Cloaking due to anomalous localized resonance in plasmonic structures

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- Introduction
- Neumann-Poincaré operator
- Spectral characterization of CALR
- Spectrum of the NP operator on confocal ellipses
- CALR on confocal ellipses

- The dielectric problem: for a given compactly supported function f, $\int_{\mathbb{R}^2} f dx = 0$,

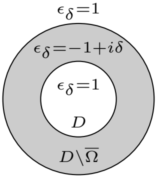
$$\nabla \cdot \epsilon_\delta \nabla V_\delta = f \quad \text{in} \quad \mathbb{R}^2 ,$$

with decay condition $V_\delta(x) \to 0$ as $|x| \to \infty$.

- The distribution of the dielectric constant (permittivity) ϵ_δ is given by

$$\epsilon_\delta = \begin{cases} 1 & \text{in} \quad \mathbb{R}^2 \setminus \overline{\Omega} , \\ -1 + i\delta & \text{in} \quad \Omega \setminus \overline{D} , \\ 1 & \text{in} \quad D \end{cases}$$

for a loss parameter $\delta > 0$.



- In quasistatic regime, the time averaged electromagnetic power produced by the source dissipated into heat approximately

$$E_\delta := \Im \int_{\mathbb{R}^2} \epsilon_\delta |\nabla V_\delta|^2 dx = \int_{\Omega \setminus D} \delta |\nabla V_\delta|^2 dx$$

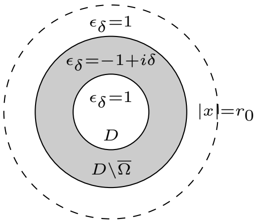
Problem of the cloaking by anomalous localized resonance (CALR):

- Characterizing the source f such that the following two conditions are satisfied:

$$\lim_{\delta \to 0} E_\delta = \infty \quad \text{as} \quad \delta \to 0\,, \tag{1}$$

and

$$|V_\delta(x)/\sqrt{E_\delta}| \to 0 \quad \text{as} \quad \delta \to 0 \quad \text{when} \quad |x| > r_0\,. \tag{2}$$



Cloaking by anomalous localized resonance:

- (1) implies that an infinite amount of energy dissipated per unit time in the limit $\delta \to 0$
- Scaling the solution V_δ by $1/\sqrt{E_\delta}$, the energy dissipation becomes constant (independent of δ)
- (2) says that the scaled solution approaches zero as $\delta \to 0$, so the effect of the source becomes negligible. (the source is essentially invisible)
- *weak* CARL: limits in (1, 2) are replaced by limsup.

The recent results and motivations

- The anomalous localized resonance was first discovered by Nicorovici, McPhedran, and Milton (Optical and dielectric properties of partially resonant composites, PRB, 1994)
- ALR is related invisibility cloaking, Milton and Nocorovici (On the cloaking effects associated with anomalous localized resonance, PRSA, 2006)
- The condition (1) results the solution V_δ oscillates very rapidly as δ tends to zero.
  - Numerical simulations[NMM, MN] shows that the oscillation takes places near the interfaces $\partial\Omega$ and ∂D .
- In [MN], if the core and the shell are concentric disks of radii r_i and r_e, and f is a polarizable dipole located at y $(f(x) = a \cdot \nabla\delta(x-y))$, then there is a critical radius

$$r_* = \sqrt{r_e^3/r_i}$$

such that for y outside r_*.

- In the recent paper of Ammari, Ciraolo, Kang, Lee, and Milton (Spectral theory of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance, ARMA, 2013), a general method to analyze CALR has been developed
  - The condition (1) can characterized in terms of the spectrum of the Neumann-Poincaré type operator associated with the double interface dielectric equation.
  - Using spectral characterization, for concentric disks, it has been shown that CALR does not occur for any source f supported outside $r_* = \sqrt{r_e^3/r_i}$.
  - Furthermore, if the Newtonian potential of f satisfies an additional mild condition on its Fourier coefficients, then CALR take place.

- These results were extended in (Kohn, Lu, Schweizer and Weinstein, A variational perspective on cloaking by anomalous localized resonance, preprint) to the case when the core is not radial by using variational approach with assumption f is supported on circles.
- *The circular structure seems the only known coated structure where CALR occurs*, and it it of interest to find such a structure other than CALR takes place.

- The single layer potential on a function $\varphi \in L^2(\Gamma)$ is defined by

$$S_\Gamma[\varphi](x) = \frac{1}{2\pi} \int_\Gamma \ln|x-y|\varphi(y)d\sigma(t), \quad x \in \mathbb{R}^2 .$$

- The Neumann-Poincaré operator on Γ is defined by

$$\mathcal{K}_\Gamma^*[\varphi](x) = \frac{1}{2\pi} \int_\Gamma \frac{\langle x-y, \nu(x)\rangle}{|x-y|^2} \varphi(y)\, d\sigma(y), \quad x \in \Gamma .$$

- The solution V_δ to the dielectric problem involves two interfaces: $\Gamma_i := \partial D$ and $\Gamma_e := \partial\Omega$ can be represented as

$$V_\delta(x) = F(x) + \mathcal{S}_{\Gamma_i}[\varphi_i](x) + \mathcal{S}_{\Gamma_e}[\varphi_e](x) \quad x \in \mathbb{R}^2 ,$$

  for a pair of potentials $(\varphi_i, \varphi_e) \in \mathcal{H}_0 (= L_0^2(\Gamma_i) \times L_0^2(\Gamma_e))$, where F is the Newtonian potential of the source term f.

- From the continuity of the flux along interfaces, (φ_i, φ_e) satisfies

$$(z_\delta I + \mathbb{K}^*) \begin{bmatrix} \varphi_i \\ \varphi_e \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial \nu_i} \\ -\frac{\partial F}{\partial \nu_e} \end{bmatrix},$$

where

$$z_\delta = \frac{i\delta}{2(2 - i\delta)} \quad \text{and} \quad \mathbb{K}^* = \begin{bmatrix} -\mathcal{K}^*_{\Gamma_i} & -\frac{\partial}{\partial \nu_i}\mathcal{S}_{\Gamma_e} \\ \frac{\partial}{\partial \nu_e}\mathcal{S}_{\Gamma_e} & \mathcal{K}^*_{\Gamma_e} \end{bmatrix}$$

This operator $\mathbb{K}^* : \mathcal{H} \to \mathcal{H}(= L^2(\Gamma_i) \times L^2(\Gamma_e))$ is the Neumann-Poincaré-type operator associated with interfaces problem with two interface Γ_i and Γ_e.

- Defining the operator $\mathbb{S} : \mathcal{H} \to \mathcal{H}$ by

$$\mathbb{S} = \begin{bmatrix} \mathcal{S}_{\Gamma_i} & \mathcal{S}_{\Gamma_e} \\ \mathcal{S}_{\Gamma_i} & \mathcal{S}_{\Gamma_e} \end{bmatrix}.$$

- $-\mathbb{S}$ is positive-definite on $\mathcal{H}_0$.

- $\mathbb{K}^*$ is self-adjoint and compact assuming that Γ_i and Γ_e are $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$ on $\mathcal{H}_0$ with inner product

$$\langle \varphi, \psi \rangle_{\mathbb{S}} = -\langle \varphi, \mathbb{S}[\psi] \rangle, \quad \varphi, \psi \in \mathcal{H}_0 \,.$$

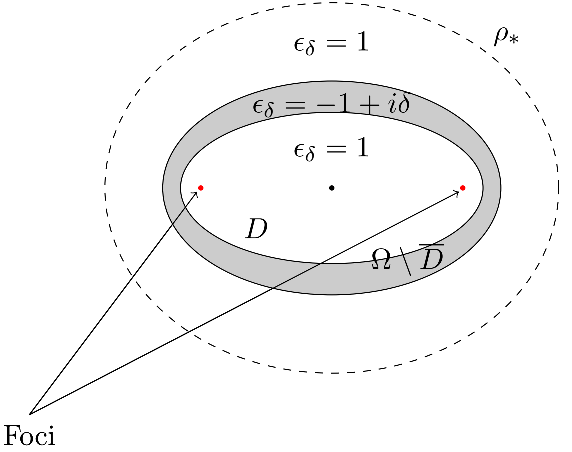
- Suppose $\ker \mathbb{K}^* = \{0\}$ and let $\lambda_1,\ \lambda_2, \cdots \ (|\lambda_1| \geq |\lambda_2| \geq \cdots)$ be the nonzero eigenvalues of $\mathbb{K}^*$ and Ψ_n be the corresponding normalized eigenfunctions. Then

$$\begin{bmatrix} \varphi_i \\ \varphi_e \end{bmatrix} = \sum_n \frac{\langle g, \Psi_n \rangle_{\mathbb{S}}}{\lambda_n + z_\delta} \Psi_n, .$$

- The spectral characterization [ACKLM]:

$$E_\delta \approx \delta \sum_n \frac{|\langle g, \Psi_n \rangle_{\mathbb{S}}|^2}{\lambda_n^2 + \delta^2} \,.$$

A geometric structure of the CALR in the confocal ellipses:



- The elliptic coordinates (ρ, ω) for $x = (x_1, x_2)$ are defined by

$$x_1 = R\cos\omega\cosh\rho, \quad x_2 = R\sin\omega\sinh\rho, \quad \rho > 0,\ 0 \le \Omega \le 2\pi\,.$$

In this coordinate, $E = \{(\rho, \omega) : \rho = \rho_0\}$ is an ellipse whose foci are $(\pm R, 0)$. The length element and the normal derivative on E are given by

$$d\sigma = \Xi d\omega \quad \text{and} \quad \frac{\partial}{\partial\nu} = \Xi^{-1}\frac{\partial}{\partial\nu}\,,$$

where $\Xi = \Xi(\rho_0, \omega) = R\sqrt{\sinh^2\rho_0 + \sin^2\omega}\,.$

- Let $E = \{(\rho, \omega) \,:\, \rho = \rho_0\}$, for a harmonic polynomial given by $h(x) = \cos n\omega (e^{n\rho} + e^{-n\rho})$,

$$\mathcal{S}_E[\nabla h \cdot \nu](x) = \begin{cases} (\alpha_n - \frac{1}{2})(e^{n\rho} + e^{-n\rho}) \cos n\omega, & \rho \le \rho_0 \\ \beta_n e^{-n\rho} \cos n\omega \,, & \rho > \rho_0 \,, \end{cases}$$

where

$$\alpha_n = \frac{1}{2e^{2n\rho_0}} \quad \text{and} \quad \beta_n = \frac{-e^{2n\rho_0} + e^{-2n\rho_0}}{2} \,.$$

Similarly, $h(x) = \sin n\omega (e^{n\rho} - e^{-n\rho})$,

$$\mathcal{S}_E[\nabla h \cdot \nu](x) = \begin{cases} (-\alpha_n - \frac{1}{2})(e^{n\rho} - e^{-n\rho}) \sin n\omega, & \rho \le \rho_0 \\ \beta_n e^{-n\rho} \sin n\omega \,, & \rho > \rho_0 \,. \end{cases}$$

- From the jump formula of the single layer potential:

$$\mathcal{K}_E^*[\Xi^{-1}\cos n\omega] = \alpha_n \Xi^{-1}\cos n\omega$$
$$\mathcal{K}_E^*[\Xi^{-1}\sin n\omega] = -\alpha_n \Xi^{-1}\sin n\omega\,.$$

Lemma

α_n and $-\alpha_n$ are eigenvalues of the NP operator $\mathcal{K}_E^*$ on the ellipse $E = \{(\rho,\omega) : \rho = \rho_0\}$ and corresponding eigenfunctions are $\Xi(\rho_0,\omega)^{-1}\cos n\omega$ and $\Xi(\rho_0,\omega)^{-1}\sin n\omega\,.$

Assuming ∂D and $\partial\Omega$ are confocal ellipses with common foci $(\pm R, 0)$.

$$\partial D = \Gamma_i = \{(\rho,\omega) \,:\, \rho = \rho_i\} \quad \text{and} \quad \partial\Omega = \Gamma_e = \{(\rho,\omega) \,:\, \rho = \rho_e\}\,.$$

For $k = i, e$ and nonnegative integers n,

$$\phi_n^{ck}(\omega) = \Xi(\rho_k,\omega)^{-1}\cos n\omega, \quad \phi_n^{sk}(\omega) = \Xi(\rho_k,\omega)^{-1}\sin n\omega\,.$$

- for $k = i, e$,

$$\mathcal{S}_{\Gamma_k}[\phi_n^{ck}](x) = \begin{cases} -\dfrac{e^{n\rho}+e^{-n\rho}}{2ne^{n\rho_k}}\cos n\omega, & \rho \le \rho_k\,, \\ -\dfrac{e^{n\rho_k}+e^{-n\rho_k}}{2ne^{n\rho}}\cos n\omega, & \rho > \rho_k \end{cases}$$

$$\mathcal{S}_{\Gamma_k}[\phi_n^{sk}](x) = \begin{cases} -\dfrac{e^{n\rho}-e^{-n\rho}}{2ne^{n\rho_k}}\sin n\omega, & \rho \le \rho_k\,, \\ -\dfrac{e^{n\rho_k}-e^{-n\rho_k}}{2ne^{n\rho}}\sin n\omega, & \rho > \rho_k \end{cases}$$

Also, we have

$$\frac{\partial}{\partial \nu_i}\mathcal{S}_{\Gamma_e}[\phi_n^{ce}] = -\frac{e^{n\rho_i} - e^{-n\rho_i}}{2e^{n\rho_e}}\phi_n^{ci},$$

$$\frac{\partial}{\partial \nu_e}\mathcal{S}_{\Gamma_i}[\phi_n^{ci}] = \frac{e^{n\rho_i} + e^{-n\rho_i}}{2e^{n\rho_e}}\phi_n^{ce},$$

$$\frac{\partial}{\partial \nu_i}\mathcal{S}_{\Gamma_e}[\phi_n^{se}] = -\frac{e^{n\rho_i} + e^{-n\rho_i}}{2e^{n\rho_e}}\phi_n^{si},$$

$$\frac{\partial}{\partial \nu_e}\mathcal{S}_{\Gamma_i}[\phi_n^{si}] = \frac{e^{n\rho_i} - e^{-n\rho_i}}{2e^{n\rho_e}}\phi_n^{se}.$$

From these formulas,

$$\mathbb{K}^* \begin{bmatrix} a\phi_n^{ci} \\ b\phi_n^{ce} \end{bmatrix} = \begin{bmatrix} \phi^{ci} & 0 \\ 0 & \phi_n^{ce} \end{bmatrix} A \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbb{K}^* \begin{bmatrix} a\phi_n^{si} \\ b\phi_n^{se} \end{bmatrix} = \begin{bmatrix} \phi^{si} & 0 \\ 0 & \phi_n^{se} \end{bmatrix} B \begin{bmatrix} a \\ b \end{bmatrix},$$

where

$$A = \begin{bmatrix} -\dfrac{1}{2e^{2n\rho_i}} & \dfrac{e^{n\rho_i} - e^{-n\rho_i}}{2e^{n\rho_e}} \\ \dfrac{e^{n\rho_i} + e^{-n\rho_i}}{2e^{n\rho_e}} & \dfrac{1}{2e^{2n\rho_e}} \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{1}{2e^{2n\rho_i}} & \dfrac{e^{n\rho_i} + e^{-n\rho_i}}{2e^{n\rho_e}} \\ \dfrac{e^{n\rho_i} - e^{-n\rho_i}}{2e^{n\rho_e}} & -\dfrac{1}{2e^{2n\rho_e}} \end{bmatrix}$$

Lemma

The eigenvalues of $\mathbb{K}^*$ *are* $\pm\lambda_{1,n}$ *and* $\pm\lambda_{2,n}$ $(n = 0, 1, 2, \dots)$ *where*

$$\lambda_{1,n} = \frac{1}{4}\left(e^{-2n\rho_e} - e^{-2n\rho_i} - \sqrt{(e^{-2n\rho_e} - e^{-2n\rho_i})^2 + 4e^{-2n(\rho_e-\rho_i)}}\right),$$

$$\lambda_{2,n} = \frac{1}{4}\left(e^{-2n\rho_e} - e^{-2n\rho_i} + \sqrt{(e^{-2n\rho_e} - e^{-2n\rho_i})^2 + 4e^{-2n(\rho_e-\rho_i)}}\right),$$

and eigenfunctions (not normalized) corresponding to $\lambda_{1,n}, -\lambda_{1,n}, \lambda_{2,n}, -\lambda_{2,n}$ *are, respectively,*

$$\Psi_n^{1+} = \begin{bmatrix} a_{1,n}\phi_n^{ci} \\ b_n\phi_n^{ce} \end{bmatrix}, \quad \Psi_n^{1-} = \begin{bmatrix} b_n\phi_n^{si} \\ a_{2,n}\phi_n^{se} \end{bmatrix}, \quad \Psi_n^{2+} = \begin{bmatrix} a_{2,n}\phi_n^{ci} \\ b_n\phi_n^{ce} \end{bmatrix}, \quad \Psi_n^{2-} = \begin{bmatrix} b_n\phi_n^{si} \\ a_{1,n}\phi_n^{se} \end{bmatrix},$$

where

$$a_{1,n} = e^{-2n\rho_e} + e^{-2n\rho_i} + \sqrt{(e^{-2n\rho_e} - e^{-2n\rho_i})^2 + 4e^{-2n(\rho_e-\rho_i)}},$$

$$a_{2,n} = e^{-2n\rho_e} + e^{-2n\rho_i} - \sqrt{(e^{-2n\rho_e} - e^{-2n\rho_i})^2 + 4e^{-2n(\rho_e-\rho_i)}},$$

$$b_n = -2e^{-n(\rho_e-\rho_i)}(1 + e^{-2n\rho_i}).$$

- $\Psi_n^{1\pm}, \Psi_n^{2\pm}$, $n = 1, 2, \ldots$, are orthogonal to each other with respect to the inner product $\langle \ , \ \rangle_{\mathbb{S}}$, and

$$\langle \Psi_n^{1+}, \Psi_n^{1+} \rangle_{\mathbb{S}} = \frac{\pi}{n} \left(a_{1,n}^2 e^{-n\rho_i} \cosh n\rho_i + 2a_{1,n} b_n e^{-n\rho_e} \cosh n\rho_i + b_n^2 e^{-n\rho_e} \cosh n\rho_e \right),$$

$$\langle \Psi_n^{1-}, \Psi_n^{1-} \rangle_{\mathbb{S}} = \frac{\pi}{n} \left(b_n^2 e^{-n\rho_i} \sinh n\rho_i + 2a_{2,n} b_n e^{-n\rho_e} \sinh n\rho_i + a_{2,n}^2 e^{-n\rho_e} \sinh n\rho_e \right),$$

$$\langle \Psi_n^{2+}, \Psi_n^{2+} \rangle_{\mathbb{S}} = \frac{\pi}{n} \left(a_{2,n}^2 e^{-n\rho_i} \cosh n\rho_i + 2a_{2,n} b_n e^{-n\rho_e} \cosh n\rho_i + b_n^2 e^{-n\rho_e} \cosh n\rho_e \right),$$

$$\langle \Psi_n^{2-}, \Psi_n^{2-} \rangle_{\mathbb{S}} = \frac{\pi}{n} \left(b_n^2 e^{-n\rho_i} \sinh n\rho_i + 2a_{1,n} b_n e^{-n\rho_e} \sinh n\rho_i + a_{1,n}^2 e^{-n\rho_e} \sinh n\rho_e \right).$$

Asymptotic behaviors as n tends to ∞.

- Γ_e and Γ_i are sufficiently close to each other $((\rho_e - \rho_i) \leq 2\rho_i)$:

$$\sqrt{(e^{-2n\rho_e} - e^{-2n\rho_i})^2 + 4e^{-2n(\rho_e-\rho_i)}} = 2e^{-n(\rho_e-\rho_i)} + o(e^{-n(\rho_e-\rho_i)}).$$

It then follows that

$$\lambda_{1,n} \sim -e^{-n(\rho_e-\rho_i)}, \quad \lambda_{2,n} \sim e^{-n(\rho_e-\rho_i)},$$

and

$$a_{1,n} \sim e^{-n(\rho_e-\rho_i)}, \quad a_{2,n} \sim -e^{-n(\rho_e-\rho_i)}, \quad b_n \sim -e^{-n(\rho_e-\rho_i)}.$$

- On the other hand $(2\rho_i < \rho_e - \rho_i)$:

$$\sqrt{(e^{-2n\rho_e} - e^{-2n\rho_i})^2 + 4e^{-2n(\rho_e-\rho_i)}} = e^{-2n\rho_i} + 2e^{-2n(\rho_e-2\rho_i)} + o(e^{-2n(\rho_e-2\rho_i)}),$$

so that

$$\lambda_{1,n} \sim -e^{-2n\rho_i}, \quad \lambda_{2,n} \sim e^{-2n(\rho_e-2\rho_i)},$$

and

$$a_{1,n} \sim e^{-2n\rho_i}, \quad a_{2,n} \sim -e^{-2n(\rho_e-2\rho_i)}, \quad b_n \sim -e^{-n(\rho_e-\rho_i)}.$$

Lemma

(i) *If* $\rho_e \leq 3\rho_i$, *then*

$$\langle \Psi_n^{1\pm}, \Psi_n^{1\pm} \rangle_{\mathbb{S}}, \ \langle \Psi_n^{2\pm}, \Psi_n^{2\pm} \rangle_{\mathbb{S}} \sim n^{-1} e^{-2n(\rho_e - \rho_i)}.$$

(ii) *If* $\rho_e > 3\rho_i$, *then*

$$\langle \Psi_n^{1+}, \Psi_n^{1+} \rangle_{\mathbb{S}}, \ \langle \Psi_n^{2-}, \Psi_n^{2-} \rangle_{\mathbb{S}} \sim n^{-1} e^{-4n\rho_i},$$

and

$$\langle \Psi_n^{1-}, \Psi_n^{1-} \rangle_{\mathbb{S}}, \ \langle \Psi_n^{2+}, \Psi_n^{2+} \rangle_{\mathbb{S}} \sim n^{-1} e^{-2n(\rho_e - \rho_i)}.$$

- Assume that the source f is located outside Ω, and write the Newtonian potential of f as

$$F(x) = c - \sum_{n\geq 1} (F_n^+ \cos n\omega \cosh n\rho + F_n^- \sin n\omega \sinh n\rho) \,,$$

- The series on RHS converges in $0 < \rho < \rho_0$ iff

$$\limsup_{n\to\infty} |F_n^\pm|^{1/n} \leq e^{-\rho_0} \,.$$

- Since f is located outside Ω the series converges in $0 < \rho < \rho_e$, and we infer that

$$\limsup_{n\to\infty} |F_n^\pm|^{1/n} \leq e^{-\rho_e}.$$

- *Gap condition* on $\{F_n^\pm\}$ for confocal ellipses: The sequence $F_n^\pm$ is said to satisfy the gap condition GC[ρ_*] for some constant ρ_* if

GC[ρ_*]: there exists a sequence $\{n_k\}$ with $n_1 < n_2 < \cdots$ such that

$$\lim_{k\to\infty} e^{-(n_{k+1}-n_k)(\rho_e-\rho_i)} e^{2n_k\rho_*} (|F_{n_k}^+|^2 + |F_{n_k}^-|^2) = \infty.$$

Theorem (Main result)

Let f be the source function supported in $\mathbb{R}^2 \setminus \overline{\Omega}$ and F be the Newtonian potential of f. Let

$$\rho_* = \begin{cases} \dfrac{3\rho_e - \rho_i}{2} & \text{if } \rho_e \leq 3\rho_i, \\ 2(\rho_e - \rho_i) & \text{if } \rho_e > 3\rho_i. \end{cases}$$

(i) *If F does not extend as a harmonic function in $\{\rho < \rho_*\}$, then*

$$\limsup_{\delta \to 0} E_\delta = \infty$$

and there is C independent of δ such that $|V_\delta(x)| \leq C$ for all x satisfying $\rho \geq \rho_0$ provided that $\rho_0 > 2\rho_e - \rho_i$ if $\rho_e \leq 3\rho_i$ and $\rho_0 > 3\rho_e - 4\rho_i$ if $\rho_e > 3\rho_i$. So weak CALR takes place.

(ii) *If, in addition, the coefficients $F_n^{\pm}$ of F satisfy $GC[\rho_*]$, then CALR takes place, i.e.,*

$$\lim_{\delta \to 0} E_\delta = \infty.$$

(iii) *If f is supported outside ρ_* (so that F extends as a harmonic function in $\{\rho \leq \rho_*\}$), then there is a constant C such that $E_\delta \leq C$ for all δ.*

Remarks

- GC$[\rho_*]$ is a condition on the gap $(n_{k+1} - n_k)$ among nonzero coefficients $F_n^{\pm}$.
- If f is a dipole source $(f(x) = a \cdot \nabla \delta_{x_0}(x))$ for some x_0, then $F(x) = a \cdot \nabla_x G(x - x_0)$ satisfies GC$[\rho_*]$ if $\rho_0 < \rho$ (the source x_0 is located inside ρ_*), where $G(x) = \frac{1}{2\pi} \ln |x|$.
- It is interesting to observe that if we put $\rho_e = \ln r_e$ and $\rho_i = \ln r_i$, then
$$\frac{3\rho_e - \rho_i}{2} = \ln \sqrt{r_e^3/r_1},$$
so the thin case is similar to the circular case.
- It is not known whether there is a source f for which only the weak CALR, not CALR, takes place.

Sketch of proof ($\rho_e \le 3\rho_i$): g is given by

$$g := \begin{bmatrix} \frac{\partial F}{\partial \nu_i} \\ -\frac{\partial F}{\partial \nu_e} \end{bmatrix} = \sum_{n \ge 1} \begin{bmatrix} nF_n^+ \sinh n\rho_i \phi_n^{ci} \\ -nF_n^+ \sinh n\rho_e \phi_n^{ce} \end{bmatrix} + \sum_{n \ge 1} \begin{bmatrix} nF_n^- \cosh n\rho_i \phi_n^{si} \\ -nF_n^- \cosh n\rho_e \phi_n^{se} \end{bmatrix}.$$

We have

$$\langle g, \Psi_n^{1+} \rangle_{\mathbb{S}} \sim F_n^+ e^{n\rho_i}, \quad \langle g, \Psi_n^{1-} \rangle_{\mathbb{S}} \sim F_n^- e^{n\rho_i},$$
$$\langle g, \Psi_n^{2+} \rangle_{\mathbb{S}} \sim F_n^+ e^{n\rho_i}, \quad \langle g, \Psi_n^{2-} \rangle_{\mathbb{S}} \sim -F_n^- e^{n\rho_i}.$$

Since

$$\begin{bmatrix}\varphi_i\\ \varphi_e\end{bmatrix} = \sum_{k=1,2}\sum_n \left[\frac{\langle g, \Psi_n^{k+}\rangle_{\mathbb{S}}}{(\lambda_{k,n}+z_\delta)\langle \Psi_n^{k+}, \Psi_n^{k+}\rangle_{\mathbb{S}}}\Psi_n^{k+} + \frac{\langle g, \Psi_n^{k-}\rangle_{\mathbb{S}}}{(-\lambda_{k,n}+z_\delta)\langle \Psi_n^{k-}, \Psi_n^{k-}\rangle_{\mathbb{S}}}\Psi_n^{k-}\right].$$

We have

$$\begin{aligned}
|\mathcal{S}_{\Gamma_i}[\varphi_i](x)| &\le \sum_{k=1,2}\sum_n \left|\frac{a_{k,n}\langle g, \Psi_n^{k+}\rangle_{\mathbb{S}}}{(\lambda_{k,n}+z_\delta)\langle \Psi_n^{k+}, \Psi_n^{k+}\rangle_{\mathbb{S}}}\mathcal{S}_{\Gamma_i}[\phi_n^{ci}](x)\right| \\
&\quad + \sum_{k=1,2}\sum_n \left|\frac{b_n\langle g, \Psi_n^{k-}\rangle_{\mathbb{S}}}{(-\lambda_{k,n}+z_\delta)\langle \Psi_n^{k-}, \Psi_n^{k-}\rangle_{\mathbb{S}}}\mathcal{S}_{\Gamma_i}[\phi_n^{si}](x)\right| \\
&\le C\sum_n n e^{n(2\rho_e-\rho_i)}\left(\left|F_n^+\mathcal{S}_{\Gamma_i}[\phi_n^{ci}](x)\right| + \left|F_n^-\mathcal{S}_{\Gamma_i}[\phi_n^{si}](x)\right|\right) \\
&\le C\sum_n e^{2n\rho_e}e^{-n\rho}(|F_n^+|+|F_n^-|)
\end{aligned}$$

for $\rho > \rho_i$. Similarly we have, for $\rho > \rho_e$,

$$|\mathcal{S}_{\Gamma_e}[\varphi_e](x)| \le C\sum_n (|F_n^+|+|F_n^-|)e^{n(3\rho_e-\rho_i)}e^{-n\rho}.$$

We conclude that

$$|V_\delta(x)| \leq |F(x)| + |\mathcal{S}_{\Gamma_i}[\varphi_i](x)| + |\mathcal{S}_{\Gamma_e}[\varphi_e](x)| < C$$

regardless of δ for any $\rho \geq \rho_0$ with $\rho_0 > 2\rho_e - \rho_i$.

Next we investigate the behavior of E_δ as $\delta \to 0$.

$$E_\delta \approx \delta \sum_{k=1,2} \sum_{n=1}^{\infty} \frac{1}{\lambda_{k,n}^2 + \delta^2} \left(\frac{|\langle g, \Psi_n^{k+} \rangle_{\mathbb{S}}|^2}{\langle \Psi_n^{k+}, \Psi_n^{k+} \rangle_{\mathbb{S}}} + \frac{|\langle g, \Psi_n^{k-} \rangle_{\mathbb{S}}|^2}{\langle \Psi_n^{k-}, \Psi_n^{k-} \rangle_{\mathbb{S}}} \right)$$
$$\approx \sum_{n=1}^{\infty} \frac{\delta n e^{2n\rho_e} ((F_n^+)^2 + (F_n^-)^2)}{e^{-2n(\rho_e - \rho_i)} + \delta^2}.$$

Suppose that F does not extend as a harmonic function in $\{(\rho, \omega) : \rho < \rho_*\}$. Then we have

$$\limsup_{n\to\infty} |F_n^+|^{1/n} > e^{-\rho_*} \quad \text{or} \quad \limsup_{n\to\infty} |F_n^-|^{1/n} > e^{-\rho_*}.$$

So, there is a subsequence, say $\{n_k\}$, such that, for all k,

$$e^{2n_k\rho_*} ((F_{n_k}^+)^2 + (F_{n_k}^-)^2) \geq 1\,.$$

Let $\delta_k = e^{-n_k(\rho_e - \rho_i)}$ for $k = 1, 2, \ldots$. Then, we have

$$E_{\delta_k} \approx \sum_{n=1}^{\infty} \frac{\delta_k n e^{2n\rho_e} ((F_n^+)^2 + (F_n^-)^2)}{e^{-2n(\rho_e - \rho_i)} + \delta^2} \geq \frac{\delta_k n_k e^{2n_k(2\rho_e - \rho_i)}}{2} ((F_k^+)^2 + (F_k^-)^2) \to \infty$$

as $k \to \infty$. This proves (i).

Suppose that $F_n^\pm$ satisfies GC[ρ_*] and $\{n_k\}$ be the subsequence appearing in the condition. For $\delta \to 0$, let $k(\delta)$ be the number such that

$$n_{k(\delta)} \le -\frac{\ln \delta}{\rho_e - \rho_i} < n_{k(\delta)+1}.$$

Then, we have

$$\delta > e^{-n_{k(\delta)+1}(\rho_e-\rho_i)},$$

and hence

$$E_\delta \approx \sum_{n=1}^{\infty} \frac{\delta n e^{2n\rho_e}((F_n^+)^2 + (F_n^-)^2)}{e^{-2n(\rho_e-\rho_i)} + \delta^2} \ge \frac{\delta n_{k(\delta)} e^{2n_{k(\delta)}\rho_e}((F_{n_{k(\delta)}}^+)^2 + (F_{n_{k(\delta)}}^-)^2)}{e^{-2n_k(\delta)(\rho_e-\rho_i)}}$$
$$\ge n_{k(\delta)} e^{-(n_{k(\delta)+1} - n_{k(\delta)})(\rho_e-\rho_i)} e^{2n_{k(\delta)}\rho_*}(|F_{n_{k(\delta)}}^+|^2 + |F_{n_{k(\delta)}}^-|^2) \to \infty$$

as $\delta \to 0$. This proves (ii).

If the source f is located outside ρ_*, then its Newtonian potential F is harmonic in a neighborhood of $\{(\rho,\omega) : \rho \leq \rho_*\}$, and hence

$$\limsup_{n\to\infty} |F_n^{\pm}|^{1/n} \leq e^{-\rho_* - \epsilon}$$

for some $\epsilon > 0$. Thus it follows that

$$E_\delta \leq C \sum_{n=1}^{\infty} n e^{2n\rho_*} ((F_n^+)^2 + (F_n^-)^2) < \infty.$$

Thank you.