On the stability of plane Couette flows[1]

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Contents

- The Navier-Stokes equations:

- The motion of an incompressible homogeneous viscous Newtonian fluid in Ω is described by the following nonlinear system of partial differential equations, named after Navier (1822) and Stokes (1845):

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = 0 & \text{in } Q \\ \operatorname{div}\mathbf{v} = 0 & \text{in } Q, \end{cases} \tag{1}$$

where

$\Omega \subset \mathbf{R}^3$: a smooth domain

$(\mathbf{x}, t) \in Q = \Omega \times (0,\infty)$

$\nu > 0$: the viscosity constant

$\mathbf{v} = (v^1(\mathbf{x},t), v^2(\mathbf{x},t), v^3(\mathbf{x},t))$: the (unknown) velocity

$p = p(\mathbf{x},t)$: the (unknown) pressure

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \left(\sum_{i=1}^{3} v^i \frac{\partial}{\partial_{x_i}}\right)\mathbf{v} = (\mathbf{v}\cdot\nabla v^1, \mathbf{v}\cdot\nabla v^2, \mathbf{v}\cdot\nabla v^3)$$

The hydrodynamic stability problem

- The stability problem:

- A very interesting problem in mathematical fluid mechanics is to prove the stability or instability of a given stationary solution of (1).

- For a given stationary solution

$$\mathbf{v}_0 = (v_0^1(\mathbf{x}), v_0^2(\mathbf{x}), v_0^3(\mathbf{x}))$$

of (1), let us consider the following IBVP

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = 0 \text{ in } Q \\ \operatorname{div} \mathbf{v} = 0 \qquad \text{in } Q \\ \mathbf{v} = \mathbf{v}_0 \quad \text{on } \partial\Omega \times (0, \infty) \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_0 + \mathbf{a} \quad \text{in } \Omega, \end{cases} \tag{2}$$

where $\mathbf{a} = (a^1(\mathbf{x}), a^2(\mathbf{x}), a^3(\mathbf{x}))$ is an initial perturbation.

- The stationary solution $\mathbf{v}_0$ is *stable* if there is a small positive number ε such that for any $\mathbf{a}$ with $||\mathbf{a}|| \le \varepsilon$, the IBVP (2) has a unique global solution $\mathbf{v} = \mathbf{v}(t)$, which tends to $\mathbf{v}_0$ as $t \to \infty$.

- A mathematical formulation of the stability problem:

- Instead of IBVP (2), we may consider the following equivalent problem for the perturbation $\mathbf{u} = \mathbf{v} - \mathbf{v}_0$:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \tilde{\Delta} \mathbf{u} + \nabla p = 0 \ \text{ in } \ Q \\ \operatorname{div} \mathbf{u} = 0 \qquad\qquad \text{ in } \ Q \\ \mathbf{u} = 0 \quad \text{on } \ \partial\Omega \times (0, \infty) \\ \mathbf{u}(0) = \mathbf{a} \quad \text{in } \ \Omega, \end{cases} \tag{3}$$

where

$$-\nu \tilde{\Delta} \mathbf{u} = -\nu \Delta \mathbf{u} + (\mathbf{v}_0 \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v}_0.$$

- Note that

$$\mathbf{a} = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \operatorname{div} \mathbf{a} = 0 \quad \text{in } \Omega.$$

- Let $\mathbf{X}(\Omega)$ be a Banach space of some vector fields in Ω such that $\left[C_0^\infty(\Omega)\right]^3 \subset \mathbf{X}(\Omega)$. Then we denote by $\mathbf{X}_{0,\sigma}(\Omega)$ the closure of the set

$$C_{0,\sigma}^\infty(\Omega) = \{\mathbf{f} \in [C_0^\infty(\Omega)]^3 : \operatorname{div} \mathbf{f} = 0\}$$

in $\mathbf{X}(\Omega)$.

The hydrodynamic stability problem

- Typical examples of $\mathbf{X}_{0,\sigma}(\Omega)$ are

$$\mathbf{L}^q_\sigma(\Omega) = \mathbf{L}^q_{0,\sigma}(\Omega) \quad \text{and} \quad \mathbf{H}^{1,q}_{0,\sigma}(\Omega)$$

for $1 < q < \infty$. Recall that

$$\mathbf{H}^{1,q}(\Omega) = \{\mathbf{u} \in \mathbf{L}^q(\Omega) : \nabla \mathbf{u} \in \mathbf{L}^q(\Omega)\}$$

and

$$\mathbf{H}^{1,q}_{0,\sigma}(\Omega) = \left\{\mathbf{u} \in \mathbf{H}^{1,q}(\Omega) : \mathbf{u}|_{\partial\Omega} = 0, \quad \operatorname{div} \mathbf{u} = 0\right\}.$$

Definition. The stationary solution $\mathbf{v}_0$ is *(exponentially) stable* in $\mathbf{X}_{0,\sigma}(\Omega)$ if there is a number $\varepsilon > 0$ such that for each $\mathbf{a} \in \mathbf{X}_{0,\sigma}(\Omega)$ with $||\mathbf{a}||_{\mathbf{X}(\Omega)} \leq \varepsilon$, the perturbation problem (3) has a unique global solution $\mathbf{u} \in C([0,\infty); \mathbf{X}_{0,\sigma}(\Omega))$, which decays (exponentially) as $t \to \infty$.

- Stability for large ν:

- The stability of $\mathbf{v}_0$ is rather trivial if the viscosity constant ν is sufficiently large. Let Ω be bounded. From (3), we have

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\mathbf{u}|^2\,dx + \nu\int_\Omega |\nabla\mathbf{u}|^2\,dx = -\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}_0\cdot\mathbf{u}\,dx.$$

In view of the Poincaré inequality

$$\int_\Omega |\mathbf{u}|^2\,dx \leq C\int_\Omega |\nabla\mathbf{u}|^2\,dx,$$

we have

$$\frac{d}{dt}||\mathbf{u}(t)||^2_{\mathbf{L}^2(\Omega)} + C^{-1}\nu||\mathbf{u}(t)||^2_{\mathbf{L}^2(\Omega)} \leq 2||\nabla\mathbf{v}_0||_{\mathbf{L}^\infty(\Omega)}||\mathbf{u}(t)||^2_{\mathbf{L}^2(\Omega)}.$$

Hence if ν is so large that

$$\delta := C^{-1}\nu - 2||\nabla\mathbf{v}_0||_{\mathbf{L}^\infty(\Omega)} > 0,$$

then

$$||\mathbf{u}(t)||^2_{\mathbf{L}^2(\Omega)} \leq e^{-\delta t}||\mathbf{a}||^2_{\mathbf{L}^2(\Omega)}.$$

- Stability for small ν:

- One major mathematical problem of the hydrodynamic stability theory is to prove the stability of a specific stationary flow v_0 for small viscosity constant ν.

- Plane Couette flows:

- In this talk, we study the stability of a *plane Couette flow*

$$\mathbf{v}_0 = (x_3, 0, 0)$$

defined in the infinite layer domain

$$\Omega = \left\{ \mathbf{x} = (\mathbf{x}', x_3) \in \mathbf{R}^3 \,:\, -1 < x_3 < 1 \right\}.$$

- The plane Couette flow is one of few known stationary flows whose stability has been proved rigorously. It is of course extremely simple.

Stability results for the plane Couette flow

- The linear stability analysis:

- Let $\mathcal{P}$ be the Helmholtz projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{L}^2_\sigma(\Omega)$:

$$\mathbf{u} = \mathcal{P}\mathbf{u} + \nabla p$$

for some $p \in H^1_{loc}(\overline{\Omega})$ with $\nabla p \in \mathbf{L}^2(\Omega)$. We consider

$$\mathcal{L} : D(\mathcal{L}) = \mathbf{H}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{H}^{2,2}(\Omega) \to \mathbf{L}^2_\sigma(\Omega),$$

defined by

$$\mathcal{L}\mathbf{u} = \mathcal{P}\left[-\nu\Delta\mathbf{u} + (\mathbf{v}_0 \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v}_0\right] \quad \text{for all } \mathbf{u} \in D(\mathcal{L}).$$

Then (3) can be reduced to the following abstract Cauchy problem in $\mathbf{L}^2_\sigma(\Omega)$:

$$\begin{cases} \partial_t \mathbf{u}(t) = -\mathcal{L}\mathbf{u}(t) - \mathcal{P}\left((\mathbf{u}(t) \cdot \nabla)\mathbf{u}(t)\right) \\ \mathbf{u}(0) = \mathbf{a}. \end{cases} \tag{4}$$

- The stability of $\mathbf{v}_0$ is closely related to the location of the spectrum $\sigma(-\mathcal{L})$ of the unbounded operator $-\mathcal{L}$ in $\mathbf{L}^2_\sigma(\Omega)$. Recall that $\sigma(-\mathcal{L})$ is the complement of the resolvent set $\rho(-\mathcal{L})$ which consists of all complex λ such that $\lambda + \mathcal{L}$ has a bounded inverse;
  (i) $\lambda + \mathcal{L} : D(\mathcal{L}) \to \mathbf{L}^2_\sigma(\Omega)$ is bijective, and
  (ii) there is a constant $C > 0$ such that

$$\| (\lambda + \mathcal{L})^{-1} \mathbf{u}\|_{\mathbf{L}^2_\sigma(\Omega)} \le C\|\mathbf{u}\|_{\mathbf{L}^2_\sigma(\Omega)} \quad \text{for all } \mathbf{u} \in \mathbf{L}^2_\sigma(\Omega).$$

- In 1973, Romanov showed that if there is a number $\delta > 0$ such that

$$\mathrm{Re}\,\lambda \le -\delta \quad \text{for all} \quad \lambda \in \sigma(-\mathcal{L}), \tag{5}$$

then $\mathbf{v}_0$ is exponentially stable in $\mathbf{H}^{1,2}_{0,\sigma}(\Omega)$: for each $\mathbf{a} \in \mathbf{H}^{1,2}_{0,\sigma}(\Omega)$ with $||\mathbf{a}||_{\mathbf{H}^{1,2}(\Omega)}$ being sufficiently small, the problem (4) has a unique global solution $\mathbf{u} \in C([0,\infty); \mathbf{H}^{1,2}_{0,\sigma}(\Omega))$ which decays exponentially in $\mathbf{H}^{1,2}_{0,\sigma}(\Omega)$ as $t \to 0$.

Stability results for the plane Couette flow

- In 1973, Romanov also showed in a nearly rigorous manner that there is a number $\delta > 0$ such that

$$\mathrm{Re}\,\lambda \leq -\delta \quad \text{for all} \quad \lambda \in \sigma_P(-\mathcal{L}), \tag{6}$$

where $\sigma_P(-\mathcal{L})$ is the point spectrum of $-\mathcal{L}$:

$$\sigma_P(-\mathcal{L}) = \{\lambda \in \mathbf{C} : \lambda + \mathcal{L} \ \text{ is not injective}\} = \{\text{all eigenvalues of } \ -\mathcal{L}\} \subset \sigma(-\mathcal{L}).$$

- A weaker version of (6) was obtained by Solopenko in 1989. He proved that

$$\mathrm{Re}\,\lambda < 0 \quad \text{for all} \quad \lambda \in \sigma_P(-\mathcal{L}).$$

- From (6), Romanov concluded that $\mathbf{v}_0$ is exponentially stable in $\mathbf{H}^{1,2}_{0,\sigma}(\Omega)$. However this famous stability result of Romanov has not been proved completely yet.

- Two gaps of Romanov's argument:

(i) His proof of (6) is based crucially on a numerical computation which has not been verified yet.

(ii) He deduced (5) from (6) without a detailed proof. But this is not trivial at all because $\sigma_P(-\mathcal{L}) \neq \sigma(-\mathcal{L})$ in general.

Stability results for the plane Couette flow

- A stability result in $\mathbf{L}^3_\sigma(\Omega)$:

- It has been shown by Abe and Shibata (2003) and Abels and Wiegner (2005), independently, that the Stokes operator $-\mathcal{A} = \nu \mathcal{P} \Delta$ generates an analytic semigroup $\{e^{-t\mathcal{A}}\}_{t \geq 0}$ on $\mathbf{L}^q_\sigma(\Omega)$ for each $q \in (1, \infty)$.

- Then Abe and Shibata proved the exponential stability of $\mathbf{v}_0$ in $\mathbf{L}^3_\sigma(\Omega)$ under the assumption that ν is sufficiently large. In this case, the operator $-\mathcal{L}$ can be regarded as a small perturbation of $-\mathcal{A}$.

- An open problem:

- It remains still open to provide a rigorous proof of the stability of the plane Couette flow in some $\mathbf{X}_{0,\sigma}(\Omega)$ for the case of small viscosity ν.

Stability results for the plane Couette flow

- Basic ideas for our setting:

- From Romanov-Solopenko's result, we have

$$\operatorname{Re}\lambda < 0 \quad \text{for all} \quad \lambda \in \sigma_P(-\mathcal{L}). \tag{7}$$

But it remains still open to deduce from (7) that

$$\delta = \sup_{\lambda \in \sigma(-\mathcal{L})} \operatorname{Re}\lambda < 0.$$

For it is possible that $\sigma_P(-\mathcal{L})$ has an accumulation point in the imaginary axis or $\sigma_P(-\mathcal{L})$ is a proper subset of $\sigma(-\mathcal{L})$.

- Such a difficulty is due to the unboundedness of the domain $\Omega = \mathbf{R}^2 \times (-1, 1)$: the Sobolev embedding $\mathbf{H}^{1,q}_{0,\sigma}(\Omega) \hookrightarrow \mathbf{L}^q_\sigma(\Omega)$ is continuous but not compact.

- To circumvent that difficulty, we consider the Sobolev spaces consisting of functions in Ω which are periodic in $\mathbf{x}' = (x_1, x_2)$.

- Note that if $1 < q < \infty$, then every $\mathbf{a} \in \mathbf{L}^q_\sigma(\Omega)$ satisfies

$$|\mathbf{a}(\mathbf{x})| \to 0 \quad \text{as} \quad |\mathbf{x}| \to 0.$$

Instead of this boundary condition on $\mathbf{a}$ at infinity, we assume that

$$\mathbf{a}(\cdot, x_3) \text{ is } \mathbf{T}\text{-periodic for a.e. } x_3 \in (-1, 1).$$

Here $\mathbf{T} = [-l, l]^2$ denotes a torus with $l > 0$ fixed.

- Then the associated Sobolev spaces have the compact embedding property.

- Function spaces:

- Spaces of test functions: Let $\mathcal{D}$ be the space of all complex-valued functions f on $\overline{\Omega}$ which can be written as

$$f(\mathbf{x}', x_3) = \sum_{\mathbf{k} \in J} g_{\mathbf{k}}(x_3)\, e^{i\omega < \mathbf{k}, \mathbf{x}' >}$$

for some finite subset J of $\mathbf{Z}^2$ and some $g_{\mathbf{k}} \in C^\infty([-1, 1])$, where $\omega = \frac{\pi}{l}$.

- Since the set $\{e^{i\omega < \mathbf{k}, \cdot >} : \mathbf{k} \in \mathbf{Z}^2\}$ is orthogonal in $L^2(\mathbf{T})$, the coefficients of each $f \in \mathcal{D}$ are given uniquely by the partial Fourier series of f:

$$g_{\mathbf{k}}(x_3) = \hat{f}_{\mathbf{k}}(x_3) := \frac{1}{(2l)^2} \int_{\mathbf{T}} f(\mathbf{x}', x_3) e^{-i\omega < \mathbf{k}, \mathbf{x}' >}\, dx' \quad (\mathbf{k} \in \mathbf{Z}^2).$$

- Let us define

$$\mathcal{D}_0 = \{f \in \mathcal{D} \,:\, f = 0 \quad \text{on} \quad \partial\Omega\}$$

and

$$\mathcal{D}_{0,\sigma} = \left\{\mathbf{f} \in [\mathcal{D}_0]^3 \,:\, \operatorname{div} \mathbf{f} = 0 \quad \text{in} \quad \Omega\right\}.$$

- The Sobolev norms: if $m \geq 1$ and $1 < q < \infty$,

$$||f||_{0,q} = ||f||_q = \left[\int_{-1}^{1} \int_{\mathbf{T}} |f|^q \, dx' dx_3 \right]^{\frac{1}{q}}$$

and

$$||f||_{m,q} = \left[\sum_{|\alpha| \leq m} ||D^\alpha f||_q^q \right]^{\frac{1}{q}}$$

for $f \in \mathcal{D}$ and similarly for $\mathbf{f} \in [\mathcal{D}]^3$.

- The Sobolev spaces:

$$H_0^{1,q} = \overline{\mathcal{D}_0}^{||\cdot||_{1,q}}, \quad H^{m,q} = \overline{\mathcal{D}}^{||\cdot||_{m,q}},$$

$$L^q = H^{0,q}, \quad \mathbf{L}^q = [L^q]^3, \quad \mathbf{H}^{m,q} = [H^{m,q}]^3,$$

$$\mathbf{L}_\sigma^q = \overline{\mathcal{D}_{0,\sigma}}^{||\cdot||_q} \quad \text{and} \quad \mathbf{H}_{0,\sigma}^{1,q} = \overline{\mathcal{D}_{0,\sigma}}^{||\cdot||_{1,q}}.$$

- Our stability result:

Theorem (Heck, Kim, Kozono (2009))

There exists a small number $\varepsilon > 0$ such that for any $\mathbf{a} \in \mathbf{L}^3_\sigma$ with $\|\mathbf{a}\|_3 \le \varepsilon$, there exists a unique strong solution $(\mathbf{u}, p)$ of the problem (3) satisfying

$$\mathbf{u} \in C([0,\infty); \mathbf{L}^3_\sigma) \cap C((0,\infty); \mathbf{H}^{1,3}_{0,\sigma} \cap \mathbf{H}^{2,3}), \quad p \in C((0,\infty); H^{1,3}), \quad \int p\, dx = 0.$$

Furthermore there are positive constants δ and C such that

$$\|\mathbf{u}(t)\|_3 + t^{\frac{1}{2}} \|\nabla \mathbf{u}(t)\|_3 \le C e^{-\delta t} \|\mathbf{a}\|_3$$

for all $t > 0$. Here the constants ε, δ and C depend only on l and ν.

- Remarks:

(i) The exponential stability in $\mathbf{L}^3_\sigma$ follows immediately from the theorem.

(ii) It should be noted that the (exponential) stability of $\mathbf{v}_0$ is proved for any viscosity constant ν.

The Helmholtz decomposition

- The Helmholtz projection $\mathcal{P}_q$:

Theorem

Let $1 < q < \infty$. Then for each $\mathbf{u} \in \mathbf{L}^q$, there exists a unique $\mathbf{v} \in \mathbf{L}^q_\sigma$ such that

$$\mathbf{u} = \mathbf{v} + \nabla p \quad \textit{for some} \quad p \in H^{1,q}.$$

Moreover we have

$$||\mathbf{v}||_q + ||\nabla p||_q \leq C(q)||\mathbf{u}||_q.$$

- By this theorem, the mapping

$$\mathbf{u} \in \mathbf{L}^q \quad \mapsto \quad \mathbf{v} = \mathcal{P}_q \, \mathbf{u} \in \mathbf{L}^q_\sigma$$

defines a bounded linear operator $\mathcal{P}_q$ (called *the Helmholtz projection*) of $\mathbf{L}^q$ onto $\mathbf{L}^q_\sigma$.

The Helmholtz decomposition

- The crucial step of the proof of the theorem is to show that for each $\mathbf{u}$ in $[\mathcal{D}_0]^3$, there exist $\mathbf{v} \in \mathcal{D} \cap L^q_\sigma$ and $p \in \mathcal{D}$ such that

$$\mathbf{u} = \mathbf{v} + \nabla p \quad \text{and} \quad \|\mathbf{v}\|_q + \|\nabla p\|_q \leq C(q)\|\mathbf{u}\|_q.$$

- Or equivalently, it suffices to prove the existence of $p \in \mathcal{D}$ such that

$$\begin{cases} -\Delta p = \operatorname{div} \mathbf{u} \quad \text{in} \quad \Omega \\ \partial_{x_3} p = 0 \qquad \text{on} \quad \partial\Omega \\ \quad \|\nabla p\|_q \leq C(q)\|\mathbf{u}\|_q. \end{cases}$$

- Our major tools are the partial Fourier series and the Marcinkiewicz multiplier theorem.

- The Marcinkiewicz multiplier theorem

- A complex sequence $a = (a_{\mathbf{k}})_{\mathbf{k}\in\mathbf{Z}^2}$ is a *Fourier Multiplier* on $L^q((-\pi,\pi)^2)$ if

$$\left\| \sum_{\mathbf{k}\in\mathbf{Z}^2} a_{\mathbf{k}} c_{\mathbf{k}} e^{i\langle \mathbf{k},\cdot\rangle} \right\|_q \le C \left\| \sum_{\mathbf{k}\in\mathbf{Z}^2} c_{\mathbf{k}} e^{i\langle \mathbf{k},\cdot\rangle} \right\|_q$$

for any complex sequence $c = (c_{\mathbf{k}})_{\mathbf{k}\in\mathbf{Z}^2}$ with $c_{\mathbf{k}} \neq 0$ for finitely many $\mathbf{k} \in \mathbf{Z}^2$.

- Let $a = (a_{\mathbf{k}})_{\mathbf{k}\in\mathbf{Z}^2}$ be a Fourier multiplier on $L^q((-\pi,\pi)^2)$. Then the mapping

$$\sum_{\mathbf{k}\in\mathbf{Z}^2} c_{\mathbf{k}} e^{i\langle \mathbf{k},\cdot\rangle} \quad \mapsto \quad \sum_{\mathbf{k}\in\mathbf{Z}^2} a_{\mathbf{k}} c_{\mathbf{k}} e^{i\langle \mathbf{k},\cdot\rangle}$$

extends uniquely to a bounded operator T_a on $L^q((-\pi,\pi)^2)$.

- From a classical multiplier theorem due to Marcinkiewicz (1939), we obtain

Theorem

Let $a = (a_{\mathbf{k}})_{\mathbf{k}\in\mathbf{Z}^2}$ be a complex sequence such that

$$a_{\mathbf{k}} = m(\mathbf{k}) \qquad (\mathbf{k} \in \mathbf{Z}^2 \setminus \{0\})$$

for some $m \in C^2(\mathbf{R}^2 \setminus \{0\})$. Suppose that

$$[m] := \sup_{\gamma\in\{0,1\}^2} \sup_{\xi\neq 0} |\xi^\gamma D^\gamma m(\xi)| < \infty. \tag{8}$$

Then for any $q \in (1,\infty)$, the sequence $a = (a_{\mathbf{k}})_{\mathbf{k}\in\mathbf{Z}^2}$ is a Fourier multiplier on $L^q((-\pi,\pi)^2)$ and

$$\|T_a\|_{L^q\to L^q} \leq C(q) \max\{[m], |a_0|\}.$$

- Proof of the Helmholtz decomposition theorem:

- We have to show that for each $\mathbf{u} \in [\mathcal{D}_0]^3$ there exists $p \in \mathcal{D}$ satisfying

$$\begin{cases} -\Delta p = \operatorname{div} \mathbf{u} \quad \text{in} \quad \Omega \\ \partial_{x_3} p = 0 \qquad \text{on} \quad \partial\Omega \\ \|\nabla p\|_q \le C(q)\|\mathbf{u}\|_q. \end{cases} \tag{9}$$

- The given vector field $\mathbf{u}$ can be written as

$$\mathbf{u}(\mathbf{x}', x_n) = \sum_{\mathbf{k} \in \mathbf{Z}^2} \hat{\mathbf{u}}_{\mathbf{k}}(x_3)\, e^{i\omega<\mathbf{k},\mathbf{x}'>}$$

for some $\hat{\mathbf{u}}_{\mathbf{k}}$ with $\hat{\mathbf{u}}_{\mathbf{k}} \neq 0$ for only finitely many $\mathbf{k} \in \mathbf{Z}^2$. Then $p \in \mathcal{D}$ is a solution to (9) if and only if each partial Fourier coefficient $\hat{p}_{\mathbf{k}}$ of p satisfies

$$\begin{cases} (\mu^2 - \partial_{x_3}^2)\, \hat{p}_{\mathbf{k}} = i\omega\mathbf{k} \cdot \hat{u}'_{\mathbf{k}} + \partial_{x_3} \hat{u}^3_{\mathbf{k}}, \quad -1 < x_n < 1 \\ \partial_{x_3} \hat{p}_{\mathbf{k}}(\pm 1) = 0, \end{cases}$$

where $\mu = |\omega\mathbf{k}|$.

- Solving this boundary value problem, we derive an explicit solution:

$$\hat{p}_0(x_3) = \int_{-1}^{x_3} \hat{u}_0^3(y_3)\,\mathrm{d}y_n$$

and

$$\hat{p}_{\mathbf{k}}(x_3) = \int_{-1}^{1} G(\mu, x_3, y_3)\left(i\omega\mathbf{k}\cdot\hat{\mathbf{u}}'_{\mathbf{k}}(\mathbf{k}, y_3) + \partial_{y_3}\hat{u}^n_{\mathbf{k}}(y_3)\right)\,\mathrm{d}y_3$$

for $\mathbf{k} \neq 0$, where

$$G(\mu, x_3, y_3) = \frac{e^{-\mu(2+x_3+y_3)} + e^{-\mu(2-x_3-y_3)} + e^{-\mu|x_3-y_3|} + e^{-\mu(4-|x_3-y_3|)}}{2\mu(1-e^{-4\mu})}.$$

- Define the function p by

$$p(\mathbf{x}', x_n) = \sum_{\mathbf{k}\in\mathbf{Z}^2} \hat{p}_{\mathbf{k}}(x_n)\, e^{i\omega<\mathbf{k},\mathbf{x}'>}.$$

Then p is obviously a solution in $\mathcal{D}$ to the Neumann problem (9).

- Let $\alpha = (\alpha', \alpha_3)$ be a fixed multi-index such that $\alpha' \in \mathbf{N}_0^2$, $\alpha_3 \in \mathbf{N}_0$ and $|\alpha| = |\alpha'| + \alpha_3 = 1$. Then for each $\mathbf{k} \in \mathbf{Z}^2 \setminus \{0\}$, the $\mathbf{k}$-th partial Fourier coefficient of $D^\alpha p$ is given by

$$
\begin{aligned}
&(i\omega\mathbf{k})^{\alpha'} \partial_{x_3}^{\alpha_3} \hat{p}_{\mathbf{k}}(x_3) \\
&\quad = (i\omega\mathbf{k})^{\alpha'} \partial_{x_3}^{\alpha_3} \int_{-1}^{1} G(\mu, x_3, y_3) \left(i\omega\mathbf{k} \cdot \hat{\mathbf{u}}'_{\mathbf{k}}(\mathbf{k}, y_3) + \partial_{y_3} \hat{u}^n_{\mathbf{k}}(y_3) \right) \mathrm{d}y_3 \\
&\quad = \int_{-1}^{1} \partial_{x_3}^{\alpha_3} G(\mu, x_3, y_3) (i\omega\mathbf{k})^{\alpha'} (i\omega\mathbf{k}) \cdot \hat{\mathbf{u}}'_{\mathbf{k}}(\mathbf{k}, y_3) \, \mathrm{d}y_3 \\
&\qquad\qquad - \int_{-1}^{1} \partial_{y_3} \partial_{x_3}^{\alpha_3} G(\mu, x_3, y_3) (i\omega\mathbf{k})^{\alpha'} \hat{u}^3_{\mathbf{k}}(y_3) \, \mathrm{d}y_3
\end{aligned}
$$

- By the Marcinkiewicz multiplier theorem,

$$\|D^\alpha p(\cdot, x_3)\|_{L^q(\mathbf{T})} \le C \int_{-1}^{1} \left[\partial_{x_3}^{\alpha_3} G(\mu, x_3, y_3)(i\omega\mathbf{k})^{\alpha'}(i\omega\mathbf{k})\right] \|\mathbf{u}'(\cdot, y_3)\|_{L^q(\mathbf{T})}\, \mathrm{d}y_3$$
$$+ C \int_{-1}^{1} \left[\partial_{y_3}\partial_{x_3}^{\alpha_3} G(\mu, x_3, y_3)(i\omega\mathbf{k})^{\alpha'}\right] \|\mathbf{u}^3(\cdot, y_3)\|_{L^q(\mathbf{T})}\, \mathrm{d}y_3.$$

- By a direct calculation,

$$\|D^\alpha p(\cdot, x_3)\|_{L^q(\mathbf{T})} \le C \int_{-1}^{1} \left(\frac{1}{2 + x_3 + y_3} + \frac{1}{2 - x_3 - y_3} + \cdots \right) \|\mathbf{u}(\cdot, y_3)\|_{L^q(\mathbf{T})}\, \mathrm{d}y_3.$$

- Hence by the L^q-boundedness of the Hilbert transform, we obtain

$$\|D^\alpha p\|_q \le C\|\mathbf{u}\|_q.$$

Analyticity of the Stokes semigroup

- Analytic semigroups:

- Let X be a complex Banach space. Then a family $(T(t))_{t\geq 0}$ of bounded linear operators on X is called a *(one-parameter) semigroup* on X if it satisfies the functional equation:

$$\text{(FE)} \qquad \begin{cases} T(t+s) = T(t)T(s) & \text{for all } t,s \geq 0 \\ T(0) = I. \end{cases}$$

The semigroup $(T(t))_{t\geq 0}$ is said to be *strongly continuous* if for each $x \in X$ the function $T(\cdot)x : [0,\infty) \to X$ is continuous. A strongly continuous semigroup is also called a C_0*-semigroup*.

- Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X. Then its *(infinitesimal) generator* is a linear operator in X, defined by

$$\mathcal{A}x = \lim_{h\to 0^+} \frac{T(h)x - x}{h}$$

for every x in

$$D(\mathcal{A}) = \{x \in X : T(\cdot)x \text{ is right differentiable at } 0\}.$$

It is easily shown that $D(\mathcal{A})$ is dense in X and $\mathcal{A}$ is a closed operator.

Analyticity of the Stokes semigroup

- Consider the abstract differential equation in X:

$$(\mathrm{DE}) \qquad \begin{cases} \frac{d}{dt}x(t) = \mathcal{A}x(t) & \text{for all } t > 0 \\ x(0) = x \in X, \end{cases}$$

where $\mathcal{A}$ is a linear operator in X.

- If $\mathcal{A}$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X, then for each $x \in D(\mathcal{A})$, there exists a unique solution $x(\cdot)$ of (DE), which is given by $x(t) = T(t)x, t \geq 0$.

- Generators of C_0-semigroups are completely characterized by the so-called Hille-Yosida generation theorem.

- For $0 < \delta < \pi$, let Σ_δ denote the sector of angle $\delta > 0$:

$$\Sigma_\delta = \{z \in \mathbf{C} \setminus \{0\} : |\arg z| < \delta\}.$$

Analyticity of the Stokes semigroup

- A family $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ of bounded linear operators on X is called an *analytic semigroup* (of angle $\delta \in (0, \pi/2]$) if

  (i) $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$,

  (ii) the map $z \mapsto T(z)$ is analytic in Σ_δ, and

  (iii) $\lim_{z\in\Sigma_{\delta'}\to 0} T(z)x = x$ for all $x \in X$ and $\delta' \in (0, \delta)$.

In addition, if

  (iv) $\|T(\cdot)\|$ is bounded in $\Sigma_{\delta'}$ for all $\delta' \in (0, \delta)$,

then $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ is called a *bounded analytic semigroup*.

- Let $\mathcal{A}$ be the generator of an analytic semigroup; that is, it is the generator of a C_0-semigroup that can be extended (uniquely) to an analytic semigroup $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$. Then for each $x \in X$, there exists a unique solution $x(\cdot)$ of (DE), which is given by $x(t) = T(t)x, t \geq 0$.

Theorem (A generation theorem)

A closed linear operator $\mathcal{A}$ in X with dense domain is the generator of a bounded analytic semigroup $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ if and only if it is sectorial of angle $\delta \in (0, \pi/2]$; that is,

(i) the sector $\Sigma_{\pi/2+\delta}$ is contained in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$, and
(ii) for each $\varepsilon \in (0, \delta)$ there exists a constant $M_\varepsilon \geq 1$ such that

$$\|(\lambda - \mathcal{A})^{-1}\| \leq \frac{M_\varepsilon}{|\lambda|} \quad \text{for all } \lambda \in \overline{\Sigma_{\pi/2+\delta-\varepsilon}} \setminus \{0\}.$$

Theorem (A perturbation theorem)

Let $\mathcal{A}$ be the generator of an analytic semigroup on X. Then there exists a constant $\delta > 0$ such that if $\mathcal{B}$ is any closed linear operator in X satisfying

$$D(\mathcal{A}) \subset D(\mathcal{B}) \quad \text{and} \quad \|\mathcal{B}x\| \leq \delta\|\mathcal{A}x\| + C\|x\| \quad \text{for all } x \in D(\mathcal{A}),$$

where C is a constant, then $\mathcal{A} + \mathcal{B}$ is the generator of an analytic semigroup on X.

- The resolvent estimate for the Laplace operator:

- Consider the resolvent problem for the Laplacian with periodic-Dirichlet boundary condition:

$$(\lambda - \Delta)u = f \quad \text{in } \Omega, \tag{10}$$

where

$$\lambda \in \mathbf{C} \setminus (-\infty, 0), \quad u \in H_0^{1,q} \cap H^{2,q} \quad \text{and} \quad f \in L^q.$$

Here $H_0^{1,q}$ denotes the closure of $\mathcal{D}_0$ in $H^{1,q}$:

$$H_0^{1,q} = \left\{ u \in H^{1,q} : u = 0 \quad \text{on } \partial\Omega \right\}.$$

Theorem

Let $1 < q < \infty$, $0 < \varepsilon < \frac{\pi}{2}$ *and* $\lambda \in \Sigma_{\pi-\varepsilon} \cup \{0\}$. *Then for any* $f \in L^q$, *there exists a unique solution* $u \in H_0^{1,q} \cap H^{2,q}$ *of the resolvent equation* (10). *Furthermore we have*

$$|\lambda| \|u\|_q + \|u\|_{2,q} \le C_\varepsilon(q) \|f\|_q.$$

Analyticity of the Stokes semigroup

- Idea of proof:

- We have to show that for any $f \in \mathcal{D}$, there exists $u \in \mathcal{D}_0$ satisfying

$$\begin{cases} (\lambda - \Delta)u = f \quad \text{in } \Omega \\ |\lambda| \|u\|_q + \|\nabla^2 u\|_q \le C_\varepsilon(q) \|f\|_q. \end{cases} \tag{11}$$

- Taking the partial Fourier series, we have

$$\begin{cases} \left(\mu^2 - \partial_{x_n}^2\right) \hat{u}_{\mathbf{k}} &= \hat{f}_{\mathbf{k}} \qquad -1 < x_n < 1 \\ \hat{u}_{\mathbf{k}}(\pm 1) &= 0 \end{cases},$$

where $\mu = \mu(|\omega \mathbf{k}|)$ is the unique $\mu \in \Sigma_{(\pi-\varepsilon)/2}$ such that $\mu^2 = \lambda + |\omega \mathbf{k}|^2$. This problem has a unique solution $\hat{u}_{\mathbf{k}} \in C^\infty([-1,1])$, given by

$$\hat{u}_{\mathbf{k}}(x_3) = \int_{-1}^{1} K(\mu(|\omega \mathbf{k}|), x_3, y_3) \hat{f}_{\mathbf{k}}(y_3) \, \mathrm{d}y_3$$

with the kernel K defined by

$$K(\mu, x_3, y_3) = \frac{e^{-\mu(2+x_3+y_3)} + e^{-\mu(2-x_3-y_3)} - e^{-\mu|x_3-y_3|} - e^{-\mu(4-|x_3-y_3|)}}{2\mu(1-e^{-4\mu})}.$$

- Then the Marcinkiewicz multiplier theorem can be used to deduce

$$|\lambda|\|u\|_q + \|\nabla^2 u\|_q \leq C_\varepsilon(q)\|f\|_q.$$

- Applications of the resolvent estimate:

- For $1 < q < \infty$, we define

$$\Delta_q u = \Delta u \quad \text{for all } u \in D(\Delta_q) = H_0^{1,q} \cap H^{2,q}.$$

Then Δ_q is a closed linear operator in L^q with dense domain.

- The resolvent estimate implies that Δ_q generates a C_0-semigroup $\{e^{t\Delta_q}\}_{t\geq 0}$ that can be extended to a bounded analytic semigroup.

- Hence for each $a \in L^q$, the heat equation

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0,\infty) \\ u(\cdot,0) = a & \text{in } \Omega \end{cases}$$

has a unique solution u satisfying

$$u \in C([0,\infty); L^q) \cap C((0,\infty); H_0^{1,q} \cap H^{2,q}) \quad \text{and} \quad u_t \in C((0,\infty); L^q).$$

- The Stokes operator $-\mathcal{A}_q$:

- For $1 < q < \infty$, $\mathcal{A}_q$ is an unbounded operator in $\mathbf{L}^q_\sigma$ defined by

$$\mathcal{A}_q \mathbf{u} = \mathcal{P}_q\left(-\nu \Delta \mathbf{u}\right) \quad \text{for all } \mathbf{u} \in D(\mathcal{A}_q) = \mathbf{H}^{1,q}_{0,\sigma} \cap \mathbf{H}^{2,q}.$$

- The resolvent estimate for $-\mathcal{A}_q$:

Theorem

Let $1 < q < \infty$, $0 < \varepsilon < \frac{\pi}{2}$ and $\lambda \in \Sigma_{\pi-\varepsilon} \cup \{0\}$. Then for any $\mathbf{f} \in \mathbf{L}^q_\sigma$, there exists a unique solution $\mathbf{u} \in D(\mathcal{A}_q)$ of the Stokes resolvent equation

$$(\lambda + \mathcal{A}_q)\mathbf{u} = \mathbf{f}.$$

Furthermore we have

$$|\lambda| \, \|\mathbf{u}\|_q + \|\mathbf{u}\|_{2,q} \leq C_\varepsilon(q) \|\mathbf{f}\|_q.$$

- Consequently, the Stokes operator $-\mathcal{A}_q$ generates a C_0-semigroup $\{e^{-t\mathcal{A}_q}\}_{t\geq 0}$ on $\mathbf{L}^q_\sigma$ which is analtyic and bounded in every sector $\Sigma_{\pi/2-\varepsilon}, 0 < \varepsilon < \pi/2$.

The perturbed Stokes semigroup

- The perturbed Stokes operator $-\mathcal{L}_q$:

- For $1 < q < \infty$, $\mathcal{B}_q$ and $\mathcal{L}_q$ are unbounded operators in $\mathbf{L}^q_\sigma$, defined by

$$\mathcal{B}_q \mathbf{u} = \mathcal{P}_q\left((\mathbf{v}_0 \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v}_0\right) \quad \text{for } \mathbf{u} \in D(\mathcal{B}_q) = \mathbf{H}^{1,q}_{0,\sigma}$$

and

$$\mathcal{L}_q \mathbf{u} = \mathcal{A}_q \mathbf{u} + \mathcal{B}_q \mathbf{u} \quad \text{for } \mathbf{u} \in D(\mathcal{L}_q) = D(\mathcal{A}_q) = \mathbf{H}^{1,q}_{0,\sigma} \cap \mathbf{H}^{2,q}.$$

- Let $\mathbf{u} \in D(\mathcal{A}_q) = \mathbf{H}^{1,q}_{0,\sigma} \cap \mathbf{H}^{2,q}$ be given. Recall the well-known interpolation inequality: for any $\eta > 0$,

$$\|\mathbf{u}\|_{1,q} \leq \eta \|\mathbf{u}\|_{2,q} + C_\eta \|\mathbf{u}\|_q.$$

Moreover, by the resolvent estimate,

$$\|\mathbf{u}\|_{2,q} \leq C \|\mathcal{A}_q \mathbf{u}\|_q.$$

Hence for any $\eta > 0$, we have

$$\|\mathcal{B}_q \mathbf{u}\|_q \leq C \|\mathbf{u}\|_{1,q} \leq \eta \|\mathcal{A}_q \mathbf{u}\|_q + C_\eta \|\mathbf{u}\|_q.$$

- By a standard perturbation theorem, $-\mathcal{L}_q$ *is the generator of an analytic semigroup* $\{e^{-t\mathcal{L}_q}\}_{t\geq 0}$ *on* L^q_σ*. Moreover, for each* $\varepsilon \in (0,\pi)$ *there are constants* $r > 0$ *and* $M \geq 1$ *such that*

$$\overline{\Sigma}_{\pi-\varepsilon} \cap \{z \in \mathbf{C} : |z| > r\} \subset \rho(-\mathcal{L}_q)$$

and

$$\|(\lambda + \mathcal{L}_q)^{-1}\| \leq \frac{M}{|\lambda|} \quad \textit{for all } \lambda \in \overline{\Sigma}_{\pi-\varepsilon} \cap \{z \in \mathbf{C} : |z| > r\}.$$

- Let λ be any point in $\rho(-\mathcal{L}_q)$. Then the bounded linear operator

$$\lambda + \mathcal{L}_q : \mathbf{H}^{1,q}_{0,\sigma} \cap \mathbf{H}^{2,q} \to \mathbf{L}^q_\sigma$$

is bijective. Hence by the open mapping theorem, its inverse

$$(\lambda + \mathcal{L}_q)^{-1} : \mathbf{L}^q_\sigma \to \mathbf{H}^{1,q}_{0,\sigma} \cap \mathbf{H}^{2,q}$$

is bounded. Since the embedding $\mathbf{H}^{1,q}_{0,\sigma} \hookrightarrow \mathbf{L}^q_\sigma$ is compact, it follows that $(\lambda + \mathcal{L}_q)^{-1}$ is a compact operator on $\mathbf{L}^q_\sigma$.

- Therefore, by the spectral theory of compact operators, *the spectrum* $\sigma(-\mathcal{L}_q)$ *consists entirely of isolated eigenvalues and has no accumulation points except infinity.*

- The key lemma:

- By the regularity theory of the Stokes equations, we deduce that

$$\sigma(-\mathcal{L}_q) = \sigma_P(-\mathcal{L}_q) = \sigma_P(-\mathcal{L}_2).$$

- By the Romanov-Solopenko spectral result, we already knew

$$\mathrm{Re}\,\lambda < 0 \qquad \text{for all} \quad \lambda \in \sigma_P(-\mathcal{L}_2).$$

- Recall that $\sigma(-\mathcal{L}_2) = \sigma_P(-\mathcal{L}_2)$, there is a constant $r > 0$ such that $\Sigma_{3\pi/4} \cap \{\lambda \in \mathbf{C} : |\lambda| \geq r\} \subset \rho(-\mathcal{L}_2)$ and $\sigma(-\mathcal{L}_2)$ has no accumulation points in $\{\lambda \in \mathbf{C} : |\lambda| \leq r\}$. Hence there is a positive constant $\delta = \delta(l, \nu)$ such that

$$\mathrm{Re}\,\lambda \leq -2\delta \qquad \text{for all} \quad \lambda \in \sigma(-\mathcal{L}_2).$$

Lemma

There exists a positive constant $\delta = \delta(l, \nu)$ such that

$$\mathrm{Re}\,\lambda \leq -2\delta \qquad \textit{for all} \quad \lambda \in \sigma(-\mathcal{L}_q).$$

- The $L^q - L^r$-estimates for $\{e^{-t\mathcal{L}_q}\}_{t\geq 0}$:

- Using the key lemma, we can show that $\delta - \mathcal{L}_q$ is sectorial. Hence there is a constant $C > 0$ such that

$$\|e^{t(\delta-\mathcal{L}_q)}\| \leq C_q \quad \text{for all } t \geq 0 \quad \text{and} \quad \|\mathcal{L}_q e^{t(\delta-\mathcal{L}_q)}\| \leq \frac{C_q}{t} \quad \text{for all } t > 0.$$

- Note also that $\mathcal{L}_q : \mathbf{H}^{1,q}_{0,\sigma} \cap \mathbf{H}^{2,q} \to \mathbf{L}^q_\sigma$ is bijective and bounded. Hence for all $\mathbf{a} \in \mathbf{L}^q_\sigma$, we have

$$\|e^{t(\delta-\mathcal{L}_q)}\mathbf{a}\|_q \leq C_q\|\mathbf{a}\|_q \quad \text{and} \quad \|e^{t(\delta-\mathcal{L}_q)}\mathbf{a}\|_{2,q} \leq \frac{C_q}{t}\|\mathbf{a}\|_q \quad \text{for all } t > 0.$$

Lemma

Let $1 < q \leq r < \infty$. Then

$$\|D^\alpha e^{-t\mathcal{L}_q}\mathbf{a}\|_r \leq C_r t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{|\alpha|}{2}} e^{-\delta t}\|\mathbf{a}\|_q$$

for all $\mathbf{a} \in \mathbf{L}^q_\sigma$, $|\alpha| \leq 1$ and $0 < t < \infty$.

Proof of our stability result

- The abstract Cauchy problem in $\mathbf{L}^q_\sigma$:

- The original problem (3) can be reduced to the following abstract Cauchy problem in $\mathbf{L}^q_\sigma$

$$\begin{cases} \partial_t \mathbf{u}(t) = -\mathcal{L}_q \mathbf{u}(t) - \mathcal{P}_q\left((\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\right) \\ \mathbf{u}(0) = \mathbf{a}. \end{cases} \tag{12}$$

- Then our stability result can be reformulated as follows:

Theorem

There exists a small number $\varepsilon > 0$ such that for any $\mathbf{a} \in \mathbf{L}^3_\sigma$ with $\|\mathbf{a}\|_3 \le \varepsilon$, the problem (12) has a unique strong solution

$$\mathbf{u} \in C([0,\infty); \mathbf{L}^3_\sigma) \cap C((0,\infty); \mathbf{H}^{1,3}_{0,\sigma} \cap \mathbf{H}^{2,3}).$$

Furthermore there are positive constants δ and C such that

$$\|\mathbf{u}(t)\|_3 + t^{\frac{1}{2}}\|\nabla \mathbf{u}(t)\|_3 \le Ce^{-\delta t}\|\mathbf{a}\|_3$$

for all $t > 0$. Here the constants ε, δ and C depend only on l and ν.

Proof of our stability result

- Proof of the theorem:

- To solve the differential equation

$$\mathbf{u}_t = -\mathcal{L}_3\mathbf{u} - \mathcal{P}_3\left((\mathbf{u}\cdot\nabla)\mathbf{u}\right) \quad \mathbf{u}(0) = \mathbf{a},$$

we consider the integral equation

$$\text{(IE)} \qquad \mathbf{u}(t) = e^{-t\mathcal{L}_3}\mathbf{a} + \int_0^t e^{-(t-s)\mathcal{L}_3}\left[-\mathcal{P}_3\left((\mathbf{u}\cdot\nabla)\mathbf{u}\right)\right](s)\,\mathrm{d}s.$$

- Let X be the Banach space of all vector fields $\mathbf{v}\in C([0,\infty);\mathbf{L}^3_\sigma)$ such that

$$t^{1-\frac{3}{2q}}\nabla\mathbf{v}\in C([0,\infty);\mathbf{L}^q), \quad \lim_{t\to 0} t^{1-\frac{3}{2q}}||\nabla\mathbf{v}(t)||_q = 0 \quad \text{for} \quad 3\le q\le \frac{9}{2}$$

and

$$||\mathbf{v}||_X = \sup_{0\le t<\infty} e^{\delta t}\left(||\mathbf{v}(t)||_3 + t^{\frac{1}{2}}||\nabla\mathbf{v}(t)||_3 + t^{\frac{2}{3}}||\nabla\mathbf{v}(t)||_{\frac{9}{2}}\right) < \infty.$$

It follows easily from the $L^q - L^r$-estimates that if $\mathbf{v}(t) = e^{-t\mathcal{L}_3}\mathbf{a}$ for $t\ge 0$, then $\mathbf{v}\in X$ and $||\mathbf{v}||_X \le C||\mathbf{a}||_3$.

- Applying the Banach fixed point theorem, we can solve (IE) in X for small $\|\mathbf{a}\|_3$.