dxdt + \frac{C}{\epsilon r^4}\|u\|^2_{L^2(Q_{4r})},
\end{aligned}
$$

and

$$
II \le \frac{C}{r^2}\int_{t_0-4r^2}^{t_0-r^2}\int_\Omega \phi_r^2|\nabla\times u|^2dxdt \le \frac{C}{r^2}\|\nabla u\|^2_{L^2(Q_{2r})} \le \frac{C}{r^4}\|u\|^2_{L^2(Q_{4r})}.
$$

Therefore (2.14) reduces to the inequality

$$\frac{1}{2}\int_\Omega \phi_r^2 |\nabla \times u(t_0)|^2 dx + \int_0^{t_0} \psi(t) \int_\Omega \phi_r^2 |\Delta u|^2 dx dt \tag{2.17}$$

$$\leq \epsilon \int_0^{t_0} \psi(t) \int_\Omega \phi_r^2 |\Delta u|^2 dx dt + \frac{C_\epsilon}{r^4} \|u\|^2_{L^2(Q_{4r})}. \tag{2.18}$$

Taking $\epsilon = 1/2$ we have

$$\int_\Omega \phi_r^2 |\nabla \times u(t_0)|^2 dx + \int_0^{t_0} \psi(t) \int_\Omega \phi_r^2 |\Delta u|^2 dx dt \frac{C_\epsilon}{r^4} \|u\|^2_{L^2(Q_{4r})}. \tag{2.19}$$

This implies that

$$\|\nabla \times u(t_0)\|^2_{L^2(B_r)} + \|\Delta u\|^2_{L^2(Q_r)} \leq \frac{C_\epsilon}{r^4} \|u\|^2_{L^2(Q_{4r})}.$$

$\square$

Take inner product by $\nabla \times (\phi_r v_t)$ to the equation (2.1) and integrate over Ω. By the orthogonality of ∇ and $\nabla\times$, we have the identity

$$\begin{aligned} 0 &= \int_\Omega (u_t - \Delta u + \nabla p) \cdot \nabla \times (\phi_r v_t) dx \\ &= \int_\Omega u_t \cdot \nabla \times (\phi_r v_t) dx - \int_\Omega u \cdot \Delta[\nabla \times (\phi_r v_t)] dx. \end{aligned} \tag{2.20}$$

Since $\nabla \times (\phi_r v_t) = \nabla \phi_r \times v_t + \phi_r \nabla \times v_t$ and $\Delta v_t = -\phi_r \nabla \times u_t$,

$$\Delta[\nabla \times (\phi_r v_t)] = \nabla \times [(\Delta \phi_r) v_t + 2(\nabla \phi_r) \cdot \nabla v_t - \phi_r^2 \nabla \times u_t]$$

Hence (2.20) reduces to

$$\begin{aligned} 0 &= \int_\Omega u_t \cdot \nabla \times (\phi_r v_t) dx + \int_\Omega u \cdot \nabla \times [\phi_r^2 \nabla \times u_t] dx \\ &\quad - \int_\Omega u \cdot \nabla \times [(\Delta \phi_r) v_t + 2(\nabla \phi_r) \cdot \nabla v_t] dx. \end{aligned} \tag{2.21}$$

By the definition of v note that

$$\int_\Omega u_t \cdot \nabla \times (\phi_r v_t) dx = \int_{\mathbb{R}^n} |\nabla v_t|^2 dx.$$

Note also that

$$\begin{aligned} &\int_\Omega u \cdot \nabla \times [\phi_r^2 \nabla \times u_t] dx \\ &= \int_\Omega \phi_r^2 \nabla \times u \cdot \nabla \times u_t dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \phi_r^2 |\nabla \times u|^2 dx \end{aligned}$$

and

$$\int_\Omega u \cdot \nabla \times [(\Delta\phi_r) v_t + 2(\nabla\phi_r)\cdot\nabla v_t] dx = \int_\Omega \nabla \times u \cdot [(\Delta\phi_r) v_t + 2(\nabla\phi_r)\cdot\nabla v_t] dx$$

Hence (2.21) reduces to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \phi_r^2 |\nabla \times u|^2 dx + \int_\Omega |\nabla v_t|^2 dx = \int_\Omega \nabla \times u \cdot [(\Delta\phi_r) v_t + 2(\nabla\phi_r)\cdot\nabla v_t] dx. \tag{2.22}$$

Now multiply ψ_r to (2.22) and integrate over $(0, t_0)$, then we have the identity

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \phi_r^2 |\nabla \times u(t_0)|^2 dx + \int_0^{t_0} \psi(t) \int_{\mathbb{R}^n} |\nabla v_t|^2 dx dt \\
&= \int_0^{t_0} \psi(t) \int_\Omega \nabla \times u \cdot [(\Delta \phi_r) v_t + 2(\nabla \phi_r) \cdot \nabla v_t] dx dt \\
&\quad + \frac{1}{2}\int_0^{t_0} \psi'(t) \int_\Omega \phi_r^2 |\nabla \times u|^2 dx dt \\
&= I + II.
\end{aligned}
\tag{2.23}
$$

Recall the previous lemma and its corollary that

$$\|\nabla u\|_{L^2(Q_r)} \leq Cr^{-1}\|u\|_{L^2(Q_{2r})}.$$

By the same reasoning for the estimate of v and w of (**??**)-(2.24), we have

$$\|w_t\|_{L^2} \leq C\|\nabla w_t\|_{L^{2n/(n+2)}} \leq C\|u_t\|_{L^2(B_{2r})}, \tag{2.24}$$

$$\|\nabla w_t\|_{L^2} \leq \frac{1}{r}\|u_t\|_{L^2(B_{2r})}. \tag{2.25}$$

$$\|\nabla^2 v_t\|_{L^2} \leq C\|\phi_r \nabla \times u_t\|_{L^2}, \tag{2.26}$$

$$\|v_t\|_{L^{2n/(n-2)}} \leq C\|\nabla v_t\|_{L^2} \leq C\|u_t\|_{L^2(B_{2r})}. \tag{2.27}$$

Apply Hölder inequality to the right hand side of I-IV then we have

$$
\begin{aligned}
I &\leq \int_0^{t_0} \psi(t) \int_{B_{2r}} |\nabla u|[\frac{1}{r^2}|v_t| + \frac{1}{r}|\nabla v_t|] dx dt \\
&\leq \int_0^{t_0} \psi(t) \|\nabla u\|_{L^2(B_{2r})}[\frac{1}{r}\|v_t\|_{L^{2n/(n-2)}} + \frac{1}{r}\|\nabla v_t\|_{L^2}] dt \\
&\leq \int_0^{t_0} \psi(t) \|\nabla u\|_{L^2(B_{2r})}[\frac{1}{r}\|\nabla v_t\|_{L^2}] dt \\
&\leq \epsilon \int_0^{t_0} \psi(t) \int_{\mathbb{R}^n} |\nabla v_t|^2 dx dt + \frac{C}{\epsilon r^2}\|\nabla u\|^2_{L^2(Q_{2r})} \\
&\leq \epsilon \int_0^{t_0} \psi(t) \int_{\mathbb{R}^n} |\nabla v_t|^2 dx dt + \frac{C}{\epsilon r^4}\|u\|^2_{L^2(Q_{4r})},
\end{aligned}
$$

$$
II \leq \frac{C}{r^2} \int_{t_0-4r^2}^{t_0-r^2} \int_{B_{2r}} |\nabla u|^2 dx \leq \frac{C}{r^2}\|\nabla u\|^2_{L^2(Q_{2r})} \leq \frac{C}{r^4}\|u\|^2_{L^2(Q_{4r})}.
$$

Therefore (2.23) reduces to the inequality

$$\frac{1}{2}\int_\Omega \phi_r^2|\nabla\times u(t_0)|^2dx+\int_0^{t_0}\psi(t)\int_{\mathbb{R}^n}|\nabla v_t|^2dxdt$$
$$\leq \epsilon\int_0^{t_0}\psi(t)\int_{\mathbb{R}^n}|\nabla v_t|^2dxdt+\frac{C_\epsilon}{r^4}\|u\|^2_{L^2(Q_{4r})}. \qquad (2.28)$$

Take $\epsilon = 1/2$, then we have

$$\int_\Omega \phi_r^2|\nabla\times u(t_0)|^2dx+\int_0^{t_0}\psi(t)\int_{\mathbb{R}^n}|\nabla v_t|^2dxdt\leq\frac{C}{r^4}\|u\|^2_{L^2(Q_{4r})}. \qquad (2.29)$$

Contents

§ 3. L^2 solvability of Dirichlet and Neumann problems for the stationary Stokes equations.

In this section, we prove the existence and uniqueness of solution of the stationary Stokes equations for the general Lipschitz domain. After Verchota found the way the existence and uniqueness theorem for the Laplace equations, there have been considerable efforts to extend the solvability theorems to more general systems incuding the Stokes equations, Elastostatics and Maxwell equations. The main ingredient of the argument by Verchota is Rellich identity which provide the closed rafe theorem for the layer potential operators. The same idea to the Stokes systems were applied by Faber-Kerig-Verchota in 1988. We present their proof here with main modification.

First we note the L^2 bounded property of Cauchy integral operator can be extend to systems. The argument does not have any maximum principle and differentiate single equation and systems. But, unlikely C^1 domain, the integral operators like double layer is not compact and hence we can not apply Fredhorem theory directly. Verchota was akle to replace the compactness by Rellich identity in the Lipschitz domain case.

The main results are as follows: Let $D \subset \mathbf{R}^n, n \geq 3$ be a bounded lipschitz domain with connected boundary $S = \partial D$. Suppose

$$g \in L^2_N(S) = \{f \in L^2(S) \mid \int g(Q)N(Q)dQ = 0\}.$$

Then the Difichlet problem

$$(D)\begin{cases} \Delta u - \nabla p = 0, \quad \nabla \cdot u = 0 \quad \text{in} \quad D \\ u = g \quad \text{a.e in } S \text{ in the sense of nontangential convergence} \\ ||u^*||_{L^2(S)} < \infty \end{cases}$$

has a unique solution, where $u^*(p) = \sup_{Q \in \Gamma(P)} |u(Q)|$ is the nontangential maximal function. Moreover if $g \in L^2_1$, i.e., it has first derivative in $L^2(S)$, the solution verified the estimate

$$||(\nabla u)^*||_{L^2(S)} + ||(p)^*||_{L^2(S)} \leq C||g||_{L^2_1(S)}.$$

Similar result holds for the exterior domain $D_{=}\mathbf{R}^n \setminus \bar{D}$. The solution, in the case when $g \in L^2_N(S)$ can be represented in terms of double layer potential and when $g \in L^2_1(S)$ is terms of single layer potential. We also prove the unique existence of solutions to Neumann problem corresponding to connormal derivative $\frac{\partial u}{\partial \nu}$:

$$(N)\begin{cases} \Delta u - \nabla p = 0, \quad \nabla \cdot u = 0 \quad \text{in} \quad D \\ \frac{\partial u}{\partial \nu} = g \quad \text{a.e. in } S \text{ in the sense of nontangential convergence} \\ ||u^*||_{L^2} + ||(\nabla u)^*||_{L^2} + ||p^*||_{L^2} < \infty. \end{cases}$$

We show that if g verified $\int_S g = 0$, there exists unique (u, p) solving (N).
Notations: Capital X, Y, Z denote points in $D_+(= D)$ on D_-, while P, Q denote points on S.

Definition

We say that $u(X)$ converges nontangentially a.e. to $f(Q)$ if for any regular family of cones $\{T\}$, we have $\lim_{X\in\Gamma(Q)\to Q} u(X) = f(Q)$ a.e. $Q \in S$.

Contents

§ 4. Rellich estimate.

Given $f \in L^2(S)$, we define the single layer potential

$$\begin{cases} u(X) = E * f(X) = \mathcal{D}f(X) \\ p(X) = F * f(X), \end{cases}$$

(see (3.3)) then we have

$$\begin{cases} \nabla \cdot u(X) = 0, \qquad X \in \mathbf{R}^n \setminus S \\ \Delta u - \nabla p = 0 \quad \text{in} \quad D_+ \text{ or } D_- \\ |X|^{-1}|u(X)| + |\nabla u(X)| + |p(X)| + |X||\nabla p(X)| = O(|X|^{-n}) \\ \Delta p = 0 \quad \text{in} \quad \mathbf{R}^n \setminus S. \end{cases}$$

The L^2- boundedness theorem for double layer potential operator and maximal function estimates of Calderon-Zygmund singular integral operator yield

$$||(\nabla u)^*||_{L^2(S)} + ||p^*||_{L^2(S)} + ||u^*||_{L^2(S)} \leq c||f||_{L^2(S)},$$

where c depends only the Lipschitz character of D. Define the tangential gradient of u on S by

$$\nabla_t u(Q) = \nabla u(Q) - \frac{\partial u}{\partial N}(Q)N(Q), ((\nabla_t u)_{ij} = u^i_{x_j} - u^i_{x_j}N^j)$$

then by the trace formula in section 3

$$\nabla_t u_+ = \nabla_t u_-.$$

hence we obviously notate " $+$ " for interior and " $-$ " for exterior. Our goal is to show the existence of a constant $c > 0$ independent of $f \in L^2(S)$ such that

$$c^-||(\frac{1}{2}I + K)f||_{L^2(S)} \leq ||(-\frac{1}{2}I + K)f||_{L^2(S)} \leq c||(\frac{1}{2}I + K)f||_{L^2(S)} \quad (4.1)$$

Define a vector field $\alpha \in C_0^\infty(\mathbf{R}^n)$ such that on $S, \alpha \cdot N(Q) \geq c > 0$ for some C independent of Q.

From integration by parts

$$\begin{cases} \int_S \frac{\partial u_+}{\partial \nu} dQ = 0 \\ \int_S \frac{\partial u_-}{\partial \nu} dQ = -\int_S f dQ \\ \int_S u \cdot \frac{\partial u_\pm}{\partial \nu} dQ = \pm \int_{D_\pm} |\nabla u|^2 dX \end{cases} \tag{4.2}$$

$$\begin{aligned} \int_S (N\cdot\alpha) p^2 dQ &= 2\int_S \alpha^i N^j D_j u^i p - \alpha^i D_j u^i N^k D_k u^j \\ &\quad + \alpha^j D_k u^i N^j D_k u^j dQ \\ &\quad \pm \int_{D_\pm} D_i \alpha^i p^2 - 2D_j \alpha^i D_j u^i p + 2D_k \alpha^i D_j u^i D_k u^j \\ &\quad - 2D_j \alpha^i D_k u^i D_k u^i dX \end{aligned} \tag{4.3}$$

$$\begin{aligned} \int_S N\cdot\alpha |\nabla u|^2 dQ &= 2\int_S \alpha^i D_i u^j (N^k D_k u^j - N^j p) dQ \\ &\quad \pm \int_{D_\pm} D_i \alpha^i |\nabla u|^2 - 2D_j \alpha^i D_i u^k D_j u^k + 2D_k \alpha^i D_i u^k p dx. \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
\int_S N \cdot \alpha |\nabla u|^2 dQ \quad = \quad & 2 \int_S \alpha^i D_j u^k (N^i D_j u^k - N^j D_i u^k) \\
& + \alpha^i p (N^k D_i u^k - N^i D_k u^k) dQ \\
& \pm \int_{D_\pm} D_i \alpha^i |\nabla u|^2 + 2 D_j \alpha^i D_i u^k D_j u^k - 2 D_k \alpha^i D_i u^k p\alpha
\end{aligned} \quad (4
$$

The integral identities (4.4),(4.5),(4.6) are Rellich -Necas identities. From definition of conormal derivative, we have:

Lemma 5.1

$$
||p_\pm||_{L^2(S)} \leq ||\frac{\partial u_\pm}{\partial \nu}||_{L^2(S)} + ||\nabla u_\pm||_{L^2(S)} \leq ||p_\pm||_{L^2(S)} + 2||\nabla u_\pm||_{L^2(S)}
$$

Lemma 5.2

$$
\begin{aligned}
&(i)\ ||\nabla u||^2_{L^2(D_+)} \leq c||\nabla_t u||_{L^2(S)}||\tfrac{\partial u_+}{\partial \nu}||_{L^2(S)} \\
&(ii)\ ||\nabla u||^2_{L^2(D_-)} \leq c||\tfrac{\partial u_-}{\partial \nu}||_{L^2(S)} \left(||\nabla_t u||_{L^2(S)} + |\textstyle\int_S u dQ|\right)
\end{aligned}
$$

Lemma 5.2 follows from (5,10) and Poincare inequality.

Lemma 5.3

$$(i)\ ||\nabla u_+||_{L^2(S)} \leq c||\frac{\partial u_+}{\partial \nu}||_{L^2(S)}$$
$$(ii)\ ||\nabla u_-||_{L^2(S)} \leq c\left(||\frac{\partial u_-}{\partial \nu}||_{L^2(S)} + |\int_S udQ|\right)$$

Proof

In (5,13),

$$\begin{aligned}
\int_S N\cdot\alpha|\nabla u|^2 dQ &= 2\int_S \alpha\cdot N|\nabla u|^2\frac{\partial u^k}{\partial N} \\
&\quad +(\alpha\cdot\nabla u^k pN^k - \alpha\cdot pD_k u^k)dQ \\
&\quad \pm\int_{D_\pm} D_i\alpha^i|\nabla u|^2 + 2D_j\alpha^i D_i u^k D_j u^k - 2D_k\alpha^i D_i u^k pdX \\
&= 2\int_S(\alpha\cdot n|\nabla u|^2 - \alpha\cdot\nabla^k\frac{\partial u_\pm^k}{\partial\nu})dQ \\
&\quad \pm\int_{D_\pm} D_i\alpha^i|\nabla u|^2 + 2D_j\alpha^i D_i u^k D_j u^k - 2D_k\alpha^i D_i u^k pdX.
\end{aligned}$$

Thus

$$-\int_S N\cdot\alpha|\nabla u|^2 dQ = -2\int_S \alpha\cdot N|\nabla u|^2\frac{\partial u^k}{\partial N}$$
$$\mp\int_{D_\pm} D_i\alpha^i|\nabla u|^2 + 2D_j\alpha^i D_i u^k D_j u^k - 2D_k\alpha^i D_i u^k p dX.$$

Let

$$A = ||\nabla u_-||_{L^2(S)}, \quad B = ||\frac{\partial u_-}{\partial \nu}||_{L^2(S)}, \quad D = |\int_S u dQ|.$$

Using Schwirz inequality, We obtain that

$$A^4 \le c\left(A^2B^2 + ||\nabla u||^4_{L^2(D_-)} + ||\nabla u||^2_{L^2(D_-)}||p||^2_{D_-\cap supp\alpha}\right).$$

Then norm on p may be controlled lay the late side of Lemma 4.1 by the estimate $||p^*||_{L^2(S)} \le c||p||_{L^2(S)}$. then, by Lemma 4.1 and (ii) of Lemma 4.2, we have

$$A^4 \le c(A^2B^2 + B^2D^2 + AB^3 + A^3B + B^3D + A^2BD)$$

Now, by considering the two case $A \le B + D$ and $B + D \le A$, (ii) follows. Part(i) follows similarly by (i) of Lemma4.2. $\square$

Lemma 5.4

(i) $||\nabla u_+||^4_{L^2(S)} \leq c\left(||\frac{\partial u_+}{\partial \nu}||^3_{L^2(S)}||\nabla_t u||_{L^2(S)}\right)$

(ii) $||\nabla u_-||^4_{L^2(S)} \leq c||\frac{\partial u_-}{\partial \nu}||_{L^2(S)} + (|\int_S udQ|)^3\left(||\nabla_t u||_{L^2(S)} + |\int_S udQ|\right)$.

Proof

We let A, B, D be as in the proof of Lemma 4.3 and $E = ||\nabla_t u||_{L^2(S)}$.
Then (4.3) and (4.6) yield

$$A^4 \leq c\left(A^2E^2 + ||p_-||^2_{L^2(S)}E^2 + ||\nabla u||^4_{L^2(D_-)} + ||\nabla u||^2_{L^2(D_-)}||p||^2_{L^2(D_-\cap supp\alpha)}\right).$$

Applying Lemma4.1, Lemma 4.2 and $||p^*||_{L^2(S)} \leq c||p||_{L^2(S)}$,

$$A^4 \leq c\left(A^2E^2 + B^2E^2 + B^2(E+D)^2 + B^3(E+D) + A^2B(E+D)\right).$$

By Lemma4.3, $E \leq A \leq c(B+D)$ and (ii) follows. (i) follows similarly.
Now we state two Lemmas without proof.(see[FKV]). □

Lemma 5.5

Given $\eta > 0$ small enough

$$||p||_{L^2(D_+)} \leq c\left(\eta||p_+||_{L^2(S)} + |p(0)|\right) + c_\eta||\nabla u||_{L^2(D_+)}$$
$$||P||_{L^2(D_-)} \leq c\eta||p_-||_{L^2(S)} + c_\eta||\nabla u||_{L^2(D_-)}$$

where c depends only on S, c_η depends only on η and $\partial\omega$

Lemma 5.6

$$||p_+||_{L^2(S)} \leq c\left(\nabla u_+||_{L^2(S)} + |p(0)|\right)$$
$$||p_-||_{L^2(S)} \leq c(||\nabla u_-||_{L^2(S)} + |\int_{\partial\Omega} udQ|)$$

where c depends only on S.

Now we can prove $||\frac{\partial u_\pm}{\partial \nu}||_{L^2(S)}$ are essentially equivalent to $||\nabla_t u||_{L^2(S)}$,

Lemma 5.7

$$||\frac{\partial u_+}{\partial \nu}||_{L^2(S)} \leq c\left(\nabla_t u_+||_{L^2(S)} + |p(0)|\right)$$
$$||\frac{\partial u_-}{\partial \nu}||_{L^2(S)} \leq c\left(||\nabla_t u_+||_{L^2(S)} + |\int_S udQ|\right)$$

Proof

From Lemma4.1, Lemma 4.6 and Lemma 4.4, we have

$$||\frac{\partial u_+}{\partial \nu}||^4_{L^2(S)} \leq c\left(||\frac{\partial u_+}{\partial \nu}||^3_{L^2(S)}||\nabla_t u||_{L^2(S)} + |p(0)|^4\right)$$

and the first claim immediately follows. The second claim can be proved exactly same way. The continuity of tangential derivative yield:

Theorem

$$||\frac{\partial u_+}{\partial \nu}||_{L^2(S)} \leq c\left(||\frac{\partial u_-}{\partial \nu}||_{L^2(S)} + |\int_S udQ| + |p(0)|\right)$$
$$||\frac{\partial u_-}{\partial \nu}||_{L^2(S)} \leq c\left(||\frac{\partial u_+}{\partial \nu}||_{L^2(S)} + |\int_S udQ|\right)$$

Contents

§5. The Neumann problem and functional analysis.

From the trace formula, we have

$$\frac{\partial u_\pm}{\partial \nu}(Q) = \pm\frac{1}{2}f(Q) + Kf(Q),$$

where the kernel of K is defined in Theorem 3.3.

Definition

$$L_0^2(S) = \{f \in L^2(S) \mid \int_S f dQ = 0\}$$

and so it is the subspace of $L^2(S)$ of codimension n. Define $L_N^2(S)$ by the codimension 1 set so that

$$L_N^2(S) = \{f \in L^2(S) \mid \int_S f \cdot N dQ = 0\}.$$

Lemma 5.2

$$(i)\tfrac{1}{2}I + K : L_0^2(S) \to L_0^2(S) \quad \text{is one-to-one}$$
$$(i) -\tfrac{1}{2}I + K : L_N^2(S) \to L^2(S) \quad \text{is one-to-one}$$

Lemma 5.3

$$(i)\tfrac{1}{2}I + K : L_0^2(S) \to L_0^2(S) \quad \text{has closed range}$$
$$(ii) -\tfrac{1}{2}I + K : L_N^2(S) \to L^2(S) \quad \text{has closed range}$$

Theorem 5.4

$(i)\frac{1}{2}I + K$ is invertible from $L_0^2(S)$ to $L_0^2(S)$.
$ii) -\frac{1}{2}I + K$ is invertible from $L_N^2(S)$ to a subspace of $L^2(S)$ of codimen

Now, we can solve Neumann problem in D_+ or D_-.

$$(N_\pm)\begin{cases} \Delta u - \nabla p = 0, \quad \nabla \cdot u = 0 \quad \text{in} \quad D_\pm \\ \frac{\partial u}{\partial \nu} = g \quad \text{a.e. on } S \text{ in the sense of nontangential convergence} \\ ||(\nabla u)^*||_{L^2(S)} + ||u^*||_{L^2(S)} + ||p^*||_{L^2(S)} < \infty. \end{cases}$$

Theorem 5.5

(i) If $g_+ \in L^2_0(S)$, there exists, up to constant, unique u_+ and p_+ satisfying (N_+) in D_+.
(ii) If $g_- \in R(-\frac{1}{2}I + K)$, there exists, up to constant, unique u_- and p_- satisfying (N_-) in D_-.
(iii) There is c depending only on the Lipschitz character of S satisfying

$$||(u_\pm)^*||_{L^2(S)} + ||(\nabla u_\pm)^*||_{L^2(S)} + ||(P_\pm)^*||_{L^2(S)} \leq c||g_\pm||_{L^2}$$

Contents

§6. Dirichlet problem.

The solution to Dirichlet problem (D) is defined by the double layer potential of a density function f such that

$$u^j(X) = \int_S \frac{\partial E^{ij}}{\partial N(Q)}(Q-X)f^j(Q) - F^j(Q-X)N^j(Q)\cdot f^j(Q)dQ.$$

Then, again by the trace theorem, we have

$$u_\pm(Q) = \mp\frac{1}{2}f(Q) + K^* f(Q) \quad \text{for a.e. } Q \in S,$$

where K^* is the adjoint operator of K in the Neumann problem. The $L^2(S)$ estimate of nontangential maximal function $||u^*||_{L^2(S)} \leq c||f||_{L^2(S)}$ is the same as the single layer potential. We let $\mathcal{D}$ denote the orthogonal complement of $ker(\frac{1}{2}I + K)$ in $L^2(S)$.

Then by Theorem 6.4

$$\frac{1}{2}I + K^* : L_0^2 \to \mathcal{D} \quad \text{and} \quad -\frac{1}{2}I + K^* : R \to L_N^2$$

are invertible, where $R = R(-\frac{1}{2}I + K)$. Then,
$T_+ = \frac{1}{2}I + K, T_- = -\frac{1}{2}I + K,$

$$\begin{cases} u_+(X) = T((-\frac{1}{2}I + K^*)^{-1}g)(X) \\ u_-(X) = T((\frac{1}{2}I + K^*)g)(X) \end{cases}$$

are solution to Dirichlet problem in $\mathcal{D}_\pm$ respectively. The following theorem states the existence of Green's tensor for Dirichlet problem:

Theorem

Given $X \in D_+(D_-)$ there exists a matrix Green's function, G^X_{jk}, $1 \leq j, k \leq n$ and harmonic function P^X_j defined in $D_+(D_-)$ so that
(i), $G^X_{jk}(Q) = 0$ for a.e. $Q \in S$ in the sense of nontangential convergence.
(ii), $\Delta_Y(G^X_{jk}(Y) - E^{jk}(Y - X)) = \frac{\partial}{\partial Y_k}(P^X_j(Y) = 0$
$\frac{\partial}{\partial Y_k}(G^X_{jk}(Y) - E^{jk}(Y - X)) = 0 \quad \text{in} \quad D_+(D_-)$
(iii), $\frac{\partial}{\partial N(Q)}(G^X_{jk}(Q) - E^{jk}(Q - X)) - N^k(Q)P^X_j(Q)$ exists for a.e. $Q \in S$ in the sense of nontangential convergence.
(iv), $||(\nabla G^X_{jk})^*||_{L^2(S)} + ||(P^X_j)^*||_{L^2(S)} \leq c < \infty$,
where the nontangential cone is taken with respect to a family of cone excluding X.

With the Green's tensor, we can state the unique existence theorem for Dirichlet problem.

Theorem

(i) If $g \in L^2_N(S)$, there is a unique u and unique p satisfying the Dirichlet problem (D) in D_+
(ii) If $g \in L^2(S)$, there is a unique u and unique p satisfying (D) in D_-
(iii) $||(u_\pm)^*||_{L^2(S)} \leq c||g||_{L^2(S)}$.
(iv) If $g \in L^2_1(S) \cap L^2_N$ for D_+ and $g \in L^2_1(S)$ for D_-, u satisfies

$$||(u_\pm)^*||_{L^2(S)} + ||(\nabla u_\pm)^*||_{L^2(S)} + ||(p_\pm)^*||_{L^2(S)} \leq c||g||_{L^2_1(S)}.$$

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§7. L^∞- estimate Via the estimate of the Poisson kernel.

In this section we study the L^p-Dirichlet problem. Optimal estimates are obtained when the dimension $n = 3$. In the case of $n \geq 4$, we establish a weak estimate of solutions for certain range of p. We consider the Dirichlet problem for the Stokes system

$$\begin{cases} -\Delta u + \nabla p = 0 \quad \text{in} \quad D \\ \text{div } u = 0 \quad \text{in} \quad D \\ u = g \quad \text{on} \quad S. \end{cases}$$

In section 2, it is shown that there is $\epsilon > 0$ such that if $g \in L^q_\sigma(S), 2 - \epsilon < q < 2 + \epsilon$, then there exists a unique u and a unique q(modulo constant) satisfying the-Dirichlet problem and $u^* \in L^q(S)$, where u^* is the nontangetial maximal function of u. Our main results are as follows:

Therem 5.1

Let D be a bounded Lipschitz domain in $\mathbf{R}^3$, with connected boundary, Given $g \in L^\infty_\sigma(S)$, there exists a unique solution u and p(modulo constant) to the Stokes Dirichlet problem. In fact

$$||u||_{L^\infty(S)} \leq ||g||_{L^\infty(S)}.$$

Moreover, if $g \in C^\alpha(S), 0 < \alpha < \alpha_0$, then $u \in C^\alpha(D)$ and

$$||u||_{C^\alpha(D)} + \sup_{X \in D} \delta^{1-\alpha}(X)|\nabla u(X)| \leq c||g||_{C^\alpha(S)},$$

where $\delta(X) = \text{dist}(X, S)$.

Therem 5.2

Let D be a bounded Lipschitz domain in $\mathbf{R}^3$ with connected boundary. Given $g \in L^q_\sigma(S), 2 \le q \le \infty$, there exists a unique solution (u, P) (P is unique up to constant) to Dirichlet problem and $u^* \in L^q(S)$. Moreover we have

$$||U^*||_{L^q(S)} \le c||g||_{L^q(S)}.$$

Theorem 5.2 follows from the L^∞- estimate and L^2- solvability and the real interpolation. Moreover the nontagential maximal function estimate of Theorem 5.2 implies

$$||u||_{L^q_{1/q}(\mathcal{D})} \le c||g||_{L^q(S)}.$$

Therem 5.3

Let D be a bounded Lipschitz domain in $\mathbf{R}^n, n \geq 4$, with connected boundary and $2 \leq q \leq \frac{2(n-1)(n-2)}{n(n-3)}$. Then for any $g \in L^q_\sigma(S)$ the unique $L^2(S)$ solution u, satisfies

$$||u||_{L^{q_1}(D)} \leq c||g||_{L^q(S)}, q_1 = \frac{nq}{n-1}$$

As a consequence of Theorem 5.1, we state without proof the following theorem for the homogenous Difichlet problem

$$(0.5) \begin{cases} -\Delta u + \nabla p = f \quad \text{in} \quad D \\ \text{div } u = 0 \quad \text{in} \quad D \\ u = \quad \text{on} \quad S. \end{cases}$$

Theorem 5.4

Let D be a bounded Lipschitz domain in $\mathbf{R}^3$ and $\frac{3}{2} \leq q \leq 3$. Given $f \in w^q_{-1}(D)$, there exists a unique u and a unique p(up to the constant) satisfying (0.5) and $u \in w^q_1(D)$. Moreover

$$||u||_{w^q_1(D)} + ||p||_{L^q(D)} \leq c||f||_{w^q_{-1}}(D)$$

The proof of Theorem 5.4 may be carried out using Theorem 5.1, L^2-estimate in [FKV] and the argument by Jerison and Kenig.

The key step is to show that the Green's function has certain decay when $|X - Y|$ is large in composition with $\delta(X)$. To do this, we use Rellich identity, Caccioppoli inequality and the Dirichlet problem is solvable in L^q for some $q < 2$. In the case of $n = 3$, the estimates on the Green's function yields the desired L^∞-estimates.

But when $n \geq 4$, it only gives some weak estimates. We remark that in [Pipher-Verchota], it is shown that when the dimension $n \geq 4$, the L^p Dirichlet problem for the biharmonic equation in general is not solvable in Lipschitz domain for $q > 2$ larage enough $(q > 6, n = 4; q > 4, n = 5)$. The counter example they found in given by $D = O \times \mathbf{R}_+$ where O is an open subset of the unit sphere in $\mathbf{R}^n, n \geq 4$. It is noted that in [DK2] such domain fail to produce counter -examples for the system of elastostatics. We let $(G(X,Y), P(X,Y))$ be the matrix Green's tensor defined by

$$G(X,Y) = E(X-Y) - v^X(Y)$$
$$P(X,Y) = F(X-Y) - P^X(Y),$$

where $(v^X(Y), P^X(Y))$ is the matrix valued solution to the Dirichlet problem of the stationary Stokes equations with boundary data

$$v^X(Q) = E(X-Q) \quad \text{for} \quad Q \in S$$

Lemma

Let $X \in D$ and $P \in S$ and $r = |X - P| \leq 2dist(X, S)$. Then

$$\int_{S \setminus \Delta(P,r)} |G(X,\cdot))^*(Q)|^q dq \leq cr^{(n-1)-(n-2)q} \quad \text{if} \quad 2 - \epsilon_0 \leq q \leq 2 + \epsilon_0.$$

Proof

We apply the non-tangential maximal function estimate of u^* to get

$$\begin{aligned} \int_{S \setminus \Delta(P,r)} & |G(X,\cdot))^*(Q)|^q dQ \\ & \leq c \int_{D \cap \partial B(P,r)} |G(X,Q)|^q dQ \\ & \leq c[\int_{D \cap \partial B(P,r)} |E(X-Q)|^q + |v^X(Q)|^q dQ \end{aligned}$$

Clearly,

$$\int_{D \cap \partial B(P,4r)} |E(X,Q)|^q dQ \leq cr^{(n-1)-(n-2)q}$$

Also, note that

$$
\begin{aligned}
\int_{D\cap\partial B(P,4r)} |v^X(Q)|^q dQ &\leq cr^{(n-1)(1-\frac{q}{2+\epsilon_0})} \left(\int_{D\cap\partial B(P,4r)} |v^X(Q)|^{2+\epsilon_0} dQ \right)^{\frac{q}{2+\epsilon_o}} \\
&\leq cr^{(n-1)(1-\frac{q}{2+\epsilon_0})} \left(\int_S |(v^X)^*(Q)|^{2+\epsilon_0} dQ \right)^{\frac{q}{2+\epsilon_o}} \\
&\leq cr^{(n-1)(1-\frac{q}{2+\epsilon_0})} \left(\int_S |E(X-Q)|^{2+\epsilon_0} dQ \right)^{\frac{q}{2+\epsilon_o}} \\
&\leq cr^{(n-1)(1-\frac{q}{2+\epsilon_0})} \left(\int_{cr}^{\infty} \frac{t^{n-2}}{t^{(n-2)(2+\epsilon_0)}} dt \right)^{\frac{q}{2+\epsilon_o}} \\
&\leq cr^{(n-1)-(n-2)q}.
\end{aligned}
$$

Lemma

Let $X_0 \in \bar{D}, R > 0$ be small, and $D(X_0, R) = D \cap B(X_0, R)$. Assume (u, p) is a solution to the Stokes system in $D(X_0, 3R)$ and $u = 0$ on $B(X_0, 3R) \cap S$. Then

$$\int_{D(X_0,R)} |\nabla u|^2 dX \leq \frac{c}{R^2} \int_{D(X_0,2R)} |u|^2 dX.$$

Define the conormal derivative

$$\frac{\partial u}{\partial \nu_Q} = \frac{\partial u}{\partial N(Q)} - pN(Q)$$

The following is the main lemma.

Lemma

There exists $\epsilon_1 > 0$ such that if $X \in D, P \in S$ and $r = |X - P| \leq 2dist(X, S)$ then it follows that

$$\int_{R<|Q-P|<2R} |\frac{\partial G}{\partial \nu_Q}(X, Q) - P^X(X_0)N(Q)|^2 dQ \leq c(\frac{r}{R})^{\epsilon_1} \frac{1}{R^2 r^{n-3}}$$

where $R \geq 3r$ and X_0 is some point in D which does not depend on R.

Proof
Fix $X_0 \in D$ such that X_0 is away from X and $dist(X_0, S) \geq c > 0$. For $\tau \in [1,3]$, let $D(\tau R) = \{Y \in D \mid \frac{R}{\tau} \leq |Y - P| \leq \tau R\}$. It follows from the Lemma ? in [FKV] and rescaling argument that

$$\begin{aligned}\int_{R\leq|Q-P|\leq 2R} & |\frac{\partial G}{\partial \nu_Q}(X,Q) - P^X(X_0)N(Q)|^2 dQ \\ \leq & \ c\int_{\partial D(\tau R)} |\frac{\partial G}{\partial \nu_Q}(X,Q) - P^X(X_0)N(Q)|^2 dQ \\ \leq & \ c\int_{\partial D(\tau R)} \nabla_{tan} G(X,Q)|^2 dQ + cR^{n-1}|P^X(X_R) - P^X(X_0)|\end{aligned}$$

where ∇_{tan} denotes the tangential derivative on S and X_R is a point in $D(R)$ such that $dist(X_R, S) \sim R$. Integrating in $\tau \in [1, \frac{3}{2}]$, we have

$$
\begin{aligned}
\int_{R\le|Q-P|\le 2R} & \left|\frac{\partial G}{\partial \nu_Q}(X,Q) - P^X(X_0)N(Q)\right|^2 dQ \\
& \le \frac{c}{R}\int_{D(\frac{3}{2}R)} |\nabla G(X,Y)|^2 dY + cR^{n-1}|P^X(X_R) - P^X(X_0)| \\
& \le \frac{c}{R^3}\int_{D(2R)} |G(X,Y)|^2 dY + cR^{n-1}|P^X(X_R) - P^X(X_0)| \\
& \le \frac{c}{R^2}\int_{\frac{3}{2}R\le|Q-P|\le 2R} |G(X,Y)|^2 dY + cR^{n-1}|P^X(X_R) - P^X(X_0)
\end{aligned}
$$

Contents

§8. Maximum modulus estimate for the solution of the Stokes equations.

§8.1 Introduction.

A maximum modulus estimate of the nonstationary Stokes equations is presented. In the case of the stationary flow, Maremonti and Russo[1] obtained a quasi maximum principle and Varnhorn[5] showed a maximum modulus theorem for $C^{1,\alpha}$ domain:

$$\max_{x\in\Omega}|u| \le C(\Omega)\max_{x\in\partial\Omega}|u|,$$

where u is a solution to the stationary Stokes equations in domain Ω. We also note that Maz'ja and Rossmann[2] considered the maximum modulus estimate for the stationary Navier-Stokes equations in polygonal domain. In a canonical domain like ball, Kratz[5] found the best constant $C(\Omega)$ such that

$$\max_{x\in B_1}|u| \le \frac{1}{2}n(n+1)\max_{x\in\partial B_1}|u|,$$

where B_1 is the unit ball in $\mathbf{R}^n$. For a more general domain like Lipschitz in $\mathbf{R}^3$, Shen[3] obtained a maximum modulus estimate and the higher dimension problem is still unresolved.
The maximum modulus estimate of the nonstationary problem is heavily entangled with the structural form of Poisson kernel and the solvability of the boundary value problem is essential. As a classical result, Solonnikov[1] solved the initial-boundary problem in C^2 domain for the isotropic Sobolev spaces and later he[2] extended the solvability to the anisotropic Sobolev spaces. The L^2 solvability for the Lipschitz domain was obtained by Shen[**?**] for any dimension and Choe and Kozono[3] considered the case for the mixed norm potential spaces.

To be more specific, we state the nonstationary Stokes equations:

$$\begin{aligned} & u_t - \nu \Delta u + \nabla p = 0 && \text{in } \Omega \times (0,T), \\ & div\, u = 0 && \text{in } \Omega \times (0,T), \\ & u|_{t=0} = 0 && \text{in } \Omega, \\ & u|_{\partial\Omega \times (0,T)} = g && \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{8.1}$$

where Ω is C^2 bounded connected domain in $\mathbf{R}$ and $0 < T < \infty$ and ν is the viscosity which we assume 1. In addition, we assume the boundary data g satisfies the compatibility condition:

$$\int_{\partial\Omega} g \cdot N d\sigma = 0$$

for almost all t, where N is the outward unit normal vector on the boundary. Since nontrivial initial data can be treated by solving homogeneous boundary value problems, we consider only the initial-boundary value problems with zero initial data.

Contrary to the stationary case, the quasi maximum principle fails, namely, there is an unbounded solution whose boundary data is bounded. Heuristically speaking, at the boundary point where the normal component of boundary data has a jump discontinuity along an $(n-2)$-dimensional surface on the boundary passing to it, the tangential component of the velocity blows up in the neighborhood of it. So we can not expect the quasi maximum modulus theorem like the stationary case.

In this paper, we only consider the case that the space dimension is greater than or equal to 3. Dimension 2 case follows exactly the same path with logarithmic kernels. Denote E for the fundamental solution to Laplace equation and Γ for the fundamental solution to heat equation with unit conductivity. For a given boundary point $y \in \partial\Omega$ $N(y)$ is the outward unit normal vector at y.

We define the $(n-1)$-dimensional convolution

$$\mathbf{S}(f)(x) = \int_{\partial\Omega} E(x-y)f(y)d\sigma(y)$$

for real-value function $f : \mathbf{R} \to \mathbf{R}$ which is just the single layer potential of f on $\partial\Omega$. We need a composite kernel. We define a composite kernel function $\kappa(x,t)$ on $\Omega \times (0,T)$ by

$$\kappa(x,t) = \int_{\partial\Omega} \frac{\partial\Gamma}{\partial N(y)}(x-y,t)E(y)d\sigma(y)$$

and a surface potential $\mathbf{T}$ for f by

$$\mathbf{T}(f)(x,t) = 4\int_0^t \int_{\partial\Omega} \kappa(x-y,t-s)f(y,s)d\sigma(y)ds,$$

for real-value function $f : bfR^n \to \mathbf{R}$. We state our main theorem: For given $x \in \Omega$, $\bar{x}$ is the nearest point of x on $\partial\Omega$ such that $dist(x,\partial\Omega) = |\bar{x}-x|$ and for a vector valued function $v(x)$, we define the normal component and tangential component to the nearest point $\bar{x}$ by

$$v_N(x) = (v(x)\cdot N(\bar{x}))N(\bar{x}) \quad \text{and} \quad v_T(x) = v(x) - (v(x)\cdot N(\bar{x}))N(\bar{x}).$$

Theorem 6

Suppose that the domain Ω is bounded C^2 and u is a solution to (8.1) for bounded boundary data g. The normal component of the velocity u_N is bounded and there is also a constant $C(\Omega)$ such that

$$\max_{(x,t)\in\Omega\times(0,T)} |u_N(x,t)| \le C(\Omega) \max_{(y,t)\in\partial\Omega\times(0,T)} |g(y,t)|.$$

Furthermore, the tangential component of the velocity u satisfies that

$$\begin{aligned}\max_{(x,t)\in\Omega\times(0,T)} &|u_T(x,t) - \nabla\mathbf{S}(g\cdot N)_T(x,t) - \nabla\mathbf{T}(g\cdot N)_T(x,t)| \\ &\le C(\Omega) \max_{(y,t)\in\partial\Omega\times(0,T)} |g(y,t)|\end{aligned}$$

Define the modulus of continuity of f at x by $\omega(f)(r,x) = \sup_{y\in B_r(x)\cap\Omega} |f(y) - f(x)|$ and we say f is Dini-continuous in Ω if

$$||f||_{Dini,\Omega} = \sup_{x\in\Omega} \int_0^{r_0} \omega(f)(r,x)\frac{dr}{r} < \infty$$

for an $r_0 > 0$. From a direct computation, we have $\nabla\mathbf{S}(f)$ and $\nabla\mathbf{T}(f)$ are bounded if f is Dini-continuous on $\partial\Omega$ and we obtain a maximum modulus estimate:

Corollary

Suppose that the domain Ω is bounded C^2 and u is a solution to (8.1). Suppose g is bounded on $\partial\Omega$ and the normal component g_N is Dini-continuous. Then, there is a constant $C(\Omega)$ depending only on Ω such that

$$\max_{(x,t)\in\Omega\times(0,T)} |u(x,t)| \leq C(\Omega)\left(\max_{(y,t)\in\partial\Omega\times(0,T)} |g(y,t)| + ||g_N||_{Dini,\partial\Omega}\right).$$

As a separate interest, we obtain an improved L^2 theory like Lemma 4.1. When the $L^2(\partial\Omega)$ norm of the boundary data is bounded in time, then $||u(\cdot,t)||_{L^2(\Omega)}$ is bounded in time. Consequently, the local boundedness holds too(see Corollary 4.2.).

§8.2 Kernels on half plane.

To study the equation (8.1), we consider the case of $\Omega = bfR^n_+ = \{(x', x_n) \in \mathbf{R}^n, |x' \in \mathbf{R}^{n-1}, 0 < x_n < \infty\}$ and for the notational simplicity we set $D_{x_i} = \frac{\partial}{\partial x_i}$ and double indices means summation up to n. For notation, we denote $x = (x', x_n)$, that is, $x' = (x_1, x_2, \cdots, x_{n-1})$. Indeed, the symbol $'$ means the coordinate up to $n-1$ and ω_n is the surface area of the unit sphere in $\mathbf{R}^n$.
We let Γ be the fundamental solution to the heat equation such that

$$\Gamma(x,t) = \begin{cases} \frac{1}{\sqrt{2\pi t}^n} e^{-\frac{|x|^2}{2t}}, & t > 0 \\ 0, & t \le 0 \end{cases}$$

and H be the Newtonian potential of Γ such that

$$H(x,t) = \int_{\mathbf{R}^n} \Gamma(y,t) E(x-y) dy.$$

The Stokes fundamental matrix (F, γ) for $\mathbf{R}^n, n \geq 3$ is

$$
\begin{aligned}
F_{ij}(x,t) &= \delta_{ij}\Gamma(x,t) + \frac{1}{(n-2)\omega_n} D_{x_i} D_{x_j} H(x,t) \\
\gamma_i &= \frac{\delta(t)}{\omega_n} \frac{x_i}{|x|^n},
\end{aligned}
$$

where $\delta(t)$ is the Dirac delta function and δ_{ij} is the Kronecker delta function. The Green's matrix (G, θ) for the half space $\mathbf{R}^n_+$ is

$$
\begin{aligned}
G_{ij}(x,y,t) &= \delta_{ij}\big(\Gamma(x-y,t) - \Gamma(x-y^*,t)\big) \\
&\quad + 4(1-\delta_{jn}) D_{x_j} \int_0^{x_n} \int_{\mathbf{R}^{n-1}} D_{x_i} E(x-z)\Gamma(z-y^*,t)dz \\
\theta_j(x,y,t) &= (1-\delta_{jn}) \left(\int_{\mathbf{R}^{n-1}} D_{x_i} E(x'-z',x_n)\Gamma(z'-y',y_n,t)dz' \right. \\
&\quad \left. + \int_{\mathbf{R}^{n-1}} E(x'-z',x_n) D_{y_n}\Gamma(z'-y',y_n,t)dz' \right),
\end{aligned}
$$

where we denote $x^* = (x', -x_n)$.

The Poisson kernel (K, π) for the half space is defined by

$$\begin{aligned}
K_{ij}(x'-y', x_n, t) &= \frac{\partial G_{ij}(x,y,t)}{\partial y_n}\Big|_{y_n=0} - \delta_{jn}\theta_i(x,y,t)|_{y_n=0} \\
&= -2\delta_{ij} D_{x_n}\Gamma(x'-y', x_n, t) + 4L_{ij}(x'-y', x_n, t) \\
&\quad -\delta_{jn}\delta(t) D_{x_i} E(x'-y', x_n), \\
\pi_j(x'-y', x_n, t) &= -2\delta(t) D_{x_j} D_{x_n} E(x'-y', x_n) + 4 D_{x_n} D_{x_n} A(x'-y', x_n \\
&\quad +4 D_t D_{x_j} A(x'-y', x_n, t),
\end{aligned}$$

where we defined that

$$\begin{aligned}
L_{ij}(x,t) &= D_{x_j}\int_0^{x_n}\int_{R^n} D_{z_n}\Gamma(z,t) D_{x_i} E(x-z)\,dz, \\
A(x,t) &= \int_{R^n}\Gamma(z', 0, t) E(x'-z', x_n)\,dz'.
\end{aligned}$$

L_{ij} and A satisfy the estimates

$$|D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} L_{ij}(x,t)| \le \frac{c}{t^{m_0+\frac{1}{2}}(|x|^2+t)^{\frac{1}{2}n+\frac{1}{2}k_0}(x_n^2+t)^{\frac{1}{2}l_0}}, \tag{8.2}$$

$$|D_x^j D_t^m A(x,t)| \le \frac{c}{t^{m+\frac{1}{2}}(|x|^2+t)^{\frac{n-2+|j|}{2}}}, \tag{8.3}$$

where $1 \le i \le n$ and $1 \le j \le n-1$ (see [3] and [1]). The estimates (8.2) of L_{ij} and the estimate of the fundamental solution to heat equation Γ imply that

$$|D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} K_{ij}(x,t)| \le \frac{c}{t^{m_0+\frac{1}{2}}(|x|^2+t)^{\frac{1}{2}n+\frac{1}{2}k_0}(x_n^2+t)^{\frac{1}{2}l_0}}. \tag{8.4}$$

The solution (u,p) of the Stokes system (8.1) in $\Omega = R_+^n$ with boundary data g is expressed by

$$\begin{aligned} u^i(x,t) &= \sum_{j=1}^n \int_0^t \int_{R^n} K_{ij}(x'-y', x_n, t-s) g_j(y',s) dy' ds, \\ p(x,t) &= \sum_{j=1}^n \int_0^t \int_{R^n} \pi_j(x'-y', x_n, t-s) g_j(y',s) dy' ds. \end{aligned} \tag{8.5}$$

We have relations among L and A such that

$$\sum_{1\le i\le n} L_{ii} = -2D_{x_n}\Gamma, \qquad L_{in} = L_{ni} + B_{in}, \tag{8.6}$$

where $B_{in}(x,t) = \int_{\mathbf{R}^{n-1}} D_{x_n}\Gamma(x'-y', x_n, t)D_{y_i}E(y',0)dy' = \frac{\partial}{x_i}\kappa(x,t)$ if $i \neq n$ and $B_{nn} = 0$. For further computation, we introduce Lebesgue spaces and Sobolev spaces:

$$L^p(\Omega) = \{f; \int_\Omega |f|^p dx < \infty\}, \quad W^{1,p}(\Omega) = \{f; \int_\Omega |f|^p + |\nabla f|^p dx < \infty\},$$

$$L^p(0,T;W^{1,p}(\Omega)) = \{f; \int_0^T \int_\Omega |f|^p + |\nabla f|^p dxdt < \infty\}.$$

§8.3 Maximum modulus estimate in the half space.

In this section, we consider the maximum modulus estimate in the half space. The normal derivative $D_{x_n}\Gamma$ has uniformly bounded L^1 norm with respect to x_n on $\partial\mathbf{R}^n_+ \times (0,T)$(see (8.15)) and hence we focus only on the kernel function L_{ij}. By introducing a composite kernel κ we are able to identify the singular kernels. The following lemma is a key stone for the maximum modulus estimate.

Lemma

Let $1 \le i \le n$ *and* $1 \le j \le n-1$. *Then*

$$\int_0^\infty \int_{\mathbf{R}^{n-1}} |L_{ij}(x', x_n, t)| dx' dt < C, \tag{8.7}$$

where $C > 0$ *is independent of* $x_n > 0$ *and hence it follows that*

$$\int_0^\infty \int_{\mathbf{R}^{n-1}} |L_{in}(x', x_n, t) - B_{in}(x', x_n, t)| dx' dt < C, \tag{8.8}$$

where $C > 0$ *is independent of* $x_n > 0$.

Theorem

Let $g = (g_1, g_2, \cdots, g_n) \in L^\infty(\partial \mathbf{R}^n_+ \times (0,T))$ and (u,p) is represented by (8.5). Then,

$$\|u_T - \nabla \mathbf{S}_T(g_n) - \nabla \mathbf{T}_T(g_n)\|_{L^\infty(\mathbf{R}_+ \times (0,T))} \leq C \|g\|_{L^\infty(\partial \mathbf{R}^n_+ \times (0,T))} \tag{8.9}$$

for some $C > 0$. Furthermore, the normal component of the velocity u is bounded and there is also a constant C such that

$$\max_{(x,t) \in \mathbf{R}^n_+ \times (0,T)} |u_n(x,t)| \leq C \max_{(y,t) \in \partial \mathbf{R}^n_+ \times (0,T)} |g(y,t)|.$$

To show the L^1 boundedness of L_{ij}, we note that

$$L_{ij}(x,t) = 2^{\frac{3}{2}} \pi^{\frac{1}{2}} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{|y_n|^2}{t}} \int_{\mathbf{R}^{n-1}} D_{y_j} \Gamma'(y',t) D_{y_i} E(x'-y', x_n - y_n) dy' dy_n \tag{8.10}$$

where Γ' is Gaussian kernel in R^n.

Lemma

For $1 \leq j \leq n-1$, *we get*

$$
\begin{aligned}
&|\int_{|x'-y'|\leq \frac{1}{2}|x'|} D_{y_j}\Gamma'(y',t) D_{y_n} E(x'-y',y_n)dy'| \leq Ct^{-\frac{n-1}{2}} e^{-\frac{|x'|^2}{t}} |x'|^{-1} + Ct^{-\frac{n}{2}-\frac{1}{2}}|x'|e^{-\frac{|x'}{t}} \\
&|\int_{\frac{1}{2}|x'|\leq|y'|\leq 2|x'|, |x'-y'|\geq\frac{1}{2}|x'|} D_{y_j}\Gamma'(y',t) D_{y_n} E(x'-y',y_n)dy'| \leq Ct^{-\frac{n}{2}-\frac{1}{2}}|x'|e^{-\frac{|x'|^2}{t}}, \\
&|\int_{|y'|\leq\frac{1}{2}|x'|} D_{y_j}\Gamma'(y',t) D_{y_n} E(x'-y',y_n)dy'| \leq C|x'|^{-n}\int_{|y'|\leq\frac{1}{2}\frac{|x'|}{\sqrt{t}}} |y'|^2 e^{-|y'|^2} dy' \\
&|\int_{|y'|\geq 2|x'|} D_{y_j}\Gamma'(y',t) D_{y_n} E(x'-y',y_n)dy'| \leq Ct^{-\frac{n}{2}}\int_{\frac{2|x'|}{\sqrt{t}}\leq|y'|} |y'|^{-n+2} e^{-|y'|^2} dy',
\end{aligned}
\tag{8.11}
$$

where $C > 0$ *is independent of* x', y_n *and* t.

proof. Using integration by parts, we get

$$\begin{aligned}\int_{|x'-y'|\le\frac{1}{2}|x'|} & D_{y_j}\Gamma'(y',t)D_{x_n}E(x'-y',y_n)dy' \\ &= \int_{|x'-y'|=\frac{1}{2}|x'|}\frac{x_j-y_j}{|x'-y'|}\Gamma'(y',t)D_{y_n}E(x'-y',y_n)\sigma(dy') \\ &\quad -\int_{|x'-y'|\le\frac{1}{2}|x'|}\Gamma'(y',t)D_{y_j}D_{y_n}E(x'-y',y_n)dy'.\end{aligned} \tag{8.12}$$

For y' with $|x'-y'|=\frac{1}{2}|x'|$, we get $|\Gamma'(y',t)|\le Ct^{-\frac{n-1}{2}}e^{-\frac{|x'|^2}{2t}}$ and $|D_{y_n}E(x'-y',y_n)|\le C\frac{1}{(|x'|^2+y_n^2)^{\frac{n-1}{2}}}$. Here, the first term of the right hand side in (8.12) is dominated by

$$\begin{aligned}\int_{|x'-y'|=\frac{1}{2}|x'|}|\Gamma'(y',t)||D_{y_n}E(x'-y',y_n)|\sigma(dy') &\le Ct^{-\frac{n-1}{2}}e^{-\frac{|x'|^2}{t}}\frac{|x'|^{n-2}}{(|x'|^2+y_n^2)^{\frac{n-1}{2}}} \\ &\le Ct^{-\frac{n-1}{2}}e^{-\frac{|x'|^2}{t}}|x'|^{-1}.\end{aligned} \tag{8.13}$$

Since $\int_{|x'-y'|\le\frac{1}{2}|x'|} D_{y_j}D_{x_n}E(x'-y',y_n)dy'=0$, using the Mean value theorem, the second term of the right hand side of (8.12) satisfies

$$\begin{aligned}
&\int_{|x'-y'|\le\frac{1}{2}|x'|}(\Gamma'(y',t)-\Gamma'(x',t))D_{y_j}D_{x_n}E(x'-y',y_n)dy'\\
&\le C|x'|t^{-\frac{n-1}{2}-1}e^{-\frac{|x'|^2}{t}}\int_{|x'-y'|\le\frac{1}{2}|x'|}\frac{|x'-y'|^2y_n}{(|x'-y'|^2+y_n^2)^{\frac{n}{2}+1}}dy' \qquad (8.14)\\
&\le C|x'|t^{-\frac{n-1}{2}-1}e^{-\frac{|x'|^2}{t}}\int_{R^n}\frac{1}{(|y'|^2+1)^{\frac{n}{2}}}dy'.
\end{aligned}$$

By (8.12) - (8.14), we obtain $(8.11)_1$.
For $(8.11)_2$, note that for y' satisfying $\frac{1}{2}|x'|\le|y'|\le 2|x'|$ we have $|x'-y'|\ge\frac{1}{2}|x'|$. We have $|D_{y_j}\Gamma'(y',t)|\le Ct^{-\frac{n-1}{2}-1}|x'|e^{-\frac{|x'|^2}{t}}$ and $D_{y_n}E(x'-y',y_n)\le C|x'|^{-\frac{n-1}{2}}$, and thus we get

$$\int_{\frac{1}{2}|x'|\le|y'|\le 2|x'|,|x'-y'|\ge\frac{1}{2}|x'|} D_{y_j}\Gamma'(y',t)D_{y_n}E(x'-y',y_n)dy'\le Ct^{-\frac{n}{2}-\frac{1}{2}}|x'|e^{-\frac{|x'|^2}{t}}.$$

Hence, we obtain $(8.11)_2$.

Since $\int_{|y'|\le\frac{1}{2}|x'|} D_{y_j}\Gamma'(y',t)dy = 0$, using Mean-value Theorem , $(8.11)_3$ is proved by

$$
\begin{aligned}
&\int_{|y'|\le\frac{1}{2}|x'|} D_{y_j}\Gamma'(y',t)\Big(D_{x_n}E(x'-y',y_n) - D_{x_n}E(x',y_n)\Big)dy' \\
&\le C(|x'|^2+y_n)^{-\frac{n}{2}} \int_{|y'|\le\frac{1}{2}|x'|} t^{-\frac{n+1}{2}}|y'|^2 e^{-\frac{|y'|^2}{t}}dy' \\
&\le C(|x'|^2+y_n)^{-\frac{n}{2}} \int_{|y'|\le\frac{1}{2}\frac{|x'|}{\sqrt{t}}} |y'|^2 e^{-|y|^2}dy'.
\end{aligned}
$$

Finally, $(8.11)_4$ follows by

$$
\begin{aligned}
\int_{|y'|\ge 2|x'|} D_{y_j}\Gamma'(y',t)D_{y_n}E(x'-y',y_n)dy' &\le Ct^{-\frac{n+1}{2}} \int_{2|x'|\le|y'|} |y'|^{-n+2}e^{-\frac{|y'|^2}{t}}dy' \\
&= Ct^{-\frac{n}{2}} \int_{\frac{2|x'|}{\sqrt{t}}\le|y'|} |y'|^{-n+2}e^{-|y'|^2}dy'.
\end{aligned}
$$

end of proof.

Following a similar proof to Lemma 3.3, we get the following lemma.

Lemma

For $1 \le i,\ j \le n-1$, *we get*

$$\int_{|x'-y'|\le\frac{1}{2}|x'|} D_{y_j}\Gamma'(y',t)D_{y_i}E(x'-y',y_n)dy' \le Ct^{-\frac{n}{2}-\frac{1}{2}}|x'|e^{-\frac{|x'|^2}{t}}$$

$$\int_{\frac{1}{2}|x'|\le|y'|\le 2|x'|,|x'-y'|\ge\frac{1}{2}|x'|} D_{y_j}\Gamma'(y',t)D_{y_i}E(x'-y',y_n)dy' \le Ct^{-\frac{n}{2}-\frac{1}{2}}|x'|e^{-\frac{|x'|^2}{t}},$$

$$\int_{|y|\le\frac{1}{2}|x'|} D_{y_j}\Gamma'(y',t)D_{y_i}E(x'-y',y_n)dy' \le C|x'|^{-n}\int_{|y'|\le C\frac{1}{2}\frac{|x'|}{\sqrt{t}}}|y'|^2e^{-|y'|^2}dy',$$

$$\int_{|y'|\ge 2|x'|} D_{y_j}\Gamma'(y',t)D_{y_i}E(x'-y',y_n)dy' \le Ct^{-\frac{n}{2}}\int_{\frac{2|x'|}{\sqrt{t}}\le|y'|}|y'|^{-n+2}e^{-|y'|^2}dy'.$$

Proof of Lemma 8.2..

Note that for $1 \le i \le n$ and $1 \le j \le n-1$

$$\int_0^T \int_{R^n} |K_{ij}(x', x_n, t)| dx' dt \le \int_0^T \int_{R^n} |D_{x_n} \Gamma(x', x_n, t)| dx' dt + \int_0^T \int_{R^n} |L_{ij}(x', x_n, t)| dx' dt.$$

Here, using change of variables ($\frac{x_n^2}{t} = s$), we get

$$\begin{aligned} \int_0^T \int_{R^n} |D_{x_n} \Gamma(x', x_n, t)| dx' dt &= C \int_0^T t^{-\frac{3}{2}} x_n e^{-\frac{x_n^2}{t}} \int_{R^n} t^{-\frac{n-1}{2}} e^{-\frac{|x'|^2}{t}} dx' dt \\ &= C x_n \int_0^T t^{-\frac{3}{2}} e^{-\frac{x_n^2}{t}} \int_{R^n} e^{-|x'|^2} dx' dt \\ &= C x_n \int_0^T (\frac{x_n^2}{s})^{-\frac{3}{2}} x_n^2 s^{-2} e^{-s} ds. \end{aligned} \tag{8.15}$$

Hence, to prove Lemma 8.2., it is sufficient to show $\int_0^T \int_{R^n} |L_{ij}(x', x_n, t)| dx' dt < \infty$ for $1 \le i \le n,\ 1 \le j \le n-1$.

By $(8.2)_1$, for $1 \le i \le n$ and $1 \le j \le n-1$, we get

$$\begin{aligned}\int_0^{x_n^2}\int_{R^n}|L_{ij}(x',x_n,t)|dx'dt &\le C\int_0^{x_n^2}\int_{R^n} t^{-\frac{1}{2}}(|x'|^2+x_n^2+t)^{-\frac{n}{2}}dx'dt\\ &\le C\int_0^{x_n^2} t^{-\frac{1}{2}}(x_n^2+t)^{-\frac{1}{2}}dt = C.\end{aligned} \tag{8.16}$$

To calculate $\int_{x_n^2}^T\int_{R^n}|L_{ij}(x',x_n,t)|dx'dt$, we may assume $x_n^2 \le T$. By the representation (8.10), and Lemma 8.3. and Lemma 8.4., we have

$$\begin{aligned}&\int_{x_n^2}^T\int_{R^n}|L_{ij}(x',x_n,t)|dx'dt\\ &\le C\int_{x_n^2}^T\int_{R^n}\int_0^{x_n} t^{-\frac{3}{2}}y_n e^{-\frac{y_n^2}{t}}\Big(t^{-\frac{n-1}{2}}e^{-\frac{|x'|^2}{t}}|x'|^{-1}+t^{-\frac{n}{2}-\frac{1}{2}}|x'|e^{-\frac{|x'|^2}{t}}\\ &\qquad +|x'|^{-n}\int_{|y'|\le\frac{1}{2}\frac{|x'|}{\sqrt{t}}}|y'|^2e^{-|y'|^2}dy'+t^{-\frac{n}{2}}\int_{\frac{2|x'|}{\sqrt{t}}\le|y'|}|y'|^{-n+2}e^{-|y'|^2}dy'\Big)dy\\ &= I+II+III+IV,\end{aligned} \tag{8.17}$$

where

$$I = \int_{x_n^2}^{T} \int_{R^n} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{t}} t^{-\frac{n-1}{2}} e^{-\frac{|x'|^2}{t}} |x'|^{-1} dy_n dx' dt,$$

$$II = \int_{x_n^2}^{T} \int_{R^n} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{t}} t^{-\frac{n}{2}-\frac{1}{2}} |x'| e^{-\frac{|x'|^2}{t}} dy_n dx' dt,$$

$$III = \int_{x_n^2}^{T} \int_{R^n} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{t}} |x'|^{-n} \int_{|y'| \leq \frac{1}{2}\frac{|x'|}{\sqrt{t}}} |y'|^2 e^{-|y'|^2} dy' dy_n dx' dt,$$

$$IV = \int_{x_n^2}^{T} \int_{R^n} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{t}} t^{-\frac{n}{2}} \int_{\frac{2|x'|}{\sqrt{t}} \leq |y'|} |y'|^{-n+2} e^{-|y'|^2} dy' dy_n dx' dt.$$

Using change of variables twice, we have

$$\begin{aligned} I &= \int_{x_n^2}^{T} t^{-\frac{n}{2}-1} \int_{R^n} e^{-\frac{|x'|^2}{t}} |x'|^{-1} t \int_0^{\frac{x_n}{\sqrt{t}}} y_n e^{-y_n^2} dy_n dx' dt \\ &\leq C \int_{x_n^2}^{T} t^{-\frac{n}{2}-1} \int_{R^n} e^{-\frac{|x'|^2}{t}} |x'|^{-1} t (\frac{x_n}{\sqrt{t}})^2 dx' dt \\ &= C \int_{x_n^2}^{T} x_n^2 t^{-2} dt \\ &= C \end{aligned} \tag{8.18}$$

and

$$\begin{aligned} II &= \int_{x_n^2}^{T} t^{-\frac{n}{2}-2} \int_{R^n} e^{-\frac{|x'|^2}{t}} |x| t \int_0^{\frac{x_n}{\sqrt{t}}} y_n e^{-y_n^2} dy_n dx' dt \\ &\leq C \int_{x_n^2}^{T} t^{-\frac{n}{2}-2} \int_{R^n} e^{-\frac{|x'|^2}{t}} |x'| t (\frac{x_n}{\sqrt{t}})^2 dx' dt \\ &= C \int_{x_n^2}^{T} x_n^2 t^{-2} dt \\ &= C. \end{aligned} \tag{8.19}$$

We divide III into two parts $III = III_1 + III_2$, where

$$III_1 = \int_{x_n^2}^{T} \int_{|x'|\le\sqrt{t}} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{t}} |x'|^{-n} \int_{|y'|\le\frac{1}{2}\frac{|x'|}{\sqrt{t}}} |y'|^2 e^{-|y'|^2} dy' dy_n dx' dt,$$

$$III_2 = \int_{x_n^2}^{T} \int_{|x'|\ge\sqrt{t}} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{t}} |x'|^{-n} \int_{|y'|\le\frac{1}{2}\frac{|x'|}{\sqrt{t}}} |y'|^2 e^{-|y'|^2} dy' dy_n dx' dt.$$

Here,

$$\begin{aligned} III_1 &\le C \int_{x_n^2}^{T} t^{-\frac{3}{2}} \int_{|x'|\le\sqrt{t}} |x'|^{-n} \left(\frac{|x'|}{\sqrt{t}}\right)^{n+1} \int_0^{x_n} y_n e^{-\frac{y_n^2}{t}} dy_n dx' dt \\ &= C \int_{x_n^2}^{T} x_n^2 t^{-\frac{n}{2}-2} \int_{|x'|\le\sqrt{t}} |x'| dx' dt \\ &\le C \int_{x_n^2}^{T} x_n^2 t^{-2} dt \\ &= C, \\ III_2 &\le C \int_{x_n^2}^{T} t^{-\frac{3}{2}} \int_{|x'|\ge\sqrt{t}} |x'|^{-n} \int_0^{x_n} y_n e^{-\frac{y_n^2}{t}} dy_n dx' dt \\ &\le C \int_{x_n^2}^{T} t^{-\frac{3}{2}} x_n^2 \int_{|x'|\ge\sqrt{t}} |x'|^{-n} dx' dt \le C \int_{x_n^2}^{T} t^{-2} x_n^2 dt = C. \end{aligned}$$

Hence, we get

$$III < C. \tag{8.20}$$

Therefore, from (8.16)- (**??**), we prove

$$\int_0^T \int_{R^n} |L_{ij}(x', x_n, t)| dx' dt \le C \tag{8.21}$$

for $1 \le i \le n$ and $1 \le j \le n-1$, where C is independent of x_n. With (8.15), this implies (8.7).
By the second identity of (8.6) and (8.21), we prove (8.8) for the case $i \neq n$, and by the first identity of (8.6) and (8.21), we prove (8.8) for the case $i = n$. This ends the proof of Lemma 3.1.
□

Proof of Theorem 3.2.

We begin the proof of Theorem 3.2 by the representation (8.5) of u such that

$$\begin{aligned} u^i(x,t) &= \sum_{j=1}^{n} \int_0^t \int_{R^n} K_{ij}(x'-y',x_n,t-s)g_j(y',s)dy'ds, \\ &= \sum_{j=1}^{n-1} \int_0^t \int_{R^n} K_{ij}(x'-y',x_n,t-s)g_j(y',s)dy'ds \\ &\quad + \int_0^t \int_{R^n} K_{in}(x'-y',x_n,t-s)g_n(y',s)dy'ds \end{aligned}$$

and the last potential for g_n is written as

$$\begin{aligned} &\int_0^t \int_{R^n} K_{in}(x'-y',x_n,t-s)g_n(y',s)dy'ds \\ &= -2\delta_{in} \int_0^t \int_{R^n} D_{x_n}\Gamma(x'-y',x_n,t-s)g_n(y',s)dy'ds \\ &\quad + 4\int_0^t \int_{R^n} L_{in}(x'-y',x_n,t-s)g_n(y',s)dy'ds \\ &\quad - \frac{\partial}{\partial x_i}\int_0^t \int_{R^n} E(x'-y',x_n)g_n(y',s)dy'ds. \end{aligned}$$

Since $L_{in} = L_{ni} + B_{in}$, we have

$$\begin{aligned}\int_0^t \int_{R^n} & L_{in}(x'-y', x_n, t-s) g_n(y', s) dy' ds \\ &= \int_0^t \int_{R^n} L_{ni}(x'-y', x_n, t-s) g_n(y', s) dy' ds \\ &\quad + \frac{\partial}{\partial x_i} \int_0^t \int_{R^n} \kappa(x'-y', x_n, t-s) g_n(y', s) dy' ds\end{aligned}$$

for $1 \leq i \leq n-1$, where we defined the composite kernel function $\kappa(x,t)$ on $\mathbf{R}^n_+ \times (0,T)$ by

$$\kappa(x,t) = \int_{\mathbf{R}^{n-1}} \frac{\partial}{\partial x_n} \Gamma(x'-z', x_n, t) E(z', 0) dz'.$$

Define the surface potential $\mathbf{T}(g_n)$ by

$$\mathbf{T}(g_n)(x,t) = 4\int_0^t \int_{\mathbf{R}^{n-1}} \kappa(x'-y', x_n, t-s) g_n(y', s) dy' ds. \tag{8.22}$$

Moreover, we have that

$$\frac{\partial}{\partial x_i} \int_{R^n} E(x'-y', x_n) g_n(y', s) dy' = \frac{\partial}{\partial x_i} \mathbf{S}(g_n)$$

Therefore we conclude that the tangential part, which is associated with L_{ij}, satisfies

$$|u_T(x,t) - \nabla \mathbf{S}_T(g_n)(x,t) - \nabla \mathbf{T}_T(g_n)(x,t)| \le C||g||_{L^\infty(\mathbf{R}^{n-1}\times(0,T))}$$

for all $(x,t) \in \mathbf{R}^n_+ \times (0,T)$.

The normal velocity u_n behaves even better. First, we know that $\frac{\partial}{\partial x_n}\mathbf{S}(g_n)$ is the Poisson kernel expression of the solution for the Laplace equation in the half space and satisfies the maximum principle. In the case $i = n$, we have a relation from (8.6)

$$L_{nn} = -\sum_{1\leq i\leq n-1} L_{ii} - 2D_{x_n}\Gamma$$

which has a bounded L^1 norm on the lateral surface. This conclude the maximum modulus estimate of u_n.
$\square$

§8.4 Maximum Modulus Estimate in C^2 Domain.

We denote the Green's matrix for the domain Ω by $(G^\Omega, \theta^\Omega)$ and for a given point $x \in \Omega$ we let $\bar{x} \in \partial\Omega$ satisfy $|x - \bar{x}| = dist(x, \partial\Omega)$. The interior L^∞ bound estimate can be shown by the layer potential method in [?] and we consider separately the case that the generic point x is close enough to $\partial\Omega$. Indeed, to see the interior boundedness, we need to show the boundedness of the double layer potential in $L^\infty(0, T; L^2(\partial\Omega))$. Since the boundary data is bounded, we can represent the solution by the double layer potential in [?] from L^2 theory such that

$$\begin{aligned} u_i(x,t) = &\int_0^t \int_{\partial\Omega} \frac{\partial F_{ij}}{\partial N(y)}(x-y, t-s) h_j(y,s) d\sigma(y) ds \\ &- \int_{\partial\Omega} \frac{y_i - x_i}{\omega_n |y-x|^n} h(y,t) \cdot N(y) d\sigma(y) \\ =&(\mathbf{K}h)_i(x,t) \end{aligned} \tag{8.23}$$

and

$$g = -\frac{1}{2}h + \mathbf{K}h = (-\frac{1}{2}\mathbf{I} + \mathbf{K})h \tag{8.24}$$

in the sense of $L^2(\partial\Omega \times (0,T))$ for an $h \in L^2(\partial\Omega \times (0,T))$(see Theorem 2.3.6 and Theorem 5.1.2 in [?]).

Furthermore $-\frac{1}{2}\mathbf{I}+\mathbf{K}$ is invertible on $L^2_\sigma(\partial\Omega\times(0,T))$, where the subscript σ means solenoidal. From the representation, we have a continuity lemma in time of the density function h in (8.23).

Lemma

The inverse of the double layer potential $-\frac{1}{2}\mathbf{I}+\mathbf{K}$ is bounded in time as an operator of $L^2(\partial\Omega)$ and there is a constant $\delta>0$ such that if $|t_1-t_2|<\delta$,

$$||h(\cdot,t_2)||_{L^2(\partial\Omega)}\le C||g(\cdot,t_1)-g(\cdot,t_2)||_{L^2(\partial\Omega)}+C||h||_{L^\infty(0,t_1;L^2(\Omega))}.$$

By an iteration there is C such that

$$||(-\frac{1}{2}\mathbf{I}+\mathbf{K})^{-1}g||_{L^\infty(0,T;L^2(\Omega))}\le C||g||_{L^\infty(0,T;L^2(\Omega))}.$$

proof. We assume the boundary data $g \in L^\infty(0,T;L^2(\partial\Omega))$ and after arranging the singular integrals in the double layer potential expression we have

$$
\begin{aligned}
g_i(x,t_2)-g_i(x,t_1) = & -\frac{1}{2}(h_i(x,t_2) - h_i(x(t-1)) \\
& - \int_{\partial\Omega} \frac{y_i - x_i}{\omega_n |y-x|^n}(h(y,t_1) - h(y,t_2)) \cdot N(y) d\sigma(y) \\
& + \int_{t_1}^{t_2} \int_{\partial\Omega} \frac{\partial F_{ij}}{\partial N(y)}(x-y,t_2-s) h_j(y,s) d\sigma(y) ds \\
& + \int_0^{t_1} \int_{\partial\Omega} \left(\frac{\partial F_{ij}}{\partial N(y)}(x-y,t_2-s) - \frac{\partial F_{ij}}{\partial N(y)}(x-y,t_1-s) \right) h_j(y,s \\
= & (-\frac{1}{2}\mathbf{I} + \mathbf{H})(h(\cdot,t_2) - h(\cdot,t_1)) + \mathbf{E}_1 h + \mathbf{E}_2 h
\end{aligned}
$$

for almost all $0 < t_2 < t_2 < T$ and $x \in \partial\Omega$.

We claim $-\frac{1}{2}\mathbf{I}+\mathbf{H} : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is invertible and

$$||(-\frac{1}{2}\mathbf{I}+\mathbf{H})^{-1}e||_{L^2(\partial\Omega)} \leq C||e||_{L^2(\partial\Omega)}$$

for a constant C. First of all, if we set $e = (-\frac{1}{2}\mathbf{I}+\mathbf{H})f$ and consider the normal components, then we have

$$e_N = (-\frac{1}{2}\mathbf{I}+N\cdot\mathbf{H})f_N,$$

where $N\cdot\mathbf{H}$ is the standard double layer potential operator of Laplace equation and $-\frac{1}{2}\mathbf{I}+N\cdot\mathbf{H}$ is invertible. So given vector valued function $e \in L^2(\partial\Omega)$, there is a scalar valued function $w \in L^2(\Omega)$ satisfying

$$e_N = (-\frac{1}{2}\mathbf{I}+N\cdot\mathbf{H})w$$

with $||w||_{L^2(\partial\Omega)} \leq C||e||_{L^2(\partial\Omega)}$. Here w is the normal component of f and the tangential component v of f is obtained by

$$v = -2(e-e_NN) - 2(\mathbf{H}w - (N\cdot\mathbf{H})wN).$$

Therefore we get

$$f = v + wN$$

and f satisfies

$$||f||_{L^2(\partial\Omega)} \leq C||e||_{L^2(\partial\Omega)}.$$

It remains the estimate $\mathbf{E}_1 h$ and $\mathbf{E}_2 h$. Since Ω is C^2 domain, in the case of Gaussian kernel, there is C such that for all $(x, y, t) \in \partial\Omega \times \partial\Omega \times (0, T)$

$$|\frac{\partial}{\partial N(y)}\Gamma(x-y,t)| \leq C\frac{|x-y|^2}{\sqrt{t}^{n+2}}e^{-\frac{|x-y|^2}{2t}}.$$

Therefore we get from Minkowski inequality and Young's convolution inequality

$$
\begin{aligned}
&\left(\int_{\partial\Omega}\left|\int_{t_1}^{t_2}\int_{\partial\Omega}\frac{\partial\Gamma}{\partial N(y)}(x-y,t_2-s)h(y,s)d\sigma(y)ds\right|^2 d\sigma(x)\right)^{\frac{1}{2}}\\
&\le C\left(\int_{\partial\Omega}\left|\int_{t_1}^{t_2}\frac{1}{\sqrt{t_2-s}}\frac{1}{\sqrt{t_2-s}^{\,n-1}}\int_{\partial\Omega}\frac{|x-y|^2}{t_2-s}e^{-\frac{|x-y|^2}{2(t_2-s)}}|h(y,s)|d\sigma(y)ds\right|^2 d\sigma(x)\right.\\
&\le C\int_{t_1}^{t_2}\frac{1}{\sqrt{t_2-s}}\left(\int_{\partial\Omega}\left|\frac{1}{\sqrt{t_2-s}^{\,n-1}}\int_{\partial\Omega}\frac{|x-y|^2}{t_2-s}e^{-\frac{|x-y|^2}{2(t_2-s)}}|h(y,s)|d\sigma(y)\right|^2 d\sigma(x)\right)\\
&\le C\int_{t_1}^{t_2}\frac{1}{\sqrt{t_2-s}}||h(\cdot,s)||_{L^2(\Omega)}ds\\
&\le C\sqrt{t_2-t_1}||h||_{L^\infty(t_1,t_2;L^2(\Omega))}.
\end{aligned}
$$

By the same token, assuming $||h||_{L^\infty(0,t_1;L^2(\Omega))}$ is bounded, we have that

$$||\mathbf{E}_2 h(\cdot,t_1)||_{L^2(\partial\Omega)} \le C||h||_{L^\infty(0,t_1;L^2(\Omega))}.$$

end of proof.

We let the generic point x be away from the boundary, say $dist(x, \partial\Omega) = r_0 > 0$. Since the kernel of the double layer is bounded by $\frac{C}{r_0^{n-1-\epsilon}}$ for each $\epsilon > 0$ and the density function h of g for the double layer potential is bounded in $L^\infty(0, T; L^2(\partial\Omega))$, the interior L^∞ estimate follows.

Corollary

Suppose the boundary data $g \in L^\infty(0, T; L^2(\partial\Omega))$. If $dist(x, \partial\Omega) \geq r_0 > 0$, $\epsilon > 0$ and $t < T$, then there is C such that

$$|u(x,t)| \leq \frac{C}{r_0^{n-1-\epsilon}} ||g||_{L^\infty(0,T;L^2(\partial\Omega))}.$$

Now we start the boundary estimate. Since Stokes equations is translation and rotation invariant, we assume that $\bar{x} = 0$ and $x = (0, x_n)$, $x_n > 0$. If x is close enough to $\partial\Omega$, there is a ball $B_r(0)$ centered at origin and C^2 function $\Phi : \mathbf{R}^{n-1} \to \mathbf{R}$ such that $\Omega \cap B_r(0) = \{x_n > \Phi(x')\} \cap B_r(0)$.

Furthermore, Φ satisfies that

$$|\Phi(x')| \le C|x'|^2, \quad |\nabla'\Phi(x')| \le C|x'|, \quad |\nabla'\nabla'\Phi(x')| \le C \tag{8.25}$$

for $x' \in B_r'(0)$ and the outward unit normal vector $N(x', \Phi(x'))$ at $(x', \Phi(x')) \in \partial\Omega \cap B_r(0)$ is

$$N(x', \Phi(x')) = \frac{1}{\sqrt{1 + |\nabla'\Phi(x')|^2}}(\nabla'\Phi(x'), -1).$$

We define a transform $\mu : \Omega \cap B_r(0) \to \mathbf{R}^n_+$ such that

$$\mu(y) = \mu(y', y_n) = (y', y_n - \Phi(y'))$$

and note that $\mu^{-1}(y', y_n) = (y', y_n + \Phi(y'))$. Since our generic point x is $(0, x_n)$, we have $\mu(x) = x$. Hence the Green's matrix G on the half space can be transformed to a function μG on Ω such that

$$\mu G(x, y, t) = G(\mu(x), \mu(y), t) = G(x', x_n - \Phi(x'), y', y_n - \Phi(y'), t)$$

and satisfies the zero boundary condition

$$\mu G(x, y', \Phi(y'), t) = 0.$$

Moreover, the transformed Green's matrix $(\mu G, \mu\theta)$ satisfies a perturbed Stokes equations in $\Omega \times (0, T)$

$$\begin{aligned}
&\frac{\partial}{\partial t}(\mu G)_{ij} - \Delta_y(\mu G)_{ij} + \frac{\partial}{\partial y_j}(\mu\theta)_i \\
&= \delta_{ij}\delta(x-y)\delta(t) + D_{y_n}(\mu G)_{ij}\Delta'\Phi + 2D_{y_k}D_{y_n}(\mu G)_{ij}D_{y_k}\Phi \\
&\qquad - D_{y_n}D_{y_n}(\mu G)_{ij}D_{y_k}\Phi D_{y_k}\Phi - D_{y_n}(\mu\theta)_i D_{y_j}\Phi \\
&= \delta_{ij}\delta(x-y)\delta(t) + R(x, y, t)
\end{aligned}$$

and the solenoidal condition

$$D_{y_j}(\mu G)_{ij} = -D_{y_n}(\mu G)_{ij}D_{y_j}\Phi = S_i(x, y, t).$$

Therefore, if we let the perturbation $(J,\eta)=(G^{\Omega}-\mu G,\theta^{\Omega}-\mu\theta)$, then (J,η) satisfies the perturbation equations:

$$\frac{\partial}{\partial t}J_{ij}(x,y,t)-\Delta_y J_{ij}(x,y,t)+\frac{\partial}{\partial y_j}\eta_j(x,y,t)=R_{ij}(x,y,t) \tag{8.26}$$

$$D_{y_j}J_{ij}(x,y,t)=S_i(x,y,t), \tag{8.27}$$

where R is

$$\begin{aligned} R_{ij}=&-D_{y_n}(\mu G)_{ij}\Delta'_y\Phi-2D_{y_k}D_{y_n}(\mu G)_{ij}D_{y_k}\Phi\\ &+D_{y_n}D_{y_n}(\mu G)_{ij}D_{y_k}\Phi D_{y_k}\Phi+D_{y_n}(\mu\theta)_i D_{y_j}\Phi\\ =&I+II+III+IV. \end{aligned}$$

We have already discussed the boundedness of velocity u in the interior by double layer potential in L^2 theory, we begin to prove the boundedness near the boundary.

The plan to get L^1 bound of the perturbation J of Poisson kernel on $\partial\Omega \times (0,T)$ relies on the $L^p(0,T;W^{2,p}(\Omega))$ estimate and the trace theorem for it. Recall that the Poisson kernel is a derivative of Green's matrix and that is the reason that we need $L^p(0,T;W^{2,p}(\Omega))$ Sobolev type estimate. Therefore, we need to estimate the L^p norm of R in (8.26), $L^p(0,T;W^{1,p}(\Omega))$ norm of S and $L^p(0,T;W^{-1,p}(\Omega))$ norm of S_t in $(B_1 \cap \Omega) \times (0,T)$ in (8.27), where $W^{-1,p}(\Omega)$ is the dual space of $W^{1,p}(\Omega)$. Since the Green's matrix G is associated with the Gaussian kernel and the composite kernel H, we estimate their derivatives first. We have

$$
\begin{aligned}
|D_{y_n}\Gamma(x-y,t)| &\le \frac{C}{\sqrt{t}^n}\frac{|y-x|}{t}e^{-\frac{|y'|^2+|y_n-x_n|^2}{2t}}\\
|D_{y_n}\Gamma(x-y^*,t)| &\le \frac{C}{\sqrt{t}^n}\frac{|y^*-x|}{t}e^{-\frac{|y'|^2+|y_n+x_n|^2}{2t}}\\
|D_{y_k}D_{y_n}\Gamma(x-y,t)| &\le \frac{C}{\sqrt{t}^n}\frac{|y-x|^2}{t^2}e^{-\frac{|y'|^2+|y_n-x_n|^2}{2t}}\\
|D_{y_k}D_{y_n}\Gamma(x-y^*,t)| &\le \frac{C}{\sqrt{t}^n}\frac{|y^*-x|^2}{t^2}e^{-\frac{|y'|^2+|y_n+x_n|^2}{2t}}.
\end{aligned}
$$

Since $|D_{y_k}\Phi(y')| \le C|y'|$, $|\Delta'\Phi(y')| \le C$ and $x' = 0$, we get

$$\begin{aligned}
|D_{y_n}\Gamma(x-y,t)\Delta'\Phi(y')| &\le \frac{C}{\sqrt{t}^{n+1}}\frac{|y-x|}{\sqrt{t}}e^{-\frac{|y'|^2+|y_n-x_n|^2}{2t}} \\
&\in L^p((\Omega\cap B_r)\times(0,T)) \\
|D_{y_k}D_{y_n}\Gamma(x-y,t)\nabla'\Phi(y')| &\le \frac{C}{\sqrt{t}^{n+1}}\frac{|y-x|^3}{\sqrt{t}^3}e^{-\frac{|y'|^2+|y_n-x_n|^2}{2t}} \\
&\in L^p((\Omega\cap B_r)\times(0,T))
\end{aligned}$$

as a function of y for all $p \in [1, \frac{n+2}{n+1})$. In the same way, we have

$$D_{y_n}\Gamma(x-y^*,t)\Delta'\Phi(y'), \quad D_{y_k}D_{y_n}\Gamma(x-y,t)\nabla'\Phi(y') \in L^p((\Omega\cap B_r)\times(0,T))$$

as a function of y for all $p \in [1, \frac{n+2}{n+1})$.

Applying (8.2), we have

$$\left|D_{y_n}D_{x_j}\int_0^{x_n}\int_{\mathbf{R}^{n-1}}D_{x_i}E(x-z)\Gamma(z-y^*,t)dz\right| \le \frac{C}{t^{\frac{1}{2}}(|x'-y'|^2+|y_n+x_n|^2+t)^{\frac{1}{2}n}}$$

$$\left|D_{y_k}D_{y_n}D_{x_j}\int_0^{x_n}\int_{\mathbf{R}^{n-1}}D_{x_i}E(x-z)\Gamma(z-y^*,t)dz\right| \le \frac{C}{t^{\frac{1}{2}}(|x'-y'|^2+|y_n+x_n|^2+t)^{\frac{1}{2}(n+1)}}.$$

Hence, we have for $p\in[1,\frac{n+2}{n+1})$

$$\begin{aligned}&\left|D_{y_n}D_{x_j}\int_0^{x_n}\int_{\mathbf{R}^{n-1}}D_{x_i}E(x-z)\Gamma(z-y^*,t)dz\Delta'\Phi(y')\right|\\&\qquad\le\frac{C}{\sqrt{t}^{n+1}}\frac{1}{\sqrt{(|\frac{|y^*-x|^2}{t}+1)}^n}\in L^p((\Omega\cap B_r)\times(0,T))\end{aligned}$$

$$\begin{aligned}&\left|D_{y_k}D_{y_n}D_{x_j}\int_0^{x_n}\int_{\mathbf{R}^{n-1}}D_{x_i}E(x-z)\Gamma(z-y^*,t)dz\nabla'\Phi(y')\right|\\&\qquad\le\frac{C}{\sqrt{t}^{n+1}}\frac{1}{\sqrt{(|\frac{|y^*-x|^2}{t}+1)}^n}\in L^p((\Omega\cap B_r)\times(0,T)).\end{aligned}$$

Although there is a transformation μ of domain, these estimates imply that I, II and III are in $L^p((\Omega\cap B_r)\times(0,T))$ as a function of y for all $p\in[1,\frac{n+2}{n+1})$.
It remains to get L^p estimate of the pressure kernel θ. For each fixed time t, we have

$$\begin{aligned}|IV|\leq &C|y'|\left|\int_{\mathbf{R}^{n-1}} D_{x_i}E(x'-z',x_n)\frac{y_n}{t}\frac{1}{\sqrt{t}^n}e^{-\frac{|z'-y'|^2+y_n^2}{2t}}dz'\right|\\ &+C|y'|\left|D_{y_n}D_{y_n}\int_{\mathbf{R}^{n-1}}\frac{1}{\sqrt{|x'-z'|^2+x_n^2}^{n-2}}\frac{1}{\sqrt{t}^n}e^{-\frac{|z'-y'|^2+y_n^2}{2t}}dz'\right|.\end{aligned} \tag{8.28}$$

The first term on the right is L^p for $p\in[1,\frac{n+2}{n+1})$ by the Young's convolution inequality since the kernel $\frac{x_n}{\sqrt{|z'|^2+x_n^2}^n}$ has bounded $L^1(\mathbf{R}^{n-1})$ estimate as a function of z' independent of x_n. For the second term, we recall the following proposition by Solonnikov (Proposition 2.3 in [2]):

Lemma

Let M(x, t) be a function defined for $x \in \mathbf{R}^n_+$ *and* $t > 0$ *and having the properties*

$$M(\lambda x, \lambda^2 t) = \lambda^m M(x,t), \quad \lambda > 0,$$

$$|D_x^k D_t^s M(x,t)| \leq C t^{\frac{m-k-2s}{2}} \exp\left(-\frac{|x|^2}{2t}\right).$$

Then the integral

$$J(x, y_n, t) = \int_{\mathbf{R}^{n-1}} E(y) M(x'-y', x_n, t) dy'$$

satisfies the conditions

$$J(\lambda x, \lambda y_n, \lambda^2 t) = \lambda^{m+1} J(x, y_n, t),$$

$$|D_x^k D_{y_n}^l D_t^s J(x,y_n,t)| \leq C t^{\frac{m+n-1-2s-k_n}{2}} \left(|x'|^2 + (x_n+y_n)^2 + t\right)^{-\frac{|k'|+l+n-2}{2}} e^{-}$$

where $k = (k_1, ..., k_n)$ *and* $|k'| = k_1 + \cdots + k_{n-1}$.

So, we find the second term of (8.28) without the transformation μ is bounded by

$$C\frac{|y'|}{\sqrt{t}}\left(|\frac{y'}{\sqrt{t}}|^2+(\frac{x_n+y_n}{\sqrt{t}})^2+1\right)^{-\frac{n}{2}}\frac{1}{\sqrt{t}^n}e^{-\frac{y_n^2}{2t}}$$

$$\leq C(\frac{r}{\sqrt{t}}+1)\frac{|y'|}{\sqrt{t}}\left(|\frac{y'}{t}|^2+(\frac{x_n+y_n}{t})^2+1\right)^{-\frac{n-1}{2}}\frac{1}{\sqrt{t}^n}e^{-\frac{y_n^2}{2t}}$$

which is in $L^p((\Omega\cap B_r)\times(0,T))$ for $p\in(1,\frac{n+2}{n+1})$.
This concludes that IV is in L^p and R in (8.26) is in L^p for all $p\in(1,\frac{n+2}{n+1})$ after adjustment of the domain transformation μ.

To get $L^p(0,T;W^{1,p}(\Omega\cap B_r))$ bound of S in (8.27) we follow a similar program to R. Indeed, we have

$$\nabla_y S = -\nabla_y D_{y_n}(\mu G)_{ij} D_{y_j}\Phi(y') - D_{y_n}(\mu G)_{ij}\nabla_y D_{y_j}\Phi(y').$$

The terms in the right hand side have already been considered in the estimates of I, II and III of R except $D_{y_n}D_{y_n}(\mu G)_{ij}D_{y_j}\Phi(y')$. But, $D_{y_n}D_{y_n}(\mu G)_{ij} = \sum_{1\le j\le n-1} D_{y_k}D_{y_n}(\mu G_{ij})$ and hence $D_{y_n}D_{y_n}(\mu G)_{ij}D_{y_j}\Phi(y')$ has the form of II. Therefore, we get

$$||S||_{L^p(0,T;W^{1,p}(\Omega\cap B_r))} < C \quad \text{independent of} \quad x.$$

It remains to find $L^p(0,T;W^{-1,p}(\Omega\cap B_r))$ estimate of D_tS. Since S is defined as

$$S_i(x,y,t) = -D_{y_n}(\mu G)_{ij}D_{y_j}\Phi(y')$$

and Φ is independent of y_n, $L^p(0,T;W^{-1,p}(\Omega\cap B_r))$ norm of S_t is bounded by

$$C\int_0^T\int_{\Omega\cap B_r}|D_t(\mu G)\nabla'\Phi(y')|^p dydt$$

for a constant C.

By disregarding Φ, we have

$$D_tG_{ij} = \delta_{ij}\big(D_t\Gamma(x-y,t) - D_t\Gamma(x-y^*,t)\big) + 4(1-\delta_{jn})D_{x_j}\int_0^{x_n}\int_{\mathbf{R}^{n-1}} D_{x_i}E(x-z)D_t\Gamma(z-y^*,t)dz$$

and

$$|D_tG_{ij}| \le C\left(\frac{1}{\sqrt{t}^{n+2}} + \frac{|x-y|^2}{\sqrt{t}^{n+4}}\right)e^{-\frac{|x-y|^2}{2t}} + C\left(\frac{1}{\sqrt{t}^{n+2}} + \frac{|x-y^*|^2}{\sqrt{t}^{n+4}}\right)e^{-\frac{|x-}{2}} + C\left|D_{x_j}\int_0^{x_n}\int_{\mathbf{R}^{n-1}} D_{x_i}E(x-z)D_t\Gamma(z-y^*,t)dz\right|$$

and Proposition 2.5 in [2], we have

$$\left|D_{x_j}\int_0^{x_n}\int_{\mathbf{R}^{n-1}} D_{x_i}E(x-z)D_t\Gamma(z-y^*,t)dz\nabla'\Phi(y')\right| \le \frac{C|y'|}{t(|x-y^*|^2+t)^{\frac{n}{2}}}e$$

This implies that $D_tG_{ij}\nabla'\Phi(y') \in L^p((\Omega\cap B_r)\times(0,T))$ for all $p\in[1,\frac{n+2}{n+1})$.

Since the estimates of R and S hold only in a small ball near boundary, we need a localization. For the localization, we take a cut off function ϕ such that $\phi = 1$ in B_r and $\phi = 0$ in the complement of B_{2r} and we consider $(\phi J, \phi\eta)$ as a solution to the inhomogeneous Stokes equations. We delete the generic point x in the various expressions. Therefore by Theorem 3.1 in [2], we obtain the following lemma for the perturbation (J, η).

Lemma

There is a constant C depending on r and Ω such that

$$\begin{aligned} ||J||_{L^p(0,T;W^{2,p}(\Omega\cap B_r))} &+ ||\eta||_{L^p(0,T;W^{1,p}(\Omega\cap B_r))} \\ &\leq C(1 + ||G^\Omega||_{L^p(0,T;W^{1,p}(\Omega\cap(B_{2r}\setminus B_r)))} + ||\theta^\Omega||_{L^p(\Omega\cap(B_{2} \end{aligned}$$

for all $p \in (1, \frac{n+2}{n+1})$.

By the trace theorem in $W^{1,p}(\Omega \cap B_r)$, the following lemma also holds.

Lemma

There is a constant C depending on r, Ω such that

$$||\nabla J||_{L^p(0,T;W^{1-\frac{1}{p},p}(\partial\Omega\cap B_r))} + ||\eta||_{L^p(0,T;W^{1-\frac{1}{p},p}(\partial\Omega\cap B_r)} \leq C\big(1 + ||G^\Omega||_{L^p(0,T;W^{1,p}(\Omega\cap(B_{2r}\setminus B_r)))} + ||\theta^\Omega||_{L^p(\Omega\cap(B_{2r}\setminus B_r)\times(0,T))}\big)$$

for all $p \in (1, \frac{n+2}{n+1})$.

The generic point x is in B_r and hence the Green's matrix $(G^\Omega, \theta^\Omega)$ has no singularity in the complement of B_r as a function of (y,t). Therefore we have that for all $p \in [1, \infty]$

$$||G^\Omega||_{L^p(0,T;W^{1,p}(\Omega\cap(B_{2r}\setminus B_r)))} + ||\theta^\Omega||_{L^p(\Omega\cap(B_{2r}\setminus B_r)\times(0,T))} \leq C$$

for a constant C depending only on p, r and Ω.

Now we prove our main theorem. The Poisson kernel $K^{\Omega}(x,y,t)$ satisfies

$$K^{\Omega}(x,y,t) = \frac{\partial}{\partial N(y)} G^{\Omega}(x,y,t) - \theta^{\Omega}(x,y,t) N(y), \quad \text{for all} \quad (x,y,t) \in \Omega \times \partial\Omega$$

We have that

$$G^{\Omega} = \mu G + J, \quad \theta^{\Omega} = \mu\theta + \eta$$

and from Lemma 4.3 we know that ∇J and η have bounded $L^1((\partial\Omega \cap B_r) \times (0,T))$ norms independent of x since $L^p(0,T; W^{1-\frac{1}{p},p}(\partial\Omega \cap B_r))$ for $p \in (1, \frac{n+2}{n+1})$ is embedded in $L^1((\partial\Omega \cap B_r) \times (0,T))$. So we need to consider only $\frac{\partial}{\partial N(y)} \mu G(x,y,t)$ and $\mu\theta(x,y,t)$. The $L^1((\partial\Omega \cap B_r) \times (0,T))$ bound of $\mu\theta(x,y,t)$ as a function of y' for the generic point $x = (0, x_n)$ can be obtained by Lemma 4.1 after considering coordinate transform μ.

From the definition of the transformation of μ and the local representation of the boundary $\partial\Omega$, we have that for $y = (y', y_n) = (y', \Phi(y')) \in \partial\Omega \cap B_r$

$$\begin{aligned}\frac{\partial}{\partial N(y)} G(x,y,t) =& \frac{1}{\sqrt{1+|\nabla'\Phi(y')|^2}} \nabla'_{y'} G(x, y', y_n - \Phi(y'), t) \cdot \nabla'_{y'} \Phi(y') \\ & - \frac{1}{\sqrt{1+|\nabla'\Phi(y')|^2}} \frac{\partial}{\partial y_n} G(x, y', y_n - \Phi(y'), t) \\ =& \frac{1}{\sqrt{1+|\nabla'\Phi(y')|^2}} \nabla'_{y'} G(x, y', 0, t) \cdot \nabla'_{y'} \Phi(y') \\ & - \frac{1}{\sqrt{1+|\nabla'\Phi(y')|^2}} \frac{\partial}{\partial y_n} G(x, y', 0, t).\end{aligned}$$

Furthermore, we have already proved that the $L_1((\partial\Omega \cap B_r) \times (0,T))$ norm estimate of the first term

$$\int_0^T \int_{\partial\Omega\cap B_r} \left| \frac{1}{\sqrt{1+|\nabla'\Phi(y')|^2}} \nabla'_{y'} G(x, y', 0, t) \cdot \nabla'_{y'} \Phi(y') \right| dy' dt \le C$$

for some C independent of x since $|\nabla'\Phi(y')| \le c|y'|$.

By the expression of Poisson kernel K we have

$$
\begin{aligned}
-\frac{\partial}{\partial y_n} G_{ij}(x,y',0,t) &= K_{ij}(x'-y',x_n,t) + \delta_{jn}\eta_i(x'-y',0,t) \\
&= -2\delta_{ij} D_{x_n}\Gamma(x'-y',x_n,t) + 4(L_{ij}(x'-y',x_n,t) - \delta_{jn}B_{in}(x'-y',x_n,t)) \\
&\quad + 4\delta_{jn}B_{in}(x'-y',x_n,t) - \delta_{jn}\delta(t)D_{x_i}E(x'-y',x_n)
\end{aligned}
$$

We know already that $-2\delta_{ij} D_{x_n}\Gamma(x'-y',x_n,t) + 4(L_{ij}(x'-y',x_n,t) - \delta_{jn}B_{in}(x'-y',x_n,t))$ has L^1 bounded norm as a function of (y',t). Therefore in the solution expression for of u we can write

$$
\begin{aligned}
&\int_0^t \int_{\partial\Omega\cap B_r} \delta_{jn}\delta(t-s)D_{x_i}E(x'-y',x_n)g_j(y,s)d\sigma_y \\
&\quad = \frac{\partial}{\partial x_i}\int_{\partial\Omega\cap B_r} E(x'-y',x_n)g_n(y,t)d\sigma_y
\end{aligned}
$$

and

$$
\begin{aligned}
&4\int_0^t \int_{\partial\Omega\cap B_r} \delta_{jn}B_{in}(x'-y',x_n,t-s)g_j(y,s)d\sigma_y \\
&\quad = \frac{\partial}{\partial x_i}\mathbf{T}(g_n)(x,t).
\end{aligned}
$$

If we denote $e_n = (0, 1)$ which is the coordinate vector for y_n, we have that the component of boundary data g_n is

$$g_n = g \cdot N(y) + g \cdot (e_n - N(y)) \quad \text{for} \quad y \in \partial\Omega \cap B_r$$

where $g = (g_1, g_2, \cdots, g_n)$ is the boundary data and hence we have

$$\nabla_x \int_{\partial\Omega\cap B_r} E(x' - y', x_n) g_n(y,t) d\sigma_y = \nabla_x \mathbf{S}(g \cdot N X^{\partial\Omega\cap B_r})$$
$$+\nabla_x \int_{\partial\Omega\cap B_r} (E(x' - y', x_n) - E(x' - y', x_n - \Phi(y')))g(y,t) \cdot (e$$
$$+\nabla_x \int_{\partial\Omega\cap B_r} E(x' - y', x_n) g(y,t) \cdot (e_n - N(y)) d\sigma_y,$$

where $\mathbf{S}$ is the single layer potential operator and X is the characteristic function. Since $x' = 0$ and $\Phi(y') \leq C|y'|^2$,

$$\int_{\partial\Omega\cap B_r} \left| \nabla_x (E(x' - y', x_n) - E(x' - y', x_n - \Phi(y'))) \right| d\sigma_y \leq C.$$

Then, by observing that

$$|e_n - N(y)| \leq C|y'|$$

we find that $\nabla_x E(x'-y', x_n)\cdot(e_n - N(y))$ has bounded L^1 norm as a function of y' and we have that

$$\sup_{x\in\Omega\cap B_r}\left|\nabla_x \int_{\partial\Omega\cap B_r} E(x'-y', x_n)g(y,t)\cdot(e_n - N(y))d\sigma_y\right| \leq C||g||_{L_\infty(\partial\Omega\times(0,T))}.$$

Similarly we find that $\nabla\kappa(x'-y', x_n, t)(e_n - N(y))$ has a bounded L^1 norm as a function of (y', t) and we have that

$$\sup_{(x,t)\in\Omega\cap B_r\times(0,T)}\left|\int_0^t\int_{\partial\Omega\cap B_r}\nabla\kappa(x'-y', x_n, t-s)g(y,s)\cdot(e_n - N(y))d\sigma_y ds\right|$$

$$\leq C||g||_{L^\infty(\partial\Omega\times(0,T))}.$$

With the interior L^∞ estimate, localization with the small balls B_r and the preceding kernel estimates of L^1 bound, we prove our main Theorem 1.1. For Corollary 1.2, we observe that

$$|\nabla_x \mathbf{S}(g \cdot N X^{\partial\Omega\cap B_r})| \leq C \int_{\Omega\cap B_r} \frac{1}{|y'|^{n-1}} |g(y', \Phi(y'), t) \cdot N(y', \Phi(y')| dy',$$

where $x = (0, x_n)$. Similarly, we also have

$$|\nabla_x \mathbf{T}(g \cdot N X^{\partial\Omega\cap B_r})| \leq C \int_0^T \int_{\Omega\cap B_r} \frac{1}{|y'|^{n-1}} |g(y', \Phi(y'), t) \cdot N(y', \Phi(y'))| dy' dt.$$

The boundedness follows from the Dini-continuity of $g \cdot N$.
□

Contents

§9. Helmholtz decomposition.

In this section we are interested in establishment of Helmholtz-Wleil decomposition in L^p for Lipschitz domain. In what follows, we shall make no notational distinction between scalar valued and vector valued function with components in $L^p(\Omega)$. We define the Besov space $\mathcal{B}^p_s(\partial\Omega)$ as the collection of all measurable function f on $\partial\Omega$ such that

$$||f||_{\mathcal{B}^p_s(\partial\Omega)} =: ||f||_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(P)-f(Q)|^p}{|P-Q|^{n-1+sp}} d\sigma(P) d\sigma(Q)\right)^{\frac{1}{p}}.$$

The case $p=\infty$ corresponds to the non-homogeneous version of the space of Holder continuous functions on $\partial\Omega$. We also define $\mathcal{B}^p_{-s}(\partial\Omega)$ as the dual of $\mathcal{B}^p_s(\partial\Omega)$ satisfying $\frac{1}{p}+\frac{1}{q}=1, 0<s<1, 1<p\leq\infty$. Define $L^q_s(\mathbf{R^n})$ be the Sobolev space and $L^q_s(\Omega)$ be the restriction of $L^q_s(\mathbf{R^n})$ on Ω.

Theorem

Suppose that $u \in L^p(\Omega)$ and $div\, u \in L^p_{\frac{2}{p}}(\Omega)$. Then $u \cdot N \in \mathcal{B}^p_{-\frac{1}{p}}(\partial\Omega)$.

Proof
(1)
We commence by noting that the pairing of $L^q_{-s+\frac{1}{q}}(\Omega)$ and $L^q_{s-1+\frac{1}{p}}(\Omega)$ is well defined for any $0 < s < 1, 1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.
(2)
In fact, since $C^\infty_{comp}(\Omega)$ is dense in $L^q_\alpha(\Omega)$ for $0 \leq \alpha < \frac{1}{q}$, it is not difficult to see that $L^q_{\alpha,0}(\Omega) = L^q_\alpha(\Omega)$ for $0 \leq \alpha < \frac{1}{q}$.
(3)
In particular,

$$L^q_{-s+\frac{1}{q}}(\Omega) = \left(L^p_{s-1+\frac{1}{p}}(\Omega)\right)^* \quad \text{if} \quad -s + \frac{1}{q} \leq 0$$

and

$$L^p_{s-1+\frac{1}{p}}(\Omega) = \left(L^q_{-s+\frac{1}{q}}(\Omega)\right)^* \quad \text{if} \quad 0 \leq -s + \frac{1}{q}.$$

(4)
For any extension

$$f \in \left(L^p_{s+\frac{1}{p}}(\Omega)\right)^* = L^q_{-s-\frac{1}{p},0}(\Omega)$$

of these distribution $div\, F \in \left(C^\infty_{comp}(\Omega)\right)'$, we denote by $F \cdot N_f$ the normal component of F and define it are the linear function in $\mathcal{B}^q_{-s}(\partial\Omega)$ by

$$< F \cdot N_f, \phi >:=< f, \widetilde{\phi} > + < F, \nabla\widetilde{\phi} >$$

for all $\phi \in \mathcal{B}^p_s(\partial\Omega)$, where $\widetilde{\phi} \in L^p_{s+\frac{1}{p}}(\Omega)$ is an extension of ϕ in the trace sense.
(5)
In particular, when $S = \frac{1}{q}, \widetilde{\phi} \in L^p_1(\Omega)$.
(6)
It remains to show that $\nabla\widetilde{\phi} \in L^q_{-s+\frac{1}{q}}$. But this follows from extension lemma by Je-Ke and duality lemma by FMM.

Remark 1
From the proof, we can deduce

$$||u \cdot N||_{\mathcal{B}^p_{-\frac{1}{p}}(\partial\Omega)} \leq c(\Omega, p)\left(||u||_{L^p(\Omega)} + ||div\, u||_{L^p(\Omega)}\right).$$

Remark 2
If $u \in L^p(\Omega)$ has $div\, u = 0$, then $u \cdot N$ as a functional in $\left(\mathcal{B}^p_{1-\frac{1}{q}}(\partial\Omega)\right)^*$, annihilates all functions of the form $\chi_{\partial\Omega'}$ with Ω' connected component of Ω. We denote the collection of all such functionals by $\widetilde{\mathcal{B}}^p_{-\frac{1}{p}}(\partial\Omega)$. We introduce

$$L^p_{div,0}(\Omega) := \{u \in L^p(\Omega) : div\, u = 0 \text{ and } u \cdot N = 0\}$$

$$grad\, L^p_1(\Omega) := \{\nabla u : u \in L^p_1(\Omega)\}.$$

They are easily seen to be closed subspaces of $L^p(\Omega)$ and , for $p = 2$, we denote $\mathcal{P}, \mathcal{D}$ the corresponding orthogonal projections from $L^2(\Omega)$ onto $L^2_{div,0}(\Omega)$ and $grad\, L^2_1(\Omega)$ respectively.

Theorem

For each Lipschitz domain Ω in $\mathbf{R^n}$, with arbitrary topology, there exists a positive number ϵ depending on Ω such that $\mathcal{P}, \mathcal{D}$ extend to bounded operators from $L^p(\Omega)$ onto $L^p_{div,0}(\Omega)$ and onto $grad\, L^p_1(\Omega)$, respectively, for each $\frac{3}{2} - \epsilon < p < 3 + \epsilon$. Hence in this range

$$L^p(\Omega) = grad L^p_1(\Omega) \oplus L^p_{div,0}(\Omega)$$

where the sum is topological. In the class of Lipschitz domain the result is sharp. If however $\partial\Omega \in C^1$ then we may take $1 < p < \infty$.

To prove our theorem, we need a lemma.

Lemma

For Ω bounded Lipschitz domain, there exists a positive number $\epsilon = \epsilon(\Omega)$ with the following significance. If $\frac{3}{2} - \epsilon < p < 3 + \epsilon$, then for any $f \in L^p_{-1,0}(\Omega)$ and any $g \in \mathcal{B}^p_{-1/p}(\partial\Omega)$ satisfy the compatibility condition $< f, 1 >=< g, 1 >$, the Neumann prblem

$$\begin{cases} \Delta u = f \quad \text{in} \quad \Omega \\ \frac{\partial u}{\partial N} = g \quad \text{on} \quad \partial\Omega \\ u \in L^p_1(\Omega) \end{cases}$$

has a unique (modulo additive constants) solution u. Recall the Neumann boundary condition is interpreted in the sense that

$$\int_\Omega \nabla u(x) \cdot \nabla \phi(x) dx = - < f, \phi > + < g, Tr\phi >$$

for any $\phi \in L^q_1(\Omega)$. Moreover, ∇u satisfies the estimate

$$||\nabla u||_{L^q(\Omega)} \le c(\Omega, p) \left(||f||_{L^p_{-1,0}(\Omega)} + ||g||_{\mathcal{B}^p_{\frac{1}{p}}(\partial\Omega)} \right)$$

Proof of Theorem

(1)

We let π_Ω stands for the Newtonian potential which acts componentwise on vector fields.

(2)

We define $\widetilde{\mathcal{P}} : L^p(\Omega) \to L^p_{div,0}(\Omega) \hookrightarrow L^p(\Omega)$ by setting

$\widetilde{\mathcal{P}}_u = u - \nabla div(\pi_\Omega(u)) - \nabla\psi$, for all $u \in L^p(\Omega)$ where ψ is the unique solution to the Neumann boundary value problem

$$\spadesuit \begin{cases} \Delta\psi = 0 \quad \text{in} \quad \Omega \\ \frac{\partial\psi}{\partial N} = [u - \nabla(div\pi_\Omega(u))] \cdot N \in \widetilde{\mathcal{B}}^p_{-\frac{1}{p}}(\partial\Omega) \\ \psi \in L^p_1(\Omega). \end{cases}$$

(3)

By assumption, $\widetilde{\mathcal{P}}$ is well defined, linear and bounded and moreover $I - \widetilde{P}$ maps $L^p(\Omega)$ boundedly into $gradL^p_1(\Omega)$.

(4)
Next we claim that $\widetilde{\mathcal{P}}$ is onto $L^p_{div,0}(\Omega)$. We need only to show that $\widetilde{\mathcal{P}}|_{L^p_{div,0}(\Omega)} = I$, the identity map on $L^p_{div,0}(\Omega)$.
(5)
If $u \in L^p_{div,0}(\Omega)$ and if ψ solve $(\spadesuit)$, then the function $\psi + div\pi_\Omega(u)$ is harmonic, belongs to $L^p_1(\Omega)$ and has vanishing normal derivative.
(6)
Invoking uniqueness for the Neumann problem, it follows that $\widetilde{\mathcal{P}}_u = u - \nabla div(\pi_\Omega(u)) - \nabla\psi = u$.
(7)
The fact that on $L^2(\Omega) \cap L^p(\Omega)$ the operator $\widetilde{\mathcal{P}}$ acts as the orthogonal projection onto $L^2_{div,0}(\Omega)$ is easily seen. Thus $\widetilde{\mathcal{P}}$ extends to a bounded mapping of $L^p(\Omega)$ onto $L^p_{div,0}(\Omega)$ as desired. From this the statement about $\mathcal{D} = I - \mathcal{P}$ follows as well.
(8)
Finally, the range $p \in (\frac{3}{2} - \epsilon, 3 + \epsilon)$ with ϵ, ensures the solvability of $(\spadesuit)$.

Remark

It is clear that for any $1 < p < \infty$, the L^p-helmholtz decomposition holds if and only if the projection $\mathcal{P}$ extends to a bounded operator on $L^p(\Omega)$. Since $\mathcal{P}$ is L^2 self-adjoint, the latter condition is also equivalent to $\mathcal{P}$ being extendible to a bounded operator on $L^q(\Omega)$ for $\frac{1}{p} + \frac{1}{q} = 1$. In particular, the L^q-Helmholtz decomposition is valid if and only if the L^p -Helmholtz decomposition is valid. In a similar manner we may consider a new function class $L^p_{div}(\Omega) := \{u \in L^p(\Omega) : divu = 0\}$.

Theroem

Let Ω be a bounded Lipschitz domain in $\mathbf{R^n}$, with arbitrary topology. Then there exists $\epsilon = \epsilon(\Omega) > 0$ such that $L^p(\Omega) = gradL^p_{1,0} \oplus L^p_{div}(\Omega)$ for each $\frac{3}{2} - \epsilon < p < 3 + \epsilon$, where the direct sum is topological. Once again, in the class of Lipschitz domain, the result is sharp. If however, $\partial\Omega \in C^1$, then we may take $1 < p < \infty$.

Proof
(1)
Here the departure point is to consider the operator

$$L^p(\Omega) \ni u \to \nabla(div\pi_\Omega(u) - \psi) \in gradL^p_{1,0}(\Omega),$$

where ψ is the unique solution to the Dirichlet problem

$$\spadesuit\spadesuit \begin{cases} \Delta\psi = 0 \\ Tr\psi = Tr(div\pi_\Omega(u)) \in \mathcal{B}^p_{1-\frac{1}{p}}(\partial\Omega) \\ \psi \in L^p_1(\Omega). \end{cases}$$

The solvability of ($\spadesuit\spadesuit$) is known and the remaining proof is the same as the proof of previous theorem.
counter Examples
Now we want to show the optimality of p.

Lemma

For any bounded Lipschitz domain Ω in $\mathbf{R^n}$ and $1 < q < \infty$ and any $\xi \in \mathcal{B}^q_{-\frac{1}{q}}(\partial\Omega)$, there exists a vector field $U \in L^q(\Omega)$ such that

$$divU \in L^q(\Omega), U \cdot N = \xi$$

and

$$||divU||_{L^q(\Omega)} + ||U||_{L^q(\Omega)} \leq c(\Omega, q)||\xi||_{\mathcal{B}^q_{-\frac{1}{q}}(\partial\Omega)}.$$

Theorem

For any $p \notin [\frac{3}{2}, 3]$, there exists a bounded Lipschitz domain Ω in $\mathbf{R^n}$ for which the L^p-Helmholtz decomposition fails.

Proof

(1)

First we claim that for any bounded Lipschitz domain Ω and any $1 < p \leq 2$, the validity of L^p-Helmholtz decomposition implies the solvability of the boundary value problem

$$\heartsuit \begin{cases} u \in L^p_1(\Omega) \\ \Delta u = g - \frac{1}{|\Omega|} < g, 1 > \\ \frac{\partial u}{\partial N} = 0, \ \int_\Omega u = 0 \end{cases}$$

for any $g \in L^p_{-1,0}(\Omega)$. To see this, we fix an arbitrary g and set $g' : g - (\frac{1}{|\Omega|} < g, 1 >)$.

(2)

We know that $\pi_\Omega(g') \in L^p_1(\Omega)$.

(3)

Since, by assumption, the L^p-Helmholtz decomposition holds, we can find unique $\psi \in L^p_1(\Omega)$ and $w \in L^p_{div,0}(\Omega)$ with norms controlled in terms of the $L^p(\Omega)$ norm of U and such that

$$U - \nabla \pi_\Omega(divU) = \nabla\phi + w.$$

(5)
Consider now $f : +\frac{\partial \pi_\Omega(divU)}{\partial N} \in L^p(\partial\Omega)$ and observe that $\int_{\partial\Omega} f d\sigma = 0$.
(6)
Since $1 < p \leq 2$ there exists a unique harmonic function v in Ω so that $\frac{\partial v}{\partial n} = f$ and such that the nontangential maximal function of ∇v lies in $L^p(\partial\Omega)$.
(7)
In particular, $h := \pi_\Omega(g') - \phi - v \in L^p_1(\Omega)$ and we see easily that $u := h - \frac{1}{|\Omega|}\int_\Omega h$ solves $(\heartsuit)$.
(8)
Estimates and uniqueness follows from the previous uniqueness result and this complete the claim.
(9)
Let now $T : \left(L^2_1(\Omega)\right)^* = L^2_{-1,0}(\Omega) \to L^2_1(\Omega)$ be the solution operator of $(\heartsuit)$, mapping from g to u. Clearly this is well defined linear and bounded.

(10)
Moreover, by Green's formula, T also satisfies

$$< g, Tg_2 >=< g_2, Tg_1 >, g_1 g_2 \in L2_{-1,0}(\Omega).$$

(11)
From what we have proved so for, the solvability of the boundary problem $(\heartsuit)$ for some $p \in (1, 2]$ implies that T above extends to a bounded operator from $L^p_{-1,0}(\Omega)$ into $L^p_1(\Omega)$. Given (10), we can further conclude that under the same hypothesis, T also extends as a bounded mapping of $L^q_{-1,0}(\Omega)$ into $L^q_1(\Omega)$, where $q \geq 2$ is the conjugate exponent of p.
(12)
However, given $q > 3$, there is a bounded(cone like) Lipschitz domain Ω in $\mathbf{R^n}$ and a function $u \in L^2_1$ such that

$$\Delta u \in C^\infty(\bar{\Omega}), \ \frac{\partial u}{\partial N} = 0 \quad \text{but} \quad u \notin L^q_1.$$

(13)
In the light of our disscussion, this implies the failure of L^p-Helmholtz decomposition for $1 < p < \frac{3}{2}$ on such domains.

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Thank you for your attention!