Free Boundary Problem with Mixed type PDE: Transonic shocks

- Lecture 1. Introduction - Euler system for compressible flow / Derivation of Rankine-Hugoniot conditions / One dimensional transonic shock solution
- Lecture 2. Normal shock in a rectangular domain / Potential flow / Hyperbolic-elliptic mixed type nonlinear PDE / Derivation of a free boundary problem ([5], [6])
- Lecture 3. Multidimensional transonic shock solution, Part I ([5], [6])
- Lecture 4. Multidimensional transonic shock solution, Part II ([5], [6])
- Lecture 5. Open problems: 2-D oblique shocks

References

[1] R. Courant and K.O. Friedrichs: Supersonic Flow and Shock Waves. Springer, New York, 1948

[2] S. Friedlander and D. Serre: Handbook of Mathematical fluid dynamics Volume IV, Elsevier, 2007

[3] C. S. Morawetz, On a weak solution for a transonic flow problem, Comm. Pure Appl. Math. 38 (1985), 797– 817.

[4] Q. Han and F. Lin, Elliptic partial differential equations. Courant Institute of Math. Sci., NYU.

[5] G.-Q. Chen and M. Feldman: Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type. J. Am. Math. Soc. 16, 461– 494 (2003)

[6] G.-Q. Chen and M. Feldman:Existence and stability of multidimensional transonic flows through an infinite nozzle of arbitrary cross-sections. Arch. Ration. Mech. Anal. 184(2), 185–242 (2007)

Korea PDE Winter School 2015

2-D oblique shocks and related problems

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POSTECH

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Outline

- Introduction to the steady Euler system
  :compressible inviscid flow, constitutive relations for polytropic gas
- 2-d steady Euler system
  :real or complex eigenvalues$\Rightarrow$ Mixed type
- Introduction to shock solutions for the Euler system
  :weak formulation, the Rankine-Hugoniot conditions,
  free boundary problems
- Example: Oblique shock past a wedge
  :shock polar analysis
- Multidimensional($n \geq 2$) transonic shocks
  :potential flow

References

1. *Superconic flow and shock waves* by R. Courant, K.O. Friedrich, Springer-Verlag, 1984
2.*Finite Volume Methods for Hyperbolic Problems, Chap 14* by Randall J. LeVeque, Cambridge University Press, 2002.

Introduction to the Euler system

Conservation Laws For $q = q(x,t)$,

$$\partial_t \int_\Omega q(x,t)\,dx = \int_{\partial\Omega} \vec{F}(q,x,t)\cdot\vec{n}_{in}\,dA(x)$$

or

$$\partial_t q + div\vec{F} = 0$$

Steady State $q = q(x), \vec{F} = \vec{F}(q,x)$

$$\boxed{div\vec{F} = 0}$$

Conservation laws for compressible inviscid flow

- Conservation of mass: div[density(ρ) flux] $= 0$
- Conservation of momentum div[mommentum($\rho\vec{u}$) flux] $= 0$
- Conservation of energy div[energy(E) flux] $= 0$

Steady Euler system for compressible inviscid flow

ρ :density, $\vec{u} = (u_1, \cdots, u_n)$: velocity, p : pressure

$$div(\rho\vec{u}) = 0,$$
$$div(\rho\vec{u} \otimes \vec{u} + pI) = \vec{0}$$
$$div(\rho\vec{u}B) = div(\rho\vec{u} \underbrace{(\frac{1}{2}|\vec{u}|^2 + e + \frac{p}{\rho})}_{=:B(\text{Bernoulli's invariant})}) = 0$$

Bernoulli's law

$$div(\rho\vec{u}) = 0, \quad div(\rho\vec{u}B) = 0$$
$$\Rightarrow \vec{u} \cdot \nabla B = 0$$

The Bernoulli's invariant B is conserved along each streamline.

Constitutive relation for the ideal polytropic gas $e = \frac{p}{(\gamma-1)\rho}$ $(\gamma > 1)$

$$B = \frac{1}{2}|\vec{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho} \Rightarrow div(\rho\vec{u}B) = div(\rho\vec{u}(\frac{1}{2}|\vec{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho})) = 0$$

1. Supersonic/subsonic flow

Sound speed $c = \sqrt{\frac{\gamma p}{\rho}}$ Mach number $M = \frac{|\vec{u}|}{c}$

$(\rho, \vec{u}, p)$ is supersonic if $M > 1$, subsonic if $M < 1$, sonic if $M = 1$.

If $B = B_0$(constant),

$$\frac{1}{2}|\vec{u}|^2 + \frac{c^2}{\gamma - 1} = B_0$$

$$\Leftrightarrow |\vec{u}|^2 - c^2 = \frac{\gamma + 1}{2}(|\vec{u}|^2 - \kappa_0^2), \quad \kappa_0^2 = \frac{2(\gamma - 1)}{\gamma + 1}B_0$$

$$\Leftrightarrow \mu^2|\vec{u}|^2 + (1 - \mu^2)c^2 = \kappa_0^2, \quad \mu^2 = \frac{\gamma - 1}{\gamma + 1}$$

2. Entropy (s) Polytropic gas $\frac{p}{\rho^\gamma} = \kappa \exp(s/c_v)$

$$\vec{u} \cdot div(\rho\vec{u} \otimes \vec{u} + pI) = 0$$

$$\overset{div(\rho\vec{u})=div(\rho\vec{u}B)=0}{\Longrightarrow} \vec{u} \cdot \nabla \frac{p}{\rho^\gamma} = 0$$

The entropy is conserve along each streamline in smooth flow.

2-d Euler system

$$\vec{u} = (u, v)$$

$$\begin{cases} (\rho u)_x + (\rho v)_y = \rho(u_x + v_y) + \frac{1}{c^2}(up_x + vp_y) = 0 \\ \rho u u_x + \rho v u_y + p_x = 0 \\ \rho u v_x + \rho v v_y + p_y = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} \rho u & 0 & 1 \\ 0 & \rho u & 0 \\ 1 & 0 & \frac{u}{\rho c^2} \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix}_x + \begin{pmatrix} 0 & \rho v & 0 \\ 0 & \rho v & 1 \\ 0 & 1 & \frac{v}{\rho c^2} \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix}_y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A\partial_x q + B\partial_y q = 0 \overset{\det A \neq 0}{\Leftrightarrow} \partial_x q + A^{-1}B\partial_y q = 0$$

If $A^{-1}B$ has all real eigenvalues, then the system is hyperbolic.

$$\det(A^{-1}B - \lambda I) = 0 \Leftrightarrow \det(B - \lambda A) = 0$$

Sound speed $c = \sqrt{\frac{\gamma p}{\rho}}$ Mach number $M = \frac{|\vec{u}|}{c}$

$$\det(B - \lambda A) = 0$$

$$\lambda = \frac{v}{u}, \quad \frac{-uv \pm c^2\sqrt{M^2 - 1}}{c^2 - u^2}$$

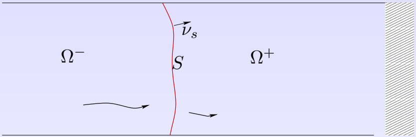
Supersonic flow$(M > 1) \Rightarrow$ Hyperbolic system
Subsonic flow$(M < 1) \Rightarrow$ Non-hyperbolic system

It is possible that M discontinuously changes across a curve (or a surface) S.

Question How to define S mathematically?

Rankine-Hugoniot conditions

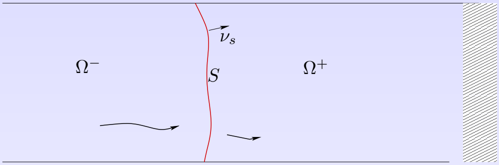
1. Observation



If $\vec{F} \in C^1(\Omega^\pm) \cap L^1_{loc}(\Omega)$ satisfies $div\vec{F} = 0$ in $\Omega^\pm$, then $\forall \phi \in C_0^\infty(\Omega)$

$$
\begin{aligned}
\int_{\Omega^+ \cup \Omega^-} \vec{F} \cdot D\phi dx &= \int_{\Omega^+} + \int_{\Omega^-} div(\vec{F}\phi) - (div\vec{F})\phi dx \\
&= \int_{\partial\Omega^+} + \int_{\partial\Omega^-} (\vec{F} \cdot \nu_{out})\phi dA \\
&= \int_S (\vec{F}^- \cdot \nu_{out}^- + \vec{F}^+ \cdot \nu_{out}^+)\phi dA \\
&= (\int_S \vec{F}^- - \vec{F}^+) \cdot \nu_s \phi dA
\end{aligned}
$$

2. Rankine-Hugoniot conditions for the Euler system



$$[\rho\vec{u}\cdot\nu_s]_S = 0, \quad [(\rho(\vec{u}\cdot\nu_s)\vec{u} + p\nu_s]_S = 0$$
$$[\rho B\vec{u}\cdot\nu_s]_S = 0 (\text{or } [B]_S = 0)$$

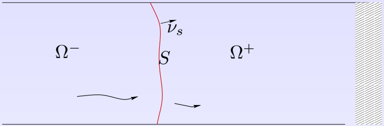
Discontinuity

$$[(\rho(\vec{u}\cdot\nu_s)\vec{u} + p\nu_s]_S = 0$$
$$\Rightarrow \rho(\vec{u}\cdot\nu_s)[(\vec{u}\cdot\tau_s)]_S = 0, \quad [\rho(\vec{u}\cdot\nu_s)^2 + p]_S = 0$$

If $\vec{u}^{\pm}\cdot\nu_s = 0, [\vec{u}\cdot\tau_s]_S \neq 0$, then S is a contact discontinuity.
If $\vec{u}\cdot\nu_s \neq 0, [(\vec{u}\cdot\tau_s)]_S = 0$, then S is a shock. Example Movie

A shock solution to the steady Euler system

$(\rho, \vec{u}, p)$ weak solution in Ω + C^1 solution in $\Omega^{\pm}$

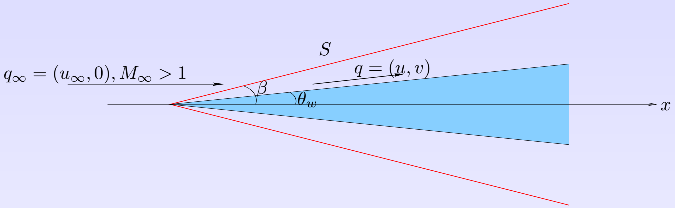


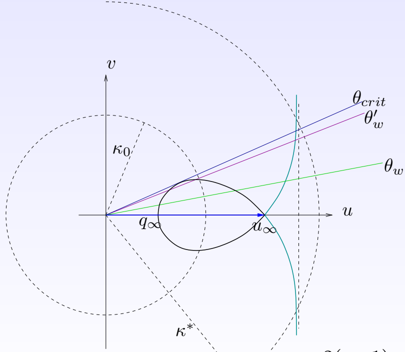
$(\rho, \vec{u}, p) \in L^1_{loc}(\Omega)$ is a (transonic) shock solution if

- $(\rho, \vec{u}, p) \in C^0(\overline{\Omega^{\pm}}) \cap C^1(\Omega^{\pm})$
- $\int_\Omega \rho\vec{u} \cdot D\phi = \int_\Omega (\rho\vec{u} \otimes \vec{u} + pI) \cdot D\phi = \int_\Omega \rho\vec{u}B \cdot D\phi = 0 \; \forall \phi \in C_0^\infty(\Omega)$
- $(0 < \vec{u}^+ \cdot \nu_s < \vec{u}^- \cdot \nu_s$ on S, and $M > 1$ in Ω^-, $M < 1$ in $\Omega^+)$
- $[\rho\vec{u} \cdot \nu_s]_S = [(\rho\vec{u} \otimes \vec{u} + pI) \cdot \nu_s]_S = [\rho B\vec{u} \cdot \nu_s]_S = 0$

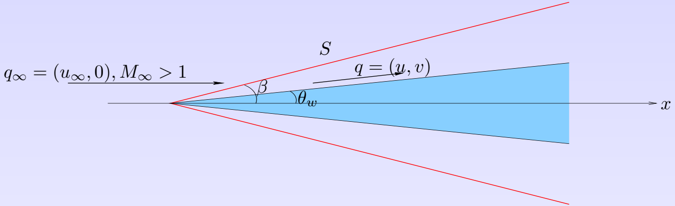
$$\Leftrightarrow \begin{cases} div(\rho\vec{u}) = div(\rho\vec{u} \otimes \vec{u} + pI) = div(\rho\vec{u}B) = 0 \text{ in } \Omega^{\pm} \; (n+2 \text{ equations}) \\ [\rho\vec{u} \cdot \nu_s]_S = [(\rho\vec{u} \otimes \vec{u} + pI) \cdot \nu_s]_S = [\rho B\vec{u} \cdot \nu_s]_S = 0, \end{cases}$$

Oblique shock past a wedge





$$\frac{2}{\gamma+1}(|\vec{q}|^2 - c^2) = |\vec{q}|^2 - \kappa_0^2, \quad \kappa_0^2 = \frac{2(\gamma-1)}{(\gamma+1)} B_0$$



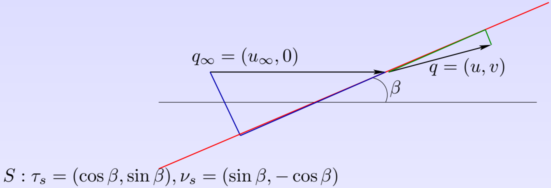
Rankine-Hugoniot conditions

$$\rho\vec{q}\cdot\nu_s = \rho_\infty$$
$$\rho(\vec{q}\cdot\nu_s)^2 + p = \rho_\infty(\vec{q}_\infty\cdot\nu_s)^2 + p_\infty$$
$$\mu^2(\vec{q}\cdot\nu_s)^2 + (1-\mu^2)\frac{\gamma p}{\rho} = \mu^2(\vec{q}_\infty\cdot\nu_s)^2 + (1-\mu^2)\frac{\gamma p_\infty}{\rho_\infty} =: \kappa^2_{\nu_s}$$

For $X = \vec{q}\cdot\nu$, $(X - \vec{q}_\infty\cdot\nu)(X - \frac{\kappa^2_\nu}{\vec{q}_\infty\cdot\nu}) = 0$

$$\boxed{\vec{q}\cdot\nu = \frac{\kappa^2_\nu}{\vec{q}_\infty\cdot\nu}}$$

The equation for $\vec{q} = (u, v)$- curve of shock polar



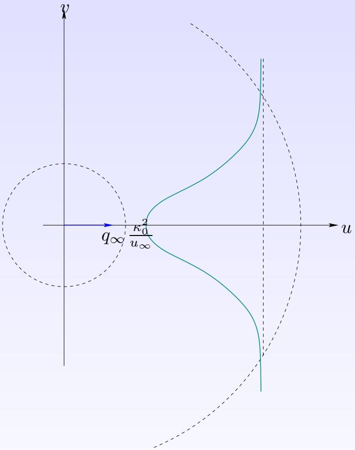
$$u = \vec{q}\cdot\hat{x} = (\vec{q}\cdot\tau)\cos\beta + (\vec{q}\cdot\nu)\sin\beta$$
$$v = \vec{q}\cdot\hat{y} = (\vec{q}\cdot\tau)\sin\beta - (\vec{q}\cdot\nu)\cos\beta$$

$$\vec{q}\cdot\tau = \vec{q}_\infty\cdot\tau = u_\infty\cos\beta, \quad \vec{q}\cdot\nu = \frac{\kappa_\nu^2}{\vec{q}_\infty\cdot\nu}$$

$$\begin{cases}\kappa_\nu^2 = \mu^2(\vec{q}_\infty\cdot\nu)^2 + (1-\mu^2)c_\infty^2\\ \kappa_0^2 = \mu^2|\vec{q}_\infty|^2 + (1-\mu^2)c_\infty^2\end{cases} \Rightarrow \frac{\kappa_\nu^2}{\vec{q}_\infty\cdot\nu} = \frac{\kappa_0^2 - \mu^2 u_\infty^2\cos^2\beta}{u_\infty\sin\beta}$$

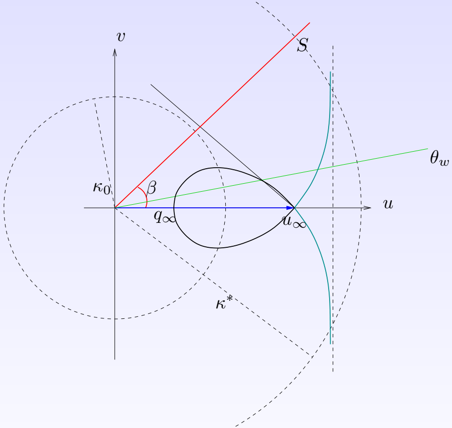
$$\Rightarrow u = (1-\mu^2)u_\infty\cos^2\beta, \quad v = (u_\infty - u)\cot\beta$$

Case 1. $u_\infty \le \kappa_0 (\Leftrightarrow u_\infty \le c_\infty)$ $v = \pm(u_\infty - u)\sqrt{\dfrac{u - \frac{\kappa_0^2}{u_\infty}}{(1-\mu^2)u_\infty + \frac{\kappa_0^2}{u_\infty} - u}}$



Entropy condition The entropy $(\sim \frac{p}{\rho^\gamma})$ must increase across a shock.
$\Rightarrow$ ρ increases, $|q|$ decreases across a shock.(admissibility)

Case 2. $u_\infty > \kappa_0 (\Leftrightarrow u_\infty > c_\infty)$ $\quad v = \pm(u_\infty - u)\sqrt{\frac{u - \frac{\kappa_0^2}{u_\infty}}{(1-\mu^2)u_\infty + \frac{\kappa_0^2}{u_\infty} - u}}$



Between strong shock and weak shock, which one is physical?

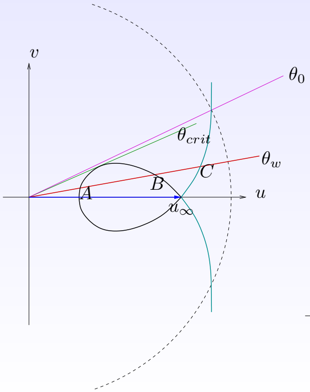
$(p,\theta)-$curve of shock polar

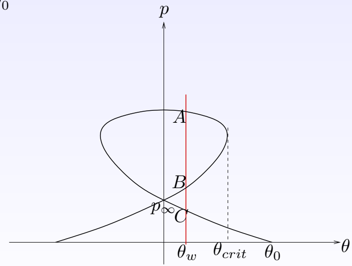
$$\rho(\vec{q}\cdot\nu)^2 + p = \rho_\infty(\vec{q}_\infty\cdot\nu)^2 + p_\infty, \quad \rho(\vec{q}\cdot\nu) = \rho_\infty(\vec{q}_\infty\cdot\nu)$$

$$\Rightarrow p - p_\infty = \rho_\infty[(\vec{q}_\infty\cdot\nu)(\vec{q}-\vec{q}_\infty)\cdot\nu + (\vec{q}_\infty\cdot\tau)(\vec{q}_\infty-\vec{q})\cdot\tau]$$

$$= \rho_\infty\vec{q}_\infty\cdot(\vec{q}_\infty-\vec{q}) = \rho_\infty u_\infty(u_\infty-u)$$

$$\tan\theta_w = \frac{v}{u} = \pm\frac{u_\infty-u}{u}\sqrt{\frac{u-\frac{\kappa_0^2}{u_\infty}}{(1-\mu^2)u_\infty+\frac{\kappa_0^2}{u_\infty}-u}}$$





Irrotational flow

Vorticity $\omega = \nabla \times \vec{u}$

Irrotational flow If $\omega = \vec{0}$, then $\vec{u} = \nabla\varphi$. (φ :velocity potential)

Isentropic Euler system $\frac{p}{\rho^\gamma} = constant$

$$\begin{cases} div(\rho\vec{u}) = 0 \\ div(\rho\vec{u}\otimes\vec{u} + pI) = 0 \end{cases}$$

Isentropic irrotational flow $\vec{u} = \nabla\varphi$

$$\begin{aligned} & div(\rho\nabla\varphi\otimes\nabla\varphi + pI) \\ & = \rho\nabla(\frac{1}{2}|\nabla\varphi|^2) + \mathcal{K}_0\rho^{\gamma-1}\nabla\rho \\ & = \rho\nabla(\frac{1}{2}|\nabla\varphi|^2 + \frac{\mathcal{K}_0}{\gamma-1}\rho^{\gamma-1}) = 0 \end{aligned}$$

$$\Rightarrow \rho^{\gamma-1} = \frac{\gamma-1}{\mathcal{K}_0}(B_0 - \frac{1}{2}|\nabla\varphi|^2)$$

$$div(\rho\vec{u}) = 0 \Rightarrow div((B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma-1}}\nabla\varphi) = 0$$

Isentropic potential flow

$$\Rightarrow \boxed{div((B_0-\frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma-1}}\nabla\varphi)=0} \tag{1}$$

Mixed type PDE If $|\nabla\varphi|^2>\kappa_0^2$ then (1) is hyperbolic, if $|\nabla\varphi|^2<\kappa_0^2$, (1) is elliptic.

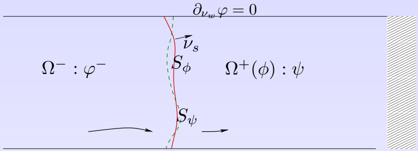
Rankine-Hugoniot conditions for (1)

$$[(B_0-\frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma-1}}\nabla\varphi\cdot\nu_s]_S=0,\quad [\partial_{\tau_s}\varphi]_S=0$$

References

S. Canić, B.L. Keyfitz, G. Lieberman(2000) CPAM **53** 484–511
G.-Q. Chen, M. Feldman(2003) JAMS **16**(3) 461–494 et al.

Framework: Transonic shocks of potential flow



Step 0. Background solution φ_0

Step 1. $div((B_0 - \frac{1}{2}|\nabla\varphi|^2)^{\frac{1}{\gamma-1}}\nabla\varphi - (B_0 - \frac{1}{2}|\nabla\varphi_0|^2)^{\frac{1}{\gamma-1}}\nabla\varphi_0) = 0$
$\Rightarrow \sum_{ij=1}^n \partial_{x_i}(a_{ij}(\nabla\varphi)\partial_{x_j}(\varphi - \varphi_0)) = 0$

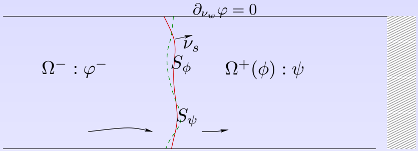
Step 2. Define a set $\mathcal{Q}$ of functions.

Step 3. Fix $\phi \in \mathcal{Q}$. Define $S_\phi = \{\phi = \varphi^-\}, \Omega_\phi^+ = \{\phi < \varphi^-\}$. Solve $\sum_{ij=1}^n \partial_{x_i}(a_{ij}(\nabla\phi)\partial_{x_j}(\psi - \varphi_0)) = 0$ for ψ in Ω_ϕ^+.

Step 4. Use ψ for the next iteration.

Step 5. Find a fixed point.

Framework: Transonic shocks of potential flow



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Step 4. Use ψ for the next iteration.

Step 5. Find a fixed point.

Mathematical issues

solvability of elliptic PDEs, regularity of solutions including shocks, compactness etc.