Mathematical analysis of the stationary motion of an incompressible viscous fluid

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Contents

The stationary Navier-Stokes equations

• Let Ω be a bounded or exterior domain in $\mathbf{R}^3$ with boundary $\partial\Omega$.

• The motion of an incompressible homogeneous viscous Newtonian fluid in Ω is described by the following nonlinear system of partial differential equations, named after Navier (1822) and Stokes (1845):

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f} & \text{in } \Omega\times(0,\infty) \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega\times(0,\infty). \end{cases}$$

• Notations:

$\mathbf{f} = (f^1(x,t), f^2(x,t), f^3(x,t))$: the external force

$\nu > 0$: the viscosity constant

$\mathbf{v} = (v^1(x,t), v^2(x,t), v^3(x,t))$: the (unknown) velocity

$p = p(x,t)$: the (unknown) pressure

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \left(\sum_{i=1}^{3} v^i \frac{\partial}{\partial_{x_i}}\right)\mathbf{v} = (\mathbf{v}\cdot\nabla v^1, \mathbf{v}\cdot\nabla v^2, \mathbf{v}\cdot\nabla v^3)$$

The stationary Navier-Stokes equations

- Assume that

$$\mathbf{f}(x,t) \to \mathbf{f}_\infty(x) = \operatorname{div} \mathbf{F}_\infty(x) \quad \text{as } t \to \infty$$

for some tensor field $\mathbf{F}_\infty$. Then the flow fields $\mathbf{v}$ and p will be stabilized for large time t, i.e.,

$$\mathbf{v}(x,t) \to \mathbf{v}_\infty(x), \quad p(x,t) \to p_\infty(x) \quad \text{as } \; t \to \infty.$$

- The limiting fields $\mathbf{v}_\infty$ and p_∞ satisfy the *stationary Navier-Stokes equations*:

$$\begin{cases} -\nu \Delta \mathbf{v}_\infty + (\mathbf{v}_\infty \cdot \nabla)\mathbf{v}_\infty + \nabla p_\infty = \operatorname{div} \mathbf{F}_\infty & \text{in } \Omega \\ \operatorname{div} \mathbf{v}_\infty = 0 & \text{in } \Omega. \end{cases}$$

- We may impose the Dirichlet (or no-slip) boundary condition on the limiting velocity $\mathbf{v}_\infty$:

$$\mathbf{v}_\infty(x) = \mathbf{0} \quad \text{for all } x \in \partial\Omega$$

- If Ω is an exterior domain, we also need the asymptotic behavior of $\mathbf{v}_\infty$ at infinity:

$$\mathbf{v}_\infty(x) \to \mathbf{c} \quad \text{as } |x| \to \infty,$$

where $\mathbf{c}$ is a constant vector.

The stationary Navier-Stokes equations

- Consider the Dirichlet problem for the stationary *Navier-Stokes equations*:

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \operatorname{div}\mathbf{F} & \text{in } \Omega \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega \end{cases} \qquad \text{(NS)}$$

if Ω is bounded, and

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \operatorname{div}\mathbf{F} & \text{in } \Omega \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega \\ \mathbf{v}(x) \to \mathbf{c} & \text{as } |x| \to \infty \end{cases} \qquad \text{(NS)}$$

if Ω is exterior.

Here

$$\nu > 0,\ \mathbf{F} : \Omega \to \mathbf{R}^{3\times 3},\ \mathbf{c} \in \mathbf{R}^3 \text{ are given;}$$

$$\mathbf{v} : \Omega \to \mathbf{R}^3,\ p : \Omega \to \mathbf{R} \text{ are unknowns.}$$

For simplicity, let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$.

Standard function spaces Let $1 < q < \infty$.

- Lebesgue spaces:

$$\|f\|_q = \|f\|_{q;\Omega} = \left[\int_\Omega |f(x)|^q \, dx\right]^{1/q},$$

$$L^q(\Omega) = \{f : \Omega \to \mathbf{R} \,|\, \|f\|_q < \infty\},$$

$$\mathbf{L}^q(\Omega) = [L^q(\Omega)]^3 \text{ or } [L^q(\Omega)]^{3\times 3}.$$

Remark. If $1 < q_1 < q_2 < \infty$, then

$$\mathbf{L}^{q_2}(\Omega) \hookrightarrow \mathbf{L}^{q_1}(\Omega)$$

$$\Updownarrow$$

$$\mathbf{L}^{q_2}(\Omega) \subset \mathbf{L}^{q_1}(\Omega) \quad \text{and} \quad \|\mathbf{v}\|_{q_1} \leq C(\Omega)\,\|\mathbf{v}\|_{q_2}.$$

- Sobolev spaces:

$$\mathbf{W}^{1,q}(\Omega) = \{\mathbf{v} \in \mathbf{L}^q(\Omega) \,|\, \nabla \mathbf{v} \in \mathbf{L}^q(\Omega)\},$$

$$\mathbf{W}_0^{1,q}(\Omega) = \left\{\mathbf{v} \in \mathbf{W}^{1,q}(\Omega) \,|\, \mathbf{v} = \mathbf{0} \ \text{ on } \partial\Omega\right\},$$

$$\mathbf{W}_{0,\sigma}^{1,q}(\Omega) = \left\{\mathbf{v} \in \mathbf{W}_0^{1,q}(\Omega) \,|\, \operatorname{div} \mathbf{v} = 0 \ \text{ in } \ \Omega\right\}.$$

Remark. By the Poincaré inequality, $\mathbf{W}_0^{1,q}(\Omega)$ is a Banach space equipped with norm $\|\nabla \cdot \|_q$.

- Spaces of test functions:

$$\mathbf{C}_0^\infty(\Omega) = \{\Phi \in \mathbf{C}^\infty(\Omega) \,|\, \Phi \text{ has compact support in } \Omega\},$$

$$\mathbf{C}_{0,\sigma}^\infty(\Omega) = \{\Phi \in \mathbf{C}_0^\infty(\Omega) \,|\, \operatorname{div} \Phi = 0 \ \text{ in } \Omega\}.$$

Remark. $\mathbf{C}_0^\infty(\Omega)$ is dense in $\mathbf{W}_0^{1,q}(\Omega)$ and $\mathbf{C}_{0,\sigma}^\infty(\Omega)$ is dense in $\mathbf{W}_{0,\sigma}^{1,q}(\Omega)$.

Definition of q-weak solutions

- Let $(\mathbf{v}, p)$ be a smooth solution of (NS). Then for all $\Phi \in \mathbf{C}_0^\infty(\Omega)$,

$$\int_\Omega (-\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p)\cdot\Phi\,dx = \int_\Omega \operatorname{div}\mathbf{F}\cdot\Phi\,dx$$

and so

$$\int_\Omega (\nu\nabla\mathbf{v} : \nabla\Phi + (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\Phi)\;dx - \int_\Omega p\operatorname{div}\Phi\,dx = -\int_\Omega \mathbf{F} : \nabla\Phi\,dx.$$

Moreover, since $\operatorname{div}\mathbf{v} = 0$ in Ω, we have

$$\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\Phi\,dx = \int_\Omega \operatorname{div}(\mathbf{v}\otimes\mathbf{v})\cdot\Phi\,dx = -\int_\Omega \mathbf{v}\otimes\mathbf{v} : \nabla\Phi\,dx,$$

where

$$\mathbf{v}\otimes\mathbf{v} = \left[v^i v^j\right]_{i,j=1,2,3}.$$

Definition. A pair $(\mathbf{v}, p)$ is called a q-*weak solution* of (NS) if

$$\mathbf{v} \in \mathbf{W}^{1,q}_{0,\sigma}(\Omega), \quad p \in L^q(\Omega)$$

and

$$\int_\Omega (\nu \nabla \mathbf{v} - \mathbf{v} \otimes \mathbf{v}) : \nabla \Phi \, dx - \int_\Omega p \operatorname{div} \Phi \, dx = -\int_\Omega \mathbf{F} : \nabla \Phi \, dx.$$

for all $\Phi \in \mathbf{C}^\infty_0(\Omega)$. A 2-weak solution of (NS) will be called simply a *weak solution*.

The fundamental L^2-result of J. Leray

Theorem. [Leray, 1933]

(a) (Existence) For each $\mathbf{F} \in \mathbf{L}^2(\Omega)$, there exists at least one weak solution of (NS).

(b) (Uniqueness) There is a small number $\delta = \delta(\Omega) > 0$ such that if $\mathbf{F} \in \mathbf{L}^2(\Omega)$ satisfies $\|\mathbf{F}\|_2 \le \delta \nu^2$, then there exists at most one weak solution of (NS).

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- Ingredients of the proof.

(i) A trilinear estimate: for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{W}_0^{1,2}(\Omega)$,

$$\int_\Omega |\mathbf{u} \otimes \mathbf{v} : \nabla \mathbf{w}| \, dx \le \|\mathbf{u}\|_4 \|\mathbf{v}\|_4 \|\nabla \mathbf{w}\|_2 \le C(\Omega) \|\nabla \mathbf{u}\|_2 \|\nabla \mathbf{v}\|_2 \|\nabla \mathbf{w}\|_2.$$

(ii) The energy (in)equality: If $(\mathbf{v}, p)$ is any weak solution of (NS), then

$$\int_\Omega (\nu \nabla \mathbf{v} - \mathbf{v} \otimes \mathbf{v}) : \nabla \Phi \, dx - \int_\Omega p \, \mathrm{div}\, \Phi \, dx = - \int_\Omega \mathbf{F} : \nabla \Phi \, dx.$$

for all $\Phi \in \mathbf{W}_0^{1,2}(\Omega)$. Taking $\Phi = \mathbf{v} \in \mathbf{W}_{0,\sigma}^{1,2}(\Omega)$, we have

$$\int_\Omega \left(\nu |\nabla \mathbf{v}|^2 - \mathbf{v} \otimes \mathbf{v} : \nabla \mathbf{v} \right) \, dx = - \int_\Omega \mathbf{F} : \nabla \mathbf{v} \, dx.$$

Note that

$$\int_\Omega \mathbf{v} \otimes \mathbf{v} : \nabla \mathbf{v} \, dx = \int_\Omega \mathbf{v} \cdot \nabla \left(\frac{1}{2} |\mathbf{v}|^2 \right) \, dx = 0.$$

Hence

$$\nu \int_\Omega |\nabla \mathbf{v}|^2 \, dx = - \int_\Omega \mathbf{F} : \nabla \mathbf{v} \, dx \quad \Rightarrow \quad \nu \|\nabla \mathbf{v}\|_2 \le \|\mathbf{F}\|_2.$$

- *Proof of the uniqueness*: Let $(\mathbf{v}_1, p_1)$ and $(\mathbf{v}_2, p_2)$ be weak solutions of (NS). Then

$$(\mathbf{v}, p) = (\mathbf{v}_1 - \mathbf{v}_2, p_1 - p_2) \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \times L^2(\Omega)$$

satisfies

$$\int_\Omega (\nu \nabla \mathbf{v} - \mathbf{v}_1 \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{v}_2) : \nabla \Phi \, dx - \int_\Omega p \operatorname{div} \Phi \, dx = 0$$

for all $\Phi \in \mathbf{W}^{1,2}_0(\Omega)$. Taking $\Phi = \mathbf{v}$, we have

$$\int_\Omega \left(\nu |\nabla \mathbf{v}|^2 - \mathbf{v} \otimes \mathbf{v}_2 : \nabla \mathbf{v} \right) \, dx = 0$$

$$\Downarrow$$

$$\nu \|\nabla \mathbf{v}\|_2^2 \le \int_\Omega \mathbf{v} \otimes \mathbf{v}_2 : \nabla \mathbf{v} \, dx \le C \|\nabla \mathbf{v}_2\|_2 \|\nabla \mathbf{v}\|_2^2.$$

$$\Downarrow$$

$$\|\nabla \mathbf{v}\|_2 = 0, \quad \text{provided that } C\|\nabla \mathbf{v}_2\|_2 < \nu.$$

• *Proof of the existence*: The Navier-Stokes equations can be regarded as a perturbation of the Stokes equations:

$$\begin{cases} -\nu\Delta \mathbf{v} + \nabla p = \operatorname{div}(\mathbf{F} - \mathbf{v}\otimes\mathbf{v}) & \text{in } \Omega \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega. \end{cases}$$

The perturbation term

$$B(\mathbf{v}) = \mathbf{v}\otimes\mathbf{v}$$

satisfies

(i) Mapping property and continuity of B:

$$B : \mathbf{W}_0^{1,2}(\Omega) \to \mathbf{L}^2(\Omega),$$

$$\|B(\mathbf{v}_1) - B(\mathbf{v}_2)\|_2 \le C\left(\|\nabla\mathbf{v}_1\|_2 + \|\nabla\mathbf{v}_2\|_2\right)\|\mathbf{v}_1 - \mathbf{v}_2\|_4;$$

(ii) Compactness of B:

$$\mathbf{v}_k \to \mathbf{v} \text{ weakly in } \mathbf{W}_0^{1,2}(\Omega) \quad \Rightarrow \quad B(\mathbf{v}_k) \to B(\mathbf{v}) \text{ strongly in } \mathbf{L}^2(\Omega).$$

Therefore, applying the Leray-Schauder principle, we can deduce that (NS) has a weak solution. The necessary a priori estimate follows from the energy equality.

Let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary $\partial\Omega$.

The linear L^q-result of L. Cattabriga

Theorem. [Cattabriga, 1961] *Let $1 < q < \infty$. Then for every $\mathbf{F} \in \mathbf{L}^q(\Omega)$, there exists a unique q-weak solution $(\mathbf{v}, p)$ of the Stokes problem*

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla p = \operatorname{div}\mathbf{F} & \text{in } \Omega \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \qquad (S)$$

Why an L^q-result for (NS)?

- More regular solutions $(\mathbf{v}, p)$: For $2 < q < \infty$,

$$\mathbf{W}^{1,q}_{0,\sigma}(\Omega) \times L^q(\Omega) \hookrightarrow \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \times L^2(\Omega).$$

- More general data $\mathbf{F}$: For $1 < q < 2$,

$$\mathbf{L}^2(\Omega) \hookrightarrow \mathbf{L}^q(\Omega).$$

- An essential approach to many important problems, for instance, the nonlinear exterior problems.

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L^q-results for bounded domains

A restriction on q:

(NS) is an $\mathbf{L}^q$-bounded perturbation of (S) in the sense that

$$B(\mathbf{v}) = \mathbf{v} \otimes \mathbf{v} \in \mathbf{L}^q(\Omega) \quad \text{for all } \mathbf{v} \in \mathbf{W}_0^{1,q}(\Omega)$$

$$\Updownarrow$$

$$\mathbf{W}_0^{1,q}(\Omega) \hookrightarrow \mathbf{L}^{2q}(\Omega)$$

$$\Updownarrow$$

$$\frac{1}{2q} \geq \frac{1}{q} - \frac{1}{3} \quad \Leftrightarrow \quad \frac{3}{2} \leq q$$

Note that if $3/2 \leq q < \infty$, then

$$\|B(\mathbf{v}_1) - B(\mathbf{v}_2)\|_q \leq C\left(\|\nabla \mathbf{v}_1\|_q + \|\nabla \mathbf{v}_2\|_q\right)\|\mathbf{v}_1 - \mathbf{v}_2\|_q$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{W}_0^{1,q}(\Omega)$. Hence applying the contraction mapping principle, we easily obtain

Lemma. *Let $3/2 \leq q < \infty$. Then there is a small number $\delta_0 = \delta_0(q, \Omega) > 0$ such that for each $\mathbf{F} \in \mathbf{L}^q(\Omega)$ with $\|\mathbf{F}\|_q \leq \delta_0 \nu^2$, there exists at least one q-weak solution (v, p) of (NS) satisfying*

$$\nu\|\nabla v\|_q + \|p\|_q \leq C\|\mathbf{F}\|_q.$$

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$$\nu\|\nabla v\|_q + \|p\|_q \leq C\|\mathbf{F}\|_q.$$

An L^q-result for smooth bounded domains

Theorem. *Let $3/2 \leq q < \infty$.*

(a) *(Existence) For each $\mathbf{F} \in \mathbf{L}^q(\Omega)$, there exists at least one q-weak solution of* (NS).

(b) *(Uniqueness) There is a small number $\delta = \delta(q, \Omega) > 0$ such that if $\mathbf{F} \in \mathbf{L}^q(\Omega)$ satisfies $\|\mathbf{F}\|_q \leq \delta\nu^2$, then there exists at most one q-weak solution of* (NS).

Remark. This theorem has been proved by

$$\begin{cases} q = 2: \ \text{Leray (1933)} \\ 2 < q < \infty: \ \text{Cattabriga (1961)} \\ \dfrac{3}{2} < q < 2: \ \text{Serre (1983)} \\ q = \dfrac{3}{2}: \ \text{Galdi-Sohr-Simader (2005), Kim (2009).} \end{cases}$$

Remark. Let $3/2 < q < \infty$. Then B is compact from $\mathbf{W}_0^{1,q}(\Omega)$ to $\mathbf{L}^q(\Omega)$. Hence the existence would follow from the Leray-Schauder principle if we could derive a priori estimates. This is possible due to the energy equality if $q \geq 2$.

An L^q-result for non-smooth bounded domains

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$.

Theorem. [Shen, 1995] *There is a number* $\varepsilon \in (0, 1]$ *satisfying the following property: Let* $3/(2+\varepsilon) < q < 3/(1-\varepsilon)$*. Then for every* $\mathbf{F} \in \mathbf{L}^q(\Omega)$*, there exists a unique* q*-weak solution of* (S)*.*

Remark. If Ω is a C^1-domain, then $\varepsilon = 1$.

Theorem. [Choe-Kim, 2011] *Let* $3/2 \le q < 3/(1-\varepsilon)$*.*

(a) *(Existence) For every* $\mathbf{F} \in \mathbf{L}^q(\Omega)$*, there exists at least one* q*-weak solution of* (NS)*.*

(b) *(Uniqueness) There is a small number* $\delta = \delta(q, \Omega) > 0$ *such that if* $\mathbf{F} \in \mathbf{L}^q(\Omega)$ *satisfies* $\|\mathbf{F}\|_q \le \delta\nu^2$*, then there exists at most one* q*-weak solution of* (NS)*.*

Remark. In fact, we obtained a similar result for the nonhomogeneous Dirichlet data in $\mathbf{L}^2(\partial\Omega)$, which extends a famous result due to Fabes, Kenig and Verchota (1988) for the Stokes equations.

• *Proof of the existence for* $3/2 \le q < 2$: Write $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, where $\|\mathbf{F}_1\|_q \le \delta_0 \nu^2$ and $\mathbf{F}_2 \in \mathbf{L}^2(\Omega)$. Then the problem

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \operatorname{div}\mathbf{F}_1 & \text{in } \Omega \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega \end{cases}$$

has a q-weak solution $(\mathbf{v}_1, p_1)$ satisfying

$$\nu\|\mathbf{v}_1\|_{3q/(3-q)} + \nu\|\nabla\mathbf{v}_1\|_q + \|p_1\|_q \le C\|\mathbf{F}_1\|_q.$$

It is easy to show that $(\mathbf{v}, p) = (\mathbf{v}_1 + \mathbf{v}_2, p_1 + p_2)$ is a q-weak solution of (NS), where $(\mathbf{v}_2, p_2)$ is given by

Lemma. *There is a small number* $\delta_2 = \delta_2(\Omega) > 0$ *such that if* $\|\mathbf{v}_1\|_3 \le \delta_2\nu$, *then the nonlinear problem*

$$\begin{cases} -\nu\Delta\mathbf{v} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v} + \mathbf{v}_1\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{v}_1) + \nabla p = \operatorname{div}\mathbf{F}_1 & \textit{in } \Omega \\ \operatorname{div}\mathbf{v} = 0 & \textit{in } \Omega \\ \mathbf{v} = \mathbf{0} & \textit{on } \partial\Omega \end{cases}$$

has at least one weak solution $(\mathbf{v}_2, p_2)$.

• *Proof of the uniqueness for* $q = 3/2$: If $\|\mathbf{F}\|_{3/2} \le \delta_0 \nu^2$, then there is a $3/2$-weak solution (v_2, p_2) of (NS) satisfying

$$\nu \|v_2\|_3 + \nu \|\nabla v_2\|_{3/2} + \|p_2\|_{3/2} \le C_0 \|\mathbf{F}\|_{3/2}.$$

Let $(\mathbf{v}_1, p_1)$ be any $3/2$-weak solutions of (NS), which is possibly different from (v_2, p_2). Then

$$(\mathbf{v}, p) = (\mathbf{v}_1 - \mathbf{v}_2, p_1 - p_2) \in \mathbf{W}^{1,3/2}_{0,\sigma}(\Omega) \times L^{3/2}(\Omega)$$

satisfies

$$\int_\Omega (\nu \nabla \mathbf{v} - \mathbf{v}_1 \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{v}_2) : \nabla \Phi \, dx - \int_\Omega p \,\mathrm{div}\, \Phi \, dx = 0$$

for all $\Phi \in \mathbf{C}^\infty_0(\Omega)$; hence

$$\int_\Omega \left(\nu \nabla \Phi : \nabla \mathbf{v} - (\mathbf{v}_1 \cdot \nabla)\Phi \cdot \mathbf{v} - (\mathbf{v}_2 \cdot \nabla \Phi) \cdot \mathbf{v} \right) dx = 0$$

for all $\Phi \in \mathbf{W}^{1,3}_{0,\sigma}(\Omega)$.

Lemma. *There is a small number* $\delta_1 = \delta_1(\Omega) > 0$ *such that if* $\|\mathbf{v}_2\|_3 \le \delta_1 \nu$, *then for every* $\mathbf{G} \in \mathbf{L}^3(\Omega)$, *there exists a unique* 3*-weak solution* (Φ, ψ) *of the dual problem*

$$\begin{cases} -\nu\Delta\Phi - (\mathbf{v}_1 \cdot \nabla)\Phi - \mathbf{v}_2 \cdot \nabla\Phi + \nabla\psi = \operatorname{div}\mathbf{G} & \textit{in } \Omega \\ \operatorname{div}\Phi = 0 & \textit{in } \Omega \\ \Phi = \mathbf{0} & \textit{on } \partial\Omega. \end{cases}$$

(*Completion of the proof*) Suppose that $\nu^{-2}\|\mathbf{F}\|_{3/2} \le \delta \equiv \min(\delta_0, \delta_1/C_0)$. Then for each $\mathbf{G} \in \mathbf{L}^3(\Omega)$,

$$-\int_\Omega \mathbf{G} : \nabla\mathbf{w}\, dx = \int_\Omega \left(\nu\nabla\Phi : \nabla\mathbf{w} - (\mathbf{v}_1 \cdot \nabla)\Phi \cdot \mathbf{w} - (\mathbf{v}_2 \cdot \nabla\Phi) \cdot \mathbf{w}\right)\, dx$$

for all $\mathbf{w} \in \mathbf{W}^{1,3/2}_{0,\sigma}(\Omega)$. Taking $\mathbf{w} = \mathbf{v}$, we have

$$-\int_\Omega \mathbf{G} : \nabla\mathbf{v}\, dx = \int_\Omega \left(\nu\nabla\Phi : \nabla\mathbf{v} - (\mathbf{v}_1 \cdot \nabla)\Phi \cdot \mathbf{v} - (\mathbf{v}_2 \cdot \nabla\Phi) \cdot \mathbf{v}\right)\, dx = 0.$$

An open problem

Note that

$$\mathbf{v} \otimes \mathbf{v} \in \mathbf{L}^1(\Omega) \quad \text{for all } \mathbf{v} \in \mathbf{W}_0^{1,q}(\Omega)$$

$$\Updownarrow$$

$$\mathbf{W}_0^{1,q}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$$

$$\Updownarrow$$

$$\frac{1}{2} \geq \frac{1}{q} - \frac{1}{3} \quad \Leftrightarrow \quad \frac{6}{5} \leq q$$

Therefore, q-weak solutions of (NS) can be defined even for $6/5 \leq q < 3/2$.

Problem. *Prove the existence and/or uniqueness of q-weak solutions of (NS) even for small $\|\mathbf{F}\|_q$, in case when $6/5 \leq q < 3/2$.*

- Let Ω be an exterior smooth domain in $\mathbf{R}^3$, and consider the exterior problem for the Navier-Stokes equations:

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \operatorname{div}\mathbf{F} & \text{in } \Omega \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega \\ \mathbf{v}(x) \to \mathbf{c} & \text{as } |x| \to \infty. \end{cases} \qquad \text{(NS)}$$

- An obvious difficulty due to the unboundedness of Ω:

$$\|\mathbf{u}\|_r \le C\|\nabla\mathbf{u}\|_2 \ \text{ for all } \mathbf{u} \in \mathbf{C}_0^\infty(\Omega) \quad \Leftrightarrow \quad r = 6.$$

More generally,

$$\|\mathbf{u}\|_r \le C\|\nabla\mathbf{u}\|_q \ \text{ for all } \mathbf{u} \in \mathbf{C}_0^\infty(\Omega) \quad \Leftrightarrow \quad 1 \le q < 3, \ \ r = q^* = \frac{3q}{3-q}.$$

- Homogeneous Sobolev spaces:

$$\mathbf{D}^{1,2}(\Omega) = \left\{\mathbf{u} \in \mathbf{L}^2_{loc}(\Omega) \,|\, \nabla\mathbf{u} \in \mathbf{L}^2(\Omega)\right\}.$$

Lemma. *For each $\mathbf{u} \in \mathbf{D}^{1,2}(\Omega)$, there exists a unique constant vector $\mathbf{u}_\infty$ such that*

$$\mathbf{u} - \mathbf{u}_\infty \in \mathbf{L}^6(\Omega) \quad \textit{and} \quad \|\mathbf{u} - \mathbf{u}_\infty\|_6 \le C(\Omega)\|\nabla\mathbf{u}\|_2.$$

Definition. Let $\mathbf{F} \in \mathbf{L}^2(\Omega)$. Then a pair $(\mathbf{v}, p)$ is called a *weak solution* of (NS) if

$$\mathbf{v} \in \mathbf{D}^{1,2}(\Omega), \quad \mathbf{v} - \mathbf{c} \in \mathbf{L}^6(\Omega), \quad p \in L^2_{loc}(\Omega),$$

$$\operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega$$

and

$$\int_\Omega (\nu \nabla \mathbf{v} : \nabla \Phi + (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \Phi)\, dx - \int_\Omega p \operatorname{div} \Phi \, dx = - \int_\Omega \mathbf{F} : \nabla \Phi \, dx.$$

for all $\Phi \in \mathbf{C}_0^\infty(\Omega)$.

The fundamental L^2-result of J. Leray

Theorem. [Leray, 1933] *For each $\mathbf{F} \in \mathbf{L}^2(\Omega)$, there exists at least one weak solution $(\mathbf{v}, p)$ of (NS) satisfying the energy inequality.*

Open questions left by Leray

• It has been long open to show that if $(\mathbf{v}, p)$ is a weak solution of (NS), then $p - p_\infty \in L^2(\Omega)$ for some constant p_∞. To understand this, assume that $\mathbf{c} = \mathbf{0}$. Then

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) \quad \text{and} \quad \mathbf{v}\otimes\mathbf{v} \in \mathbf{L}^3(\Omega) \hookrightarrow \mathbf{L}^2_{loc}(\Omega).$$

• It has been also open to show that every weak solution of (NS) satisfies the energy (in-)equality. An obvious difficulty is the failure of the trilinear estimate:

$$\int_\Omega |(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}|\, dx \le C\|\nabla\mathbf{u}\|_2\|\nabla\mathbf{v}\|_2\|\nabla\mathbf{w}\|_2 \quad \forall\ \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{C}_0^\infty(\Omega).$$

• Therefore, it remains still open to establish the uniqueness of Leray's weak solutions under sole smallness conditions on $\|\mathbf{F}\|_2$ and $|\mathbf{c}|$.

Remark. It turns out that the case $\mathbf{c} = \mathbf{0}$ is much more difficult.

An L^2-result for exterior domains with $\mathbf{c} = \mathbf{0}$

- Weak Lebesgue spaces: For $1 < q < \infty$,

$$\mathbf{L}^q_{weak}(\Omega) = \left\{ \mathbf{v} : [\mathbf{v}]_q = \sup_{t>0} t|\{x \in \Omega : |\mathbf{v}(x)| > t\}|^{1/q} < \infty \right\}.$$

Theorem. [Kim-Kozono, 2012]. *Assume that* $\mathbf{c} = \mathbf{0}$. *Then there is a small number* $\delta = \delta(\Omega) > 0$ *such that if* $\mathbf{F}$ *satisfies*

$$\mathbf{F} \in \mathbf{L}^{3/2}_{weak}(\Omega) \cap \mathbf{L}^2(\Omega) \quad \textit{and} \quad [\mathbf{F}]_{3/2} \leq \delta\nu^2,$$

then there exists a unique weak solution $(\mathbf{v}, p)$ *of* (NS) *satisfying the energy inequality*

$$\nu \int_\Omega |\nabla \mathbf{v}|^2 \, dx \leq - \int_\Omega \mathbf{F} : \nabla \mathbf{v} \, dx.$$

Remark. It remains still open to show that every weak solution of (NS) satisfies the energy (in-)equality even for suitably small $\mathbf{F}$. Hence the uniqueness of weak solutions of (NS) remains an interesting open question.

Remark. Our result improves that of Galdi and Simader (1994) who assumed that $\|(1 + |x|)^2\mathbf{F}\|_\infty$ is sufficiently small.

An open problem for $\mathbf{c} = \mathbf{0}$

Theorem. [Kozono-Yamazaki, 1998]. *Assume that* $\mathbf{c} = \mathbf{0}$. *Then there is a small number* $\delta = \delta(\Omega) > 0$ *such that if* $\mathbf{F}$ *satisfies*

$$\mathbf{F} \in \mathbf{L}^{3/2}_{weak}(\Omega) \quad \textit{and} \quad [\mathbf{F}]_{3/2} \leq \delta\nu^2,$$

then there exists at least one solution $(\mathbf{v}, p)$ *of* (NS) *satisfying*

$$\nabla\mathbf{v} \in \mathbf{L}^{3/2}_{weak}(\Omega), \quad \mathbf{v} \in \mathbf{L}^{3}_{weak}(\Omega), \quad p \in L^{3/2}_{weak}(\Omega)$$

and

$$\nu[\mathbf{v}]_3 + \nu[\nabla\mathbf{v}]_{3/2} + [p]_{3/2} \leq C(\Omega)[\mathbf{F}]_{3/2}.$$

Remark. It is impossible to replace $\mathbf{L}^{3/2}_{weak}(\Omega)$ by $\mathbf{L}^{3/2}(\Omega)$ even for linear Stokes equations or Laplace equation on exterior domains. Hence the theorem is a right substitute of the corresponding result for $3/2$-weak solutions of the Navier-Stokes equations in bounded domains.

Problem. *Assume that* $\mathbf{c} = \mathbf{0}$. *Prove the uniqueness of solutions* $(\mathbf{v}, p)$ *of* (NS) *satisfying*

$$\nabla \mathbf{v} \in \mathbf{L}^{3/2}_{weak}(\Omega), \quad \mathbf{v} \in \mathbf{L}^{3}_{weak}(\Omega), \quad p \in L^{3/2}_{weak}(\Omega)$$

for $\mathbf{F}$ *small in* $\mathbf{L}^{3/2}_{weak}(\Omega)$.

To attach the problem, we need to study the perturbed Stokes problem

$$\begin{cases} -\nu \Delta \mathbf{v} + \operatorname{div}(\mathbf{v}_1 \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{v}_2) + \nabla p = \mathbf{0} & \text{in } \Omega \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{L}^3_{weak}(\Omega)$ and $[\mathbf{v}_2]_3$ is small.

Classical L^q-estimates for the Oseen equations in $\mathbf{R}^3$

Consider the whole space problem for the Oseen equations:

$$\left\{ \begin{aligned} -\nu\Delta\mathbf{v} + (\mathbf{c}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f} \quad &\text{in } \mathbf{R}^3 \\ \operatorname{div}\mathbf{v} = 0 \quad &\text{in } \mathbf{R}^3 \\ \mathbf{v}(x) \to \mathbf{0} \quad &\text{as } |x| \to \infty, \end{aligned} \right. \qquad (\mathrm{O})_{\mathbf{R}^3}$$

where $\mathbf{c}$ is a nonzero constant vector.

By a suitable rotation, we may assume that

$$\mathbf{c} = h\,\mathbf{e}_1 = (h, 0, 0) \quad \text{and} \quad h = |\mathbf{c}| > 0.$$

Taking the Fourier transform, we then obtain a solution $(\mathbf{v}, p)$ of $(\mathrm{O})_{\mathbf{R}^3}$ given by

$$\hat{\mathbf{v}} = \frac{1}{\nu|\xi|^2 + ih\xi_1}\left(I - \frac{\xi\otimes\xi}{|\xi|^2}\right)\hat{\mathbf{f}} \quad \text{and} \quad \hat{p} = -i\frac{\xi\cdot\hat{\mathbf{f}}}{|\xi|^2}.$$

Therefore, applying the Lizorkin multiplier theorem, K. I. Babenko and G. Galdi were able to derive the following L^q-estimates:

Theorem. [Babenko, 1973; Galdi, 1991]. *Assume that* $h = |\mathbf{c}| > 0$.

(a) *For* $1 < q < 2$, *there is a constant* $C = C(q) > 0$ *such that*

$$\left(\frac{h}{\nu}\right)^{\frac{1}{2}} \|\mathbf{v}\|_{\frac{2q}{2-q};\mathbf{R}^3} + \left(\frac{h}{\nu}\right)^{\frac{1}{4}} \|\nabla \mathbf{v}\|_{\frac{4q}{4-q};\mathbf{R}^3} + \nu\|\nabla^2 \mathbf{v}\|_{q;\mathbf{R}^3} + \|\nabla p\|_{q;\mathbf{R}^3} \leq C\|\mathbf{f}\|_{q;\mathbf{R}^3}.$$

(b) *Assume that* $\mathbf{f} = \operatorname{div} \mathbf{F}$. *Then for* $1 < q < 4$, *there is a constant* $C = C(q) > 0$ *such that*

$$\left(\frac{h}{\nu}\right)^{\frac{1}{4}} \|\mathbf{v}\|_{\frac{4q}{4-q};\mathbf{R}^3} + \nu\|\nabla \mathbf{v}\|_{q;\mathbf{R}^3} + \|p\|_{q;\mathbf{R}^3} \leq C\|\mathbf{F}\|_{q;\mathbf{R}^3}.$$

Remark Using this result together with Sobolev's inequality, we have

$$\left(\frac{h}{\nu}\right)^{\frac{1}{2}} \|\mathbf{v}\|_{3;\mathbf{R}^3} + \nu\|\nabla \mathbf{v}\|_{2;\mathbf{R}^3} + \|p\|_{2;\mathbf{R}^3} \leq C\|\mathbf{f}\|_{6/5;\mathbf{R}^3}$$

and

$$\left(\frac{h}{\nu}\right)^{\frac{1}{4}} \|\mathbf{v}\|_{4;\mathbf{R}^3} + \nu\|\nabla \mathbf{v}\|_{2;\mathbf{R}^3} + \|p\|_{2;\mathbf{R}^3} \leq C\|\mathbf{F}\|_{2;\mathbf{R}^3}.$$

An L^2-result for exterior domains with $\mathbf{c} \neq \mathbf{0}$

Theorem. [Heck-Kim-Kozono, 2013]. *Assume that* $\mathbf{c} \neq \mathbf{0}$ *and* $\mathbf{F} \in \mathbf{L}^2(\Omega)$. *Then every weak solution* $(\mathbf{v}, p)$ *of* (NS) *satisfies the following properties:*

(a) *(additional integrability near infinity)*

$$\mathbf{v} \in \mathbf{L}^4(\Omega), \quad p - p_\infty \in L^2(\Omega),$$

where p_∞ *is some constant;*

(b) *(generalized energy equality)*

$$\nu \int_\Omega \nabla \mathbf{v} : \nabla(\mathbf{v} - \mathbf{A})\, dx + \int_\Omega (\mathbf{v} \cdot \nabla)\mathbf{A} \cdot (\mathbf{v} - \mathbf{A})\, dx = -\int_\Omega \mathbf{F} : \nabla(\mathbf{v} - \mathbf{A})\, dx,$$

where $\mathbf{A}$ *is any smooth vector field such that*

$$\mathbf{A} = \mathbf{0} \ \text{ near } \Omega^c, \quad \mathbf{A} = \mathbf{c} \ \text{ near infinity}, \quad \operatorname{div} \mathbf{A} = 0 \ \text{ in } \mathbf{R}^3;$$

(c) *(uniqueness)* $(\mathbf{v}, p)$ *is the unique weak solution of* (NS) *provided that*

$$\frac{1}{\nu}\|\mathbf{F}\|_2 + |\mathbf{c}| \leq \delta\, (\nu |\mathbf{c}|)^{1/2}$$

for some small number $\delta = \delta(\Omega) > 0$.

Remark. (i) Similar results were obtained by Galdi (1991) and Farwig-Sohr (1994) assuming that

$$\mathbf{f} = \operatorname{div} \mathbf{F} \in \mathbf{L}^{6/5}(\Omega) \cap \mathbf{L}^{3/2}(\Omega).$$

The key step of both Galdi and Farwig-Sohr is to prove the following additional integrability of $(\mathbf{v}, p)$ near infinity:

$$\mathbf{v} \in \mathbf{L}^3(\Omega) \quad \text{and} \quad p - p_\infty \in L^2(\Omega).$$

To do so, Galdi used a bootstrap argument originally due to Babenko (1973) while Farwig-Sohr developed an interesting functional analytic approach. Our proof is based on the approach of Farwig-Sohr, with the L^q-estimates for the linear Oseen equations being fully utilized.

(ii) Applying a degree theory, Galdi (2007) proved the existence of weak solutions of (NS) satisfying (a) and (b). Our result shows that Galdi's weak solutions in fact coincide with Leray's ones.