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History of Relativity

Newton's mechanics is defined on a family of reference frames, called the inertial frames. For example a system of point particles interacting gravitationally are

$$m_N \frac{d^2 \underline{x}_N}{dt^2} = G \sum_M \frac{m_N m_M (\underline{x}_M - \underline{x}_N)}{|\underline{x}_M - \underline{x}_N|^3}$$

where m_N is the mass of the N^{th} particle and $\underline{x}_N$ is its Cartesian position vector at time t. It is a simple exercise to check that these equations take the same form when written ~~in~~ a new set of space-time coordinates

$$\underline{x}' = \underline{\underline{R}}\,\underline{x} + \underline{v}\,t + \underline{d},$$

$$t' = t + \tau,$$

where $\underline{v}$, $\underline{d}$ and τ are any real constants and $\underline{\underline{R}}$ is any real orthogonal matrix.

The above transformation forms a 10-parameter group. (three Euler angles in $\underline{\underline{R}}$, plus three components each for $\underline{v}$ and $\underline{d}$ and plus one for τ. Today this is called the Galileo group and the invariance of Newton's laws of motion under such transformations is today called Galilean invariance, or the Principle of Galilean Relativity.

The theory of electrodynamics presented in 1864 by Maxwell clearly did not satisfy the principle of Galilean relativity. For one thing, Maxwell's equations predict that the speed of light in a vacuum is a ~~unif~~ universal constant c. But if this were true in one coordinate system x^i, t then it will not be true in the "moving" coordinate system $x^{i'}, t'$ defined by the Galilean transformation. Maxwell himself thought that electromagnetic waves were carried by a medium (similar to elasticity or fluid mechanics) called the luminiferous ether, so that the equations would hold only in a limited class of Galilean inertial frames, that is, in those coordinate frames at rest with respect to ether.

Experimentalists failed to discover effects of the earth's motion through the ether. For example if the ether was moving it should enhance the velocity of light in the direction of the earth's orbital motion and diminish the velocity of light transverse to orbital motion. The famous Michelson-Morley experiment of 1887 showed that light's velocity was the same in both cases: there is no ether.

The comprehensive solution to the problems of relativity in electrodynamics and mechanics were set out in detail in 1905 by Albert Einstein. Einstein proposed that that the Galilean transformation should be ~~rep~~ replaced by a different 10-parameter space-time transformation, called a Lorentz transformation. The Lorentz transformation leaves Maxwell's equations and speed of light ~~of~~ invariant but the equations of Newtonian mechanics such as the equations of gravitational interaction given earlier ~~[illegible]~~ are not invariant. Thus Einstein modified the existing of laws of mechanics so that they would be Lorentz invariant. The new principle of Einstein said that physics consisting of Maxwell's equations and Einstein new mechanics satisfied the principle that they were invariant under Lorentz transformations. This is called the Principle of Special Relativity.

It remained to construct a relativistic theory of gravitation. A collaboration of Einstein with his mathematician friend Marcel Grossman led Einstein to the view that gravitational field must be identified with the 10 components of the metric tensor of Riemannian space-time geometry. In this formulation the equations of physics be invariant under general coordinate transformations, not just Lorentz transformations. Einstein achievements were summarized in his 1916 paper "The Foundation of the General Theory of Relativity".

Now we will look at Lorentz transformations.
A Lorentz transformation is a transformation of space-time coordinates x^α to another system x'^α, so that

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta + a^\alpha$$

where a^α, Λ^α_β are constants restricted by the conditions

$$\Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta_{\alpha\beta} = \eta_{\gamma\beta}$$

with

$$\eta_{\alpha\beta} = \begin{cases} +1 & \alpha=\beta=1,2,3 \\ -1 & \alpha=\beta=0 \\ 0 & \alpha\neq\beta \end{cases}$$

Here α, β, γ, etc will run over the four values 1, 2, 3 and 0.
x^1, x^2, x^3 are the usual Cartesian components of the position vector $\underline{x}$ and x^0 is the time t. We take units in which the speed of light is unity, i.e $c=1$

The fundamental property that distinguishes the transformation is that they invariant the "proper time" $d\tau$ defined by

$$d\tau^2 = dt^2 - d\underline{x}^2 = -\eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

To see this just observe that

$$dx'^\alpha = \Lambda^\alpha_\beta \, dx^\beta$$

so that the new coordinate time will be

$$\begin{aligned} d\tau'^2 &= -\eta_{\alpha\beta}\, dx'^\alpha dx'^\beta \\ &= -\eta_{\alpha\beta}\, \Lambda^\alpha_\gamma \Lambda^\beta_\delta \, dx^\gamma dx^\delta \\ &= -\eta_{\gamma\delta}\, dx^\gamma dx^\delta \\ &= d\tau^2 \end{aligned}$$

It is this property that accounts for the Michelson-Morley experiment, i.e. the speed of light is the same in all inertial systems. A light front will have

$$\left|\frac{dx}{dt}\right| = \text{speed of light}$$
$$= 1 \quad (\text{in our units with } c=1)$$

Since

$$d\tau^2 = dt^2 - d\underline{x}^2$$

we have $d\tau = 0$. But as we just saw performing a Lorentz transformation gives $d\tau' = d\tau$ so $d\tau' = 0$ and

$$\left|\frac{dx'}{dt'}\right| = 1$$

i.e. the speed of light in the new primed coordinate system is still equal to one.

It is a nice exercise to show that the Lorentz transformations are the only non-singular coordinate transformations $x \to x'$ that leave $d\tau^2$ invariant.

The set of all Lorentz transformations is called the inhomogeneous Lorentz group or Poincaré group. The subset with $a^\alpha = 0$ is called the homogeneous Lorentz group. Both the ~~proper~~ homogeneous and inhomogeneous Lorentz groups have subgroups called the proper homogeneous and inhomogeneous Lorentz groups defined by imposing on $\Lambda^\alpha{}_\beta$ the additional requirements

$$\Lambda^0_0 \geq 1 , \quad \det \Lambda = +1$$

The proper homogeneous Lorentz transformations have a further subgroup consisting of rotations for which

$$\Lambda^i_j = R_{ij} \quad , \Lambda^i_0 = \Lambda^0_i = 0 \ , \Lambda^0_0 = 1$$

$(i, j = 1, 2, 3)$ and R_{ij} is a unimodular orthogonal matrix, i.e.

$$\det R = 1 , \quad R^T R = \text{Identity matrix.}$$

Now suppose one observer O sees a particle at rest, and a second observer O' sees it moving with velocity $\underline{v}$. From our relation

$$x'^{\alpha} = \Lambda^{\alpha}_{\beta} \, dx^{\beta} + a^{\alpha}$$

with $dx^1 = dx^2 = dx^3 = 0$ we have

$$dx'^{\alpha} = \Lambda^{\alpha}_0 \, dt \quad , \text{i.e.}$$

$$dx'^i = \Lambda^i_0 \, dt \ , \qquad i = 1, 2, 3$$

$$dt' = \Lambda^0_0 \, dt .$$

Since

$$\underline{v} = \frac{dx^1}{dt^1},$$

we have

$$\Lambda^i_0 = v_i \Lambda^0_0 .$$

We can get a second relation between Λ^i_0 and Λ^0_0 by setting $\gamma = \delta = 0$ in our relation

$$\Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta};$$

so that

$$\Lambda^\alpha_0 \Lambda^\beta_0 \eta_{\alpha\beta} = -1 ,$$ or

$$\sum_{i=1}^{3} (\Lambda^i)^2 - (\Lambda^0_0)^2 = -1 .$$

If we set $\Lambda^0_0 = \gamma$, so that

$$\Lambda^i_0 = \gamma v_i$$

then we see

$$\gamma^2 |\underline{v}|^2 - \gamma^2 = -1$$

and

$$\gamma = (1 - |\underline{v}|^2)^{-1/2}$$

Alternatively we take this last relation to be the definition of γ. and of course

$$\Lambda^0_0 = (1 - |\underline{v}|^2)^{-1/2}$$

The other Λ^α_β are not uniquely determined because if Λ^α_β carries a single particle from rest to velocity $\underline{v}$, then so does $\Lambda^\alpha_\gamma R^\gamma_\beta$ where R is an arbitrary rotation. One convenient choice that satisfies our defining relationship

$$\Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta}$$

is

$$\Lambda^i_j = \delta_{ij} = v_i v_j \frac{(\gamma-1)}{|\underline{v}|^2}.$$

$$\Lambda^0_j = \gamma v_j \; .$$

Now we get to crucial point:

Any proper homogeneous Lorentz transformation may be expressed as the product of a "boost" $\Lambda(\underline{v})$ defined above times a rotation R.

We have seen that under a general Lorentz transformation

$$x'^\alpha = \Lambda^\alpha_\beta \, x^\beta + a^\alpha$$

the coordinate differentials transform according to

$$dx'^\alpha = \Lambda^\alpha_\beta \, dx^\beta \,, \quad (\alpha = 1,2,3),$$
$$d\tau' = d\tau .$$

Any quantity that transforms according to the rule

$$f'^\alpha = \Lambda^\alpha_\beta \, f^\beta \qquad (\alpha, \beta = 0,1,2,3)$$

is called a four-vector.

There is a similar version for tensors

$$T'^{\gamma}{}_{\alpha\beta} = \Lambda^{\gamma}_{\delta}\,\Lambda^{\varepsilon}_{\alpha}\,\Lambda^{\zeta}_{\beta}\,T^{\delta}{}_{\varepsilon\varphi}$$

$$T'^{\alpha}{}_{\beta} = \Lambda^{\alpha}_{\gamma}\,\Lambda^{\delta}_{\beta}\,T^{\gamma}{}_{\delta}\,,$$

$$T'^{\alpha\beta} = \Lambda^{\alpha}_{\gamma}\,\Lambda^{\beta}_{\delta}\,T^{\gamma\delta}.$$

Hence if the tensor $T^{\gamma\delta}$ ~~sees a me~~ is say the Cauchy stress in a frame of reference for a fluid at rest

$$T'^{\alpha\beta} = \Lambda^{\alpha}_{\gamma}(\underline{v})\,\Lambda^{\beta}_{\delta}\,T^{\gamma\delta}$$

will give the stress for a fluid in a lab frame ($'$) where the fluid has velocity $\underline{v}$.

To see how this work for an ideal fluid we know that the energy-momentum tensor for an ideal fluid at rest taking the form

$$T^{ij} = p\,\delta_{ij}$$

$$T^{i0} = T^{0i} = 0$$

$$T^{00} = E,$$

where p is the pressure and E is the proper energy density. Now go into a reference frame at rest in the laboratory and suppose in this frame the fluid appear to be moving with velocity $\underline{v}$ and at given space-time point. Hence the energy-momentum tensor to the laboratory observer is $T'^{\alpha\beta}$. Explicitly this yields

$$T'^{ij} = p\delta_{ij} + (p+E)\frac{v_i v_j}{1-|\underline{v}|^2},$$

$$T'^{i0} = (p+E)\frac{v_i}{1-|v|^2},$$

$$T'^{00} = \frac{(E + p|\underline{v}|^2)}{1-|\underline{v}|^2}$$

where $i, j = 1,2,3$. Hence we see that the modified form of fluid mechanics arises from the needs of Lorentz invariant and special relativity.