Lecture 3: The Vlasov equation

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Depa nces

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Outline

Lecture 3

The Boltzmann equation and Vlasov equation

A singular perturbation problem

Note that small Knudsen limit $Kn \to 0$ to the Boltzmann equation corresponds to a singular perturbation problem:

$$Kn\Big(\partial_t F + \xi \cdot \nabla_x F\Big) = Q(F, F).$$

Thus, formally, as long as $\partial_t F + \xi \cdot \nabla_x F$ is uniformly bounded in the zero Knudsen limit, we may argue that

$$Q(F, F) \to 0, \quad \text{as } Kn \to 0$$

In other words, as $Kn \to 0$,

$$F \to M \quad \text{in suitable sense.}$$

Thus, zero Knudsen limit, we may hope that the Boltzmann equation behaves like the Euler equations.

Question: Can this singular perturbation limit be justified rigorously ?

The compressible Euler limit

Consider the Boltzmann equation $\varepsilon := Kn$:

$$\varepsilon\Big(\partial_t F + \xi \cdot \nabla_x F\Big) = Q(F, F).$$

• The Hilbert expansion D. Hilbert, Begrndung der kinetischen Gastheorie, Math. Ann. 72 (1912), 562-577.

Expand F as a formal power series of $\varepsilon = Kn$:

$$F = \sum_{n=0}^{\infty} \varepsilon^n F_n = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots .$$

◇ L.H.S.:

$$\varepsilon({}_t F_0 + \xi \cdot \nabla_x F_0) + \varepsilon^2({}_t F_1 + \xi \cdot \nabla_x F_1) + \varepsilon^3({}_t F_2 + \xi \cdot \nabla_x F_2) + \cdots$$

◇ R.H.S.:

$$\begin{aligned} &Q(F_0, F_0) + 2\varepsilon Q(F_1, F_0) \\ &\qquad + \varepsilon^2(2Q(F_2, F_0) + Q(F_1, F_1)) + \cdots \end{aligned}$$

Compare various orders in ε:

$$\begin{aligned}
&\mathcal{O}(1): \quad Q(F_0, F_0) = 0 \quad \Longrightarrow F_0 = M.\\
&\varepsilon: \quad 2Q(F_1, F_0) = \partial_t F_0 + \xi \cdot \nabla_x F_0 =: \mathcal{S}_0,\\
&\varepsilon^2: \quad 2Q(F_2, F_0) = \partial_t F_1 + \xi \cdot \nabla_x F_1 - Q(F_1, F_1) =: \mathcal{S}_1,
\end{aligned}$$

For the solvability of $L_M(F_1) := 2Q(F_1, F_0) = \mathcal{S}_0$, by the Fredholm alternative,

$$\mathcal{S}_0 \quad \text{is orthogonal to the kernel of } L_M^*$$

But $\mathrm{Ker} L_M^*$ is spanned by the collision invariants,

$$\langle \mathcal{S}_0, \psi_\alpha \rangle = 0, \quad \alpha = 0, 1, \cdots, 4. \ : \text{Euler equations.}$$

- Some relevant references.

  1. Russel E. Caflisch: The fluid dynamic limit of the nonlinear Boltzmann equation. CPAM. 33 (1980), no. 5, 651Ð666.
  2. Shih-Hsien Yu: Hydrodynamic limits with Shock Waves of the Boltzmann Equation. CPAM 58 (2005), 409-443.
  3. Laure Saint-Raymond: A mathematical PDE perspective on the Chapman-Enskog expansion. Bulletin of AMS, 51 (2014), 247-275.

cf. 1. The compressible Navier-Stokes limit: The Chapman-Enskog expansion

2. The incompressible Euler limit: $M \to 0$.

References

1. Earle H. Kennard: Kinetic theory of gases (1938).
2. Stephen G. Brush: The kinetic theory of gases: An anthology of classic papers with historical commentary. Imperial College Press, 2003.

The Vlasov equation

If you first hear of the Vlasov equation, you might ask the following questions:

- QA 1: What is the Vlasov equation ?
- QA 2: When do we use the Vlasov equation ?
- QA 3: What are the relations with other fluid equations ?
- ...

What is the Vlasov equation ?

The Vlasov equation = collisionless Boltzmann equation

Physical situation: Consider an ensemble of particles moving in a mean-field force fields (e.g. a many-body particle systems in mean-field setting)

Let $F = F(x, \xi, t)$ be a one-particle distribution function of particles and we assume that the collisions between particles are secondary, in contrast collisions (interactions) between particle and fields are important.

$$\frac{d}{dt}F(x(t), \xi(t), t) = \partial_t F + \dot{x}(t) \cdot \nabla_x F + \dot{\xi}(t) \cdot \nabla_\xi F = 0,$$

or equivalently

$$\partial_t F + \frac{\xi}{m} \cdot \nabla_x F + E(x, t) \cdot \nabla_\xi F = 0.$$

Let $V = V(x, y)$ be the pairwise potential between particles at position x and y, respectively. In this case, the self-consistent force field

$$\begin{aligned} E(x,t) &= -\nabla_x \iint_{\mathbb{R}^3\times\mathbb{R}^3} V(x,y)F(y,\xi_*,t)d_* dy \\ &= -\nabla_x \int_{\mathbb{R}^3} V(x,y)\rho(y,t)dy. \end{aligned}$$

e.g. 1. Electrostatic potential generated by a charge q:

$$V(x,y) = \frac{q}{4\pi}\frac{1}{|x-y|} \ : \ \text{repulsive}$$

2. Gravitational potential generated by a mass m:

$$V(x,y) = -\frac{Gm}{|x-y|} \ : \ \text{attractive}$$

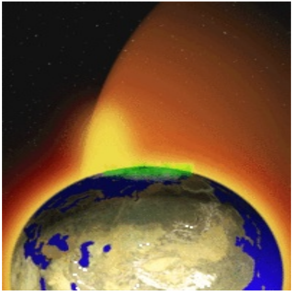
Thus, the self-consistent Vlasov equation reads as

$$\partial_t F + \frac{\xi}{m} \cdot \nabla_x F + E(x,t) \cdot \nabla_\xi F = 0,$$
$$E(x,t) = -\nabla_x \int_{\mathbb{R}^3} V(x,y)\rho(y,t)dy.$$

When do we use the Vlasov equation ?

In plasma physics, the equation was first suggested for description of plasma by Anatoly Vlasov in 1938 " A. A. Vlasov (1938). "On Vibration Properties of Electron Gas". J. Exp. Theor. Phys. (in Russian) 8 (3): 291"

- Plasma physics



Plasma = the fourth fundamental state of gases, completely ionized gases.

e.g., gas inside light bulb.

Ice $\Longrightarrow$ water $\Longrightarrow$ vapor $\Longrightarrow$ plasma.

The Vlasov-Maxwell system

Vlasov equations for electrons and ions + Maxwell equationsfor electric and magnetic force fields.

- Dynamic variables

$F_i = F_i(x, \xi, t),\ F_e = F_e(x, \xi, t)$: distribution functions for ion and electr
$E = E(x, t),\ B = B(x, t)$: electric and magentic fields density.

We set a relativistic velocity related to momentum ξ:

$$v_\alpha(\xi) = \frac{\xi}{\sqrt{m_\alpha^2 + |\xi|^2/c^2}}, \quad \alpha = i, e.$$

where c is the speed of light. Then it is easy to see that

$$|v_\alpha(\xi)| < c.$$

- The Vlasov-Maxwell system

$$
\begin{aligned}
&\partial_t F_\alpha + v_\alpha \cdot \nabla_x F_\alpha + e_\alpha \left(E + \frac{v_\alpha}{c} \times B \right) \cdot \nabla_\xi F_\alpha = 0,\\
&\partial_t E = c\nabla \times B - j, \quad \nabla \cdot E = \rho,\\
&\partial_t B = -c\nabla \times E, \quad \nabla \cdot B = 0,
\end{aligned}
$$

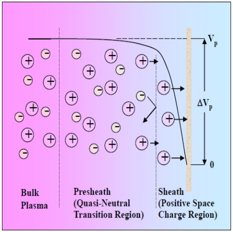
where ρ and j are the charge and current densities:

$$
\begin{aligned}
\rho &= C(d) \int \sum_\alpha e_\alpha F_\alpha d\xi, \ \text{charge density}\\
j &= C(d) \int \sum_\alpha v_\alpha e_\alpha F_\alpha d\xi \ \text{currrent density.}
\end{aligned}
$$

cf. Small data and global existence:

A single species plamsa

- Electron gun (laser) and plamsa sheath



Only one species of charged particles, say electrons or ions.

The Vlasov-Poisson system

Recall a Vlasov-Maxwell system for a single species.

$$\partial_t F + v(\xi)\cdot\nabla_x F + e\Big(E + \frac{v(\xi)}{c}\times B\Big)\cdot\nabla_\xi F = 0,$$
$$\frac{1}{c}\partial_t E = \nabla\times B - \frac{j}{c}, \quad \nabla\cdot E = \rho,$$
$$\frac{1}{c}\partial_t B = -\nabla\times E, \quad \nabla\cdot B = 0.$$

Consider a regime where

$$|v(\xi)| \ll c, \quad \partial_t B \approx 0, \quad \partial_{x_i} B_i \approx 0.$$

i.e., letting $c \to \infty$ and $B = 0$. Then, the Vlasov-Maxwell system becomes

$$\begin{aligned}
&\partial_t F + v(\xi) \cdot \nabla_x F + eE \cdot \nabla_\xi F = 0, \\
&\nabla \cdot E = \rho, \quad \nabla \times E = 0.
\end{aligned}$$

We set

$$E = \nabla_x \varphi$$

and obtain the Vlasov-Poisson system:

$$\begin{aligned}
&\partial_t F + \xi \cdot \nabla_x F + \nabla_x \varphi \cdot \nabla_\xi F = 0, \quad x, \xi \in \mathbb{R}^3, \ t \in \mathbb{R}, \\
&\Delta \varphi = \rho, \qquad \rho = \int eF d\xi.
\end{aligned}$$

cf. Rigorous justification: Degond, Ukai '80 in finite-time interval

- Astrophysics:

A galaxy is a gravitationally bound system consisting of stars, stellar remnants, an interstellar medium of gas and dust, and dark matter

cf. Size of galaxy: From 10^4 to 10^{14}, observable universe: $\geq 10^{14}$-galaxies

James Hopwood Jeans: On the theory of star-streaming and the structure of the universe, Monthly Notices of the Royal Astronomical Society, 76 (1915), 70 -84.

cf. Reinhard Rein

Inner beauties of the V-P system

don't tell me that inner beauty is more important than outer beauty, because in this society it isn't. no one gives you a chance to see if you're beautiful on the inside if you aren't on the outside.

Inner beauties of the V-P system

- The V-P system is a Hamiltonian system.

Recall that the Vlasov-Poisson (V-P) system reads as

$$\partial_t F + \frac{\xi}{m} \cdot \nabla_x F - \nabla_x \varphi \cdot \nabla_\xi F = 0, \quad x, \xi \in \mathbb{R}^3,\ t \in \mathbb{R},$$
$$\Delta \varphi = \rho, \qquad \rho = \int F d\xi.$$

Then, the V-P system is equivalent to the ODE system:

$$\frac{dx}{dt} = \frac{\xi}{m}, \quad \frac{d\xi}{dt} = E = -\nabla_x \varphi.$$

Define a particle path $[x(s) = x(s; x, v, t), \xi(s) = \xi(s; x, v, t)]$:

$$\frac{dx(s)}{ds} = \frac{\xi(s)}{m}, \quad \frac{d\xi(s)}{ds} = E(x(s), \xi(s), s),$$
$$(x(t), \xi(t)) = (x, \xi).$$

Then, along the particle path, we have

$$F(x(s), \xi(s), s) = F_0(x, \xi), \quad s > 0, \ \ x, \xi \in \mathbb{R}^3.$$

Note that $(x, \xi) \to (x(s), \xi(s))$ is a measure preserving.

We claim: **The above ODE system is a Hamilton's ODEs**

We set

$$
\begin{aligned}
H(x,\xi,t) &:= \frac{1}{2m}|\xi|^2 + \iint_{\mathbb{R}^3\times\mathbb{R}^3} V(x,y) f(y,\xi,t)\,d\xi\,dy \\
&= \frac{1}{2m}|\xi|^2 + \underbrace{\int_{\mathbb{R}^3} V(x,y)\rho_F(y,t)\,dy,}_{=:\varphi}
\end{aligned}
$$

where potential energy is mean-field.

◇ Hamilton's ODEs

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{\partial H}{\partial \xi} = \frac{\xi}{m}, \\
\frac{d\xi}{dt} &= -\frac{\partial H}{\partial x} = -\nabla_x \varphi.
\end{aligned}
$$

- Conservation laws.

Define a particle path $[x(s) = x(s; x, v, t), \xi(s) = \xi(s; x, v, t)]$:

$$\frac{dx(s)}{ds} = \frac{\xi(s)}{m}, \quad \frac{d\xi(s)}{ds} = E(x(s), \xi(s), s),$$
$$(x(t), \xi(t)) = (x, \xi).$$

Then, along the particle path, we have

$$F(x(s), \xi(s), s) = F_0(x, \xi), \quad s > 0, \quad x, \xi \in R^d.$$

Note that $(x, \xi) \to (x(s), \xi(s))$ is measure preserving.

- Conservation of L^p-norm

$$||F(t)||_{L^p} = ||F_0||_{L^p}, \quad t \geq 0.$$

Balanced laws

Consider a linear Vlasov equation:

$$\partial_t F + \xi \cdot \nabla_x F - \nabla_x \varphi \cdot \nabla_\xi F = 0.$$

- Conservtion of mass

As before, we define

$$\rho(x,t) := \int F d\xi, \quad j(x,t) = (\rho u)(x,t) = \int \xi F d\xi.$$

Using the relation

$$\xi \cdot \nabla_x F = \nabla_x \cdot (\xi F), \quad \nabla_x \varphi \cdot \nabla_\xi F = \nabla_\xi \cdot (\nabla_x \varphi F).$$

We integrate the Vlasov equation with respect to ξ-variable to obtain the continuity equation (local conservation of mass):

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad \text{i.e.,} \quad \partial_t \rho + \nabla_x \cdot j = 0.$$

- Balance of momentum

We multiply ξ to the Vlasov equation

$$\partial_t(\xi F) + \nabla_x \cdot \left(\frac{\xi \otimes \xi}{m} F\right) + \nabla_\xi \cdot (\xi \otimes \nabla_x \varphi F) = -\nabla_x \varphi F$$

and integrate the resulting relation with respect to ξ to obtain

$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u + P) = -\rho \nabla_x \varphi.$$

- Conservation of total energy

Define an energy

$$E := \int \frac{|\xi|^2}{2} F d\xi dx.$$

We multiply $\frac{|\xi|^2}{2}$ to the equation to obtain

$$\partial_t\Big(\frac{|\xi|^2}{2}F\Big) + \nabla_x\cdot\Big(\xi\frac{|\xi|^2}{2}F\Big) + \nabla_\xi\cdot\Big(-\nabla_x\varphi\frac{|\xi|^2}{2}F\Big) \\ +\nabla_x\cdot(\varphi\xi F) - \varphi\nabla_x\cdot(\xi F) = 0.$$

We integrate the above relation with $d\xi dx$ to get

$$\frac{d}{dt}\iint \frac{|\xi|^2}{2}Fd\xi dx - \iint \varphi\nabla_x\cdot(\xi F)d\xi dx = 0.$$

Note that

$$
\begin{aligned}
-\iint \varphi \nabla_x \cdot (\xi F) d\xi dx &= -\int \varphi \nabla_x \cdot j dx = \int \varphi \partial_t \rho dx \\
&= \iint V(|x-y|)\rho(y,t)\partial_t \rho(x,t) dy dx \\
&= \frac{1}{2}\iint V(|x-y|)\partial_t(\rho(x,t)\rho(y,t)) dy dx \\
&= \frac{d}{dt}\frac{1}{2}\iint V(|x-y|)\rho(x,t)\rho(y,t) dy dx \\
&= \frac{d}{dt}\frac{1}{2}\int \varphi(x,t)\rho(x,t) dx.
\end{aligned}
$$

Finally, we have the conservation of total energy:

$$\frac{d}{dt}\Big[\iint \frac{|\xi|^2}{2} F d\xi dx + \frac{1}{2}\int \varphi(x,t)\rho(x,t) dx\Big] = 0.$$

or equivalently,

$$\frac{d}{dt}\Big[\iint \frac{|\xi|^2}{2} F d\xi dx + \frac{1}{2}\int |E(x,t)|^2 dx\Big] = 0.$$

- Conservation of entropy

Note that

$$\partial_t(F \ln F) = (\partial_t F)(1 + \ln F) = -\nabla_x \cdot (\xi F \ln F) + \nabla_\xi \cdot (\nabla_x \varphi F \ln F).$$

We integrate the above relation with respect to $d\xi dx$ to get

$$\frac{d}{dt} \iint F \ln F d\xi dx = 0.$$

The V-P system is a conservative system.

Incompressible Euler limit

• Quasi-neutral limit Let $F = F(x, \xi, t)$ be a kinetic density for electron, and assume that the ion density is constant, say 1, i.e.,

$$\partial_t F + \xi \cdot \nabla_x F - \nabla_x \varphi \cdot \nabla_\xi F = 0,$$
$$\varepsilon \Delta \varphi = 1 - \int F d\xi, \quad \varepsilon \ : \ \text{Debye length.}$$

By previous argument, we have local and global balanced laws.

$$\partial_t \int F d\xi + \nabla \cdot \int \xi F d\xi = 0,$$
$$\partial_t \int \xi F d\xi + \nabla \cdot \int \xi \otimes \xi F d\xi + \nabla \varphi$$
$$= \varepsilon \nabla \cdot (\nabla \varphi \otimes \nabla \varphi) - \frac{\varepsilon}{2} \nabla (|\nabla \varphi|^2),$$
$$\frac{d}{dt}\Big[\iint \frac{1}{2} |\xi|^2 F d\xi dx + \int \frac{\varepsilon}{2} |\nabla \varphi|^2 dx \Big] = 0.$$

cf. E. Grenier, Y. Brenier

Quasi-neutral limit

We take $\varepsilon \to 0$ (quasi-neutral limit)

$$\int F d\xi = 1.$$

We again recall

$$\rho(x,t) = \int F d\xi = 1, \quad J(x,t) = \int \xi F d\xi.$$

and take an ansatz (for perfectly cold electrons)

$$F(x,\xi,t) = \delta(\xi - J(x,t)).$$

to get the incompressible Euler equation:

$$\nabla \cdot J = 0, \quad \partial_t J + \nabla \cdot J \otimes J + \nabla \varphi = 0.$$

Summary of Lecture 3

1. The Vlasov-Poisson system is a conservative system (mass, energy)
2. The compressible, incompressible fluid equations can be formally derived from the Boltzmann and Vlasov-Poisson systems.

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In this Issue

This issue of THE BIOLOGICAL PHYSICIST brings you a feature interview with Cornell University's Steven H. Strogatz, well-known nonlinear dynamicist, applied mathematician, and author, as well as one of the originators of the idea of small world networks.

On another note, our readers may have noticed a paper copy of a condensed version of recent issues of THE BIOLOGICAL PHYSICIST landing in their mailboxes recently. Since some members of the Division of Biological Physics are "off line", we are now providing all members of the Division with occasional paper editions of the most important features and announcements from recent issues. We welcome your feedback on this expansion of THE BIOLOGICAL PHYSICIST into print. And your editor asks you, if you do not plan to archive the print edition, to please recycle!

-- SB

- **Bahar**: What advice would you have for a scientist just beginning a career in interdisciplinary science ?
- **Strogatz**: First, ...
  Second, don't be afraid to work in a completely unfamiliar subject. You can come up to speed amazingly quickly if you have a collaborator in that field, and if you hang around his or her lab for a few weeks. And keep in mind that you bring many advantages as an outsider. You have a different set of tools. You will ask unusual questions. And you don't know know what's impossible.