The energy-momentum tensor for a perfect fluid is

$$T_{ij} = p\,\delta_{ij} + (p+E)\frac{v_i v_j}{1-|\underline{v}|^2} \qquad i,j = 1,2,3$$

$$T_{i0} = (p+E)\frac{v_i}{1-|\underline{v}|^2}$$

$$T_{00} = \frac{(E + p|\underline{v}|^2)}{1-|\underline{v}|^2}$$

$\underline{v}$ is the fluid velocity
E specific energy
p is pressure

The velocity four vector

$$U^i = \frac{dx^i}{d\tau} = (1-|\underline{v}|^2)^{-1/2}\,\underline{v}$$

$$U^0 = \frac{dt}{d\tau} = (1-|\underline{v}|^2)^{-1/2}$$

so that an observer in laboratory coordinates x^i sees fluid move with velocity u^i. (speed of light = 1)

The conservation laws are

$$T_{\alpha\beta;\beta} = 0$$

where ; denotes covariant differentiation.

Just to see that they agree with our usual fluid conservation laws we first write things out in a flat metric, i.e.

$$\frac{\partial T_{\alpha\beta}}{\partial x^\beta} = 0 .$$

$$\frac{\partial T_{00}}{\partial x^0} + \frac{\partial T_{01}}{\partial x^1} + \frac{\partial T_{02}}{\partial x^2} + \frac{\partial T_{03}}{\partial x^3} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{E + p|\underline{v}|^2}{1-|\underline{v}|^2}\right) + \frac{\partial}{\partial x_i}\left(\frac{(p+E)\, v_i}{1-|\underline{v}|^2} \right) = 0$$

when $|\underline{v}| << 1$ this gives to leading order

$$\frac{\partial}{\partial t}(E) + \frac{\partial}{\partial x_i}\left((p+E)\, v_i \right) = 0$$

which is the desired term in the non-relativistic Euler eqns for conservation of energy.

Next we write

$$\frac{\partial}{\partial x^0} T_{0i} + \frac{\partial}{\partial x^1} T_{1i} + \frac{\partial}{\partial x^2} T_{2i} + \frac{\partial}{\partial x^3} T_{3i} = 0$$

which gives

$$\frac{\partial}{\partial t}\left((p+E)\frac{v_i}{1-|\underline{v}|^2}\right) + \frac{\partial}{\partial x_j}\left(p\,\delta_{ij} + (p+E)\frac{v_i v_j}{1-|\underline{v}|^2}\right)$$

Again if $|\underline{v}| << 1$ the leading order approximation is

$$\frac{\partial}{\partial t}\left((p+E)\, v_i\right) + \frac{\partial}{\partial x_j}\left(p\,\delta_{ij} + (p+E)\, v_i v_j\right) = 0$$

An equation of state that links the relativistic and non-relativistic cases is

$$E = \rho + (\gamma-1)^{-1} p$$

where ρ is the gas density

A low pressure approximation is the non-relativistic case yields

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(p\,\delta_{ij} + \rho v_i v_j) = 0$$

i.e the usual non-relativistic momentum balance.

Finally the conservation of mass (sometimes called the continuity equation) if written seperately ~~[illegible]~~ is

~~[illegible]~~

$$\frac{\partial}{\partial t}(\rho(1-|\underline{v}|^2)^{-1/2}) + \operatorname{div}(\rho\underline{v}(1-|\underline{v}|^2)^{-1/2}) = 0$$

which for $|\underline{v}| << 1$ gives the standard equation

$$\frac{\partial}{\partial t}\rho + \operatorname{div}(\rho\underline{v}) = 0.$$

But note in fact it was included as our low velocity energy equation with $E \approx \rho$.

In summary the non-relativistic limit of the relativistic ideal fluid Euler equations yields the usual Euler equations of gas dynamics.

We can rewrite the equation for the energy-momentum tensor in the convenient form

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + (p+E)U^{\alpha}U^{\beta}$$

where

$$\eta^{\alpha\beta} = +1 \quad \alpha=\beta=1,2,3,$$
$$= -1 \quad \alpha=\beta=0,$$
$$= 0 \quad \alpha\neq\beta.$$

It is obvious that $\eta^{\alpha\beta}$ are just the components of the Lorentzian metric. Hence the relativistic generalization in the presence of gravitation that is need is simply

$$T^{\alpha\beta} = p g^{\alpha\beta} + (p+E)U^{\alpha}U^{\beta}$$

Furthermore if the fluid is assumed as rest with respect to the universe with

$v_i = 0$ we have

$$U^0 = 1,$$
$$U^i = 0.$$

To study the conservation laws for energy-momentum tensor even more carefully let us assume metric g for general relativity takes the form Friedmann – Robertson-Walker metric

$$d\tau^2 = -dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right\}$$

k is a constant which takes on the values $+1$, 0, or -1.

So

$$g_{00} = -1, \quad g_{11} = \frac{R^2(t)}{1-kr^2}, \quad g_{22} = r^2 R^2(t), \quad g_{33} = r^2 \sin^2\theta\, R^2(t)$$

$x_0 = t,\ x_1 = r,\ x_2 = \theta,\ x_3 = \varphi$

A useful formula that simplifies the computations is the formula for the covariant divergence of a tensor:

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu}\left(\sqrt{g}\, T^{\mu\nu}\right) + \Gamma^{\nu}_{\mu\lambda} T^{\mu\lambda}$$

where $-g$ denotes the determinant of the metric g. ~~Another useful formula is for the Christoffel~~ symbols

* Of course this just gives

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu}\left[\sqrt{g}\, p\, g^{\mu\nu} + (p+E)\, U^\mu U^\nu\right] + \Gamma^{\nu}_{\mu\lambda}\left[p\, g^{\mu\lambda} + (p+E)\, U^\mu U^\lambda\right]$$

and so the conservation law is

$$\frac{\partial p}{\partial x^\mu} g^{\mu\nu} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu}\left(\sqrt{g}\,(p+E)\, U^\mu U^\nu\right)$$

$$+ \Gamma^\nu_{\mu\lambda}\left((p+E)\, U^\mu U^\lambda \right.$$

$$+ p\left(\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu}\left(\sqrt{g}\, g^{\mu\nu}\right) + \Gamma^\nu_{\mu\lambda} g^{\mu\lambda} \right) = 0$$

The terms in parenthesis are rewritten as

$$\frac{\partial}{\partial x^\mu} g^{\mu\nu} + \Gamma^\lambda_{\lambda\mu} g^{\mu\nu} + \Gamma^\nu_{\mu\lambda} g^{\mu\lambda}$$

which is the covariant derivative of the metric tensor $\cdot\, g^{\mu\nu}{}_{;\mu}$ and hence identically zero.

The the conservation law is

$$\frac{\partial p}{\partial x^\mu} g^{\mu\nu} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu}\left(\sqrt{g}\,(p+E)\,U^\mu U^\nu\right)$$

$$+\Gamma^{\nu}_{\mu\lambda}\left((p+E)\,U^\mu U^\lambda\right) = 0$$

(This is just equation (14.2.18) in the monograph of S. Weinberg, Gravitation and Cosmology, John Wiley (1972).)

In the case $\nu = 1,2,3$, $U^\nu = 0$, $\Gamma^{\nu}_{00} = 0$ and all we have is

$$\frac{\partial p}{\partial x^\mu} g^{\mu\nu} = 0 \quad , \text{i.e.}$$

$$\left(\frac{\partial p}{\partial x^1}\right) g^{11} = 0 \qquad \left(\frac{\partial p}{\partial x^2}\right) g^{22} = 0 \qquad \left(\frac{\partial p}{\partial x^3}\right) g^{33} = 0$$

Hence $\frac{\partial p}{\partial x^1} = \frac{\partial p}{\partial x^2} = \frac{\partial p}{\partial x^3} = 0$

and p can only depend on $x^0 = t$.

In the case $\nu = 0$ the equation is

$$-\frac{\partial p}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial t}\left(\sqrt{g}\,(p+E)\right)$$

$$+ \Gamma^0_{00}\,(p+E) = 0$$

Again $\Gamma^0_{00} = 0$ and we have

$$\frac{\partial p}{\partial t} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial t}\left(\sqrt{g}\,(p+E)\right)$$

Since

$$g = \frac{R^6(t)\, r^4 \sin^2\theta}{1 - kr^2}$$

in the special case $k = 0$ we have

$$\sqrt{g} = R^3(t)\, r^2 \sin\theta$$

and the conservation law is now

$$R^3(t) \frac{\partial p}{\partial t} = \frac{\partial}{\partial t}\left(R^3(t)\,(p+E)\right)$$

which is (14.2.19) in the book of S. Weinberg

Again an equation of state is needed

which we take as ~~p~~

$$p = \frac{E}{3}$$

(with the speed of light taken to be unity).

(This is just our previous equation

of state with ρ neglected and $\gamma - 1 = \frac{1}{3}$,

i.e. $\gamma = \frac{4}{3}$. The case $\gamma = \frac{4}{3}$

is ~~just~~ the choice noted by Weinberg

in equation (2.10.28) for ~~the~~ the

extremely relativistic case)

We substitute the equation of state

$p = \frac{E}{3}$ into the conservation law and

find

$$R^3(t) \frac{\partial p}{\partial t} = 4 \frac{\partial}{\partial t} \left(R^3(t)\, p(t) \right)$$

and so

$$R^3(t)\,\cancel{\frac{dp}{dt}(t)}$$

$$R^3(t)\,\frac{dp}{dt}(t) = 4R^3(t)\,\frac{dp}{dt} + 12R^2(t)\,\frac{dR}{dt}(t)\,p(t)$$

$$-3R(t)\,\frac{dp}{dt} = 12\,\frac{dR(t)}{dt}\,p(t),$$

$$-\frac{1}{4}\,\frac{1}{p}\,\frac{dp}{dt} = \frac{1}{R}\,\frac{dR}{dt},$$

$$\log R(t) + \frac{1}{4}\log p(t) = \text{const},$$

$$R(t)^4\,p(t) = \text{const}.$$

One solution is $R(t) = \text{const}\; t^{1/2}$ so that

$$p(t) = \frac{\text{const}}{t^2}.$$ Notice

$$\frac{\frac{dR(t)}{dt}}{R(t)} = \frac{1}{2t} \equiv H(t)$$

where $H(t)$ is the Hubble "constant" which is obviously not really a constant.

Notice that with $p(t) = \frac{\text{const}}{t^2}$ and

$E = 3p = \frac{3 \cdot \text{const}}{t^2}$ the

~~stress~~ energy-momentum tensor is proportional

to $\frac{1}{t^2}$ for this special solution.

For convenience take the equation

for $R(t)$ to be simply

$$R(t) = t^{1/2}$$

(the constant can be scaled out). Then

the F-R-W metric is just (with $k=0$):

$$d\tau^2 = -dt^2 + t\left\{ dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right\}$$

Next we make a change of independent as Smoller and Temple [PNAS, vol 106, no 34, Aug 25, 2009, 14213-14218] variables to change the above F-R-W metric

into one possessing a self-similar form.

We make the change of variables

$$\bar{t} = \psi_0 \left\{ 1 + \left(\frac{t^{-1/2} r}{2} \right)^2 \right\} t$$

$$\bar{r} = t^{1/2} r$$

so with $\psi_0 = 1$ (again it can be scaled out)

$$\bar{t} = \left\{ 1 + \frac{1}{4} \frac{r^2}{t} \right\} t = t + \frac{1}{4} r^2,$$

$$\bar{r} = t^{1/2} r.$$

If we solve for r and then eliminate r

we see

$$\bar{t} = t + \frac{\bar{r}^2}{4t}$$

and hence t satisfies the quadratic

equation

$$t^2 - t\bar{t} + \frac{\bar{r}^2}{4} = 0$$

so that

$$t = \frac{\bar{t} \pm (\bar{t}^2 - \bar{r}^2)^{1/2}}{2}, \text{ or}$$

$$\frac{t}{\bar{t}} = \frac{1}{2}\left(1 \pm \left(1 - \left(\frac{\bar{r}}{\bar{t}}\right)^2\right)^{1/2}\right)$$

and

$$r = \frac{\bar{r}}{t^{1/2}} = \frac{\bar{r}}{\left(\frac{\bar{t}}{2} \pm \frac{(\bar{t}^2 - \bar{r}^2)^{1/2}}{2}\right)^{1/2}}$$

An elementary calculation shows

$$2\,dt = d\bar{t} \pm \frac{1}{2}(\bar{t}^2 - \bar{r}^2)^{-1/2}(2\bar{t}\,d\bar{t} - 2\bar{r}\,d\bar{r})$$

$$2\,dt = d\bar{t} \pm (\bar{t}^2 - \bar{r}^2)^{-1/2}(\bar{t}\,d\bar{t} - \bar{r}\,d\bar{r})$$

$$2\,dt = d\bar{t} \pm \left(1 - \left(\frac{\bar{r}}{\bar{t}}\right)^2\right)^{-1/2}\left(d\bar{t} - \frac{\bar{r}}{\bar{t}}\,d\bar{r}\right)$$

and

$$t^{1/2} r = \bar{r}$$

$$\frac{1}{2} t^{-1/2} dt\, r + t^{1/2} dr = d\bar{r}$$

$$t\, dr^2 = \left(d\bar{r} - \frac{1}{2} t^{-1/2} dt\, r\right)^2$$

$$= \left(d\bar{r} - \frac{1}{2} t^{-1/2} dt\, \bar{r}\, t^{-1/2}\right)^2$$

$$= \left(d\bar{r} - \frac{1}{2}\frac{\bar{r}}{t}\, dt\right)^2$$

$$= \left(d\bar{r} - \left(\frac{\bar{r}\, dt}{\bar{t} \pm (\bar{t}^2 - \bar{r}^2)^{1/2}}\right)\right)^2$$

$$= \left(d\bar{r} - \left(\frac{\left(\frac{\bar{r}}{\bar{t}}\right) dt}{1 \pm \left(1 - \left(\frac{\bar{r}}{\bar{t}}\right)^2\right)^{1/2}}\right)\right)^2$$

Finally set the similarity variable

$$\xi = \frac{\bar{r}}{\bar{t}}$$

to see

$$2\, dt = d\bar{t} \pm (1 - \xi^2)^{-1/2} (d\bar{t} - \xi\, d\bar{r}),$$

$$t\, dr^2 = \left(d\bar{r} - \left(\frac{\xi\, dt}{1 \pm (1 - \xi^2)^{1/2}}\right)\right)^2$$

$$t^{1/2}\, d\bar{r} = \left(1 \pm \frac{\xi(1-\xi^2)^{-1/2}}{2\nu}\right) d\bar{r} \mp \frac{\xi(1-\xi^2)^{-1/2}}{2}\, d\bar{t}$$

$$= (1-\xi^2)^{-1/2}\left(\left((1-\xi^2)^{1/2} \pm \frac{\xi}{2\nu}\right) d\bar{r} \mp \frac{\xi}{2}\, d\bar{t}\right)$$

$$= \pm(1-\xi^2)^{-1/2}\left(\left(\pm(1-\xi^2)^{1/2} + \frac{\xi}{2\nu}\right) d\bar{r} - \frac{\xi}{2}\, d\bar{t}\right)$$

$$t^{1/2}\, dr = \pm\xi(1-\xi^2)^{-1/2}\left(\left(\frac{\pm(1-\xi^2)^{1/2}}{\xi} + \frac{1}{2\nu}\right) d\bar{r} - \frac{d\bar{t}}{2}\right)$$

$$t^{1/2}\, dr = \pm\xi(1-\xi^2)^{-1/2}\left(\left(\frac{\nu\xi - 1}{\xi} + \frac{1}{2\nu}\right) d\bar{r} - \frac{d\bar{t}}{2}\right)$$

$$= \pm\xi(1-\xi^2)^{-1/2}\left(\left(\nu - \frac{1}{\xi} + \frac{1}{2\nu}\right) d\bar{r} - \frac{d\bar{t}}{2}\right)$$

$$\nu - \frac{1}{\xi} + \frac{1}{2\nu} = \frac{\nu^2 - \frac{\nu}{\xi} + \frac{1}{2}}{\nu} = \frac{\nu^2 - \frac{1}{2} - \frac{\nu^2}{2} + \frac{1}{2}}{\nu}$$

$$= \frac{\nu^2}{2\nu} = \frac{\nu}{2}$$

In summary, we have

$$t^{1/2}\, dr = \pm\xi(1-\xi^2)^{-1/2}\left(\frac{\nu}{2}\, d\bar{r} - \frac{d\bar{t}}{2}\right)$$

$$dt = \pm\xi(1-\xi^2)^{-1/2}\left(\nu\frac{d\bar{t}}{2} - \frac{d\bar{r}}{2}\right)$$

$$2\,dt = \left(\left(1 \pm (1-\xi^2)^{-1/2}\right) d\bar{t} \mp \xi (1-\xi^2)^{-1/2}\, d\bar{r}\right)$$

$$2\,dt = (1-\xi^2)^{-1/2} \left(\left((1-\xi^2)^{1/2} \pm 1\right) d\bar{t} \mp \xi\, d\bar{r}\right)$$

$$(1+\nu^2) = \frac{2\nu}{\xi}$$

$$\nu^2 - \frac{2\nu}{\xi} + 1 = 0$$

$$\nu = \frac{\frac{2}{\xi} \pm \left(\frac{4}{\xi^2} - 4\right)^{1/2}}{2} = \frac{1 \pm (1-\xi^2)^{1/2}}{\xi}$$

$$\nu\xi = 1 \pm (1-\xi^2)^{1/2}$$

$$2\,dt = (1-\xi^2)^{-1/2} \left(\pm \nu\xi\, d\bar{t} \mp \xi\, d\bar{r}\right)$$

$$= \pm (1-\xi^2)^{-1/2} \left(\nu\xi\, d\bar{t} - \xi\, d\bar{r}\right)$$

$$2\,dt = \pm \xi (1-\xi^2)^{-1/2} \left(\nu\, d\bar{t} - d\bar{r}\right)$$

$$t^{1/2}\,dr = d\bar{r} - \frac{\xi\, dt}{1 \pm (1-\xi^2)^{1/2}}$$

$$= d\bar{r} \mp \frac{\xi^2 (1-\xi^2)^{-1/2} (\nu\, d\bar{t} - d\bar{r})}{2\left(1 \pm (1-\xi^2)^{1/2}\right)}$$

$$= d\bar{r} \mp \frac{\xi^2 (1-\xi^2)^{-1/2} (\nu\, d\bar{t} - d\bar{r})}{2\nu\xi}$$

$$t^{1/2}\,dr = d\bar{r} \mp \frac{\xi (1-\xi^2)^{-1/2}}{2\nu} \left(\nu\, d\bar{t} - d\bar{r}\right)$$

Hence the metric is

$$ds^2 = -dt^2 + t\, d\iota^2 + \bar{r}^2\, d\Omega^2$$

$$= \frac{\xi^2}{4(1-\xi^2)} \left\{ (1-\nu^2)\, d\bar{t}^2 + (\nu^2-1)\, d\bar{\iota}^2 \right\}$$

$$ds^2 = \frac{(\nu^2-1)\xi^2}{4(1-\xi^2)} \left\{ - d\bar{t}^2 + d\bar{\iota}^2 \right\}$$

The coefficient may be rewritten as

$$\frac{(\nu^2-1)}{4\left(\frac{1}{\xi^2}-1\right)} = \frac{(\nu^2-1)}{4\left(\frac{(1+\nu^2)^2}{4\nu^2}-1\right)}$$

$$= \frac{\nu^2-1}{\frac{(1+\nu^2)^2}{\nu^2}-4} = \frac{\nu^2(\nu^2-1)}{(1+\nu^2)^2-4\nu^2}$$

$$= \frac{\nu^2(\nu^2-1)}{1+2\nu^2+\nu^4-4\nu^2}$$

$$= \frac{\nu^2(\nu^2-1)}{(1-\nu^2)^2} = \frac{\nu^2}{\nu^2-1}$$

So

$$ds^2 = \frac{\nu^2}{\nu^2-1} \left\{ - d\bar{t}^2 + d\bar{\iota}^2 \right\}$$

The metric now takes the form

$$ds^2 = -\frac{d\bar{t}^2}{1-v(\xi)^{2}} + \frac{d\bar{r}^2}{1-v(\xi)^{2}} + \bar{r}^2\, d\Omega^2$$

where v is defined by

$v($

$$\xi = \frac{2v(\xi)}{1+v(\xi)^2}$$

and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$.

Thus the metric has the form

$$ds^2 = -B(\bar{t},\bar{r})\, d\bar{t}^2 + \frac{1}{A(\bar{t},\bar{r})}\, d\bar{r}^2 + \bar{r}^2\, d\Omega^2$$

where A, B depend only on $\frac{\bar{r}}{\bar{t}} = \xi$.

{ The form of A, B given in these notes has v replaced by $\frac{1}{v}$ from Temple–Smoller eqn (2.6). So there may be a typo in the derivation)

The above calculations have produced a special solution of the Einstein equations. This motivates us to follow Smoller and Temple [ ] and search for the equations satisfied by A, B which depend only on $\frac{r}{t}$. First we substitute the form of the metric into the Einstein equations to obtain three partial differential equations

$$\left\{ -r\frac{A_r}{A} + \frac{1-A}{A} \right\} = \frac{\kappa B}{A} r^2 T^{00}$$

$$\cancel{r\frac{B_r}{B} + \frac{1-A}{A}}$$

$$\frac{A_t}{A} = \frac{\kappa B}{A} r T^{01}$$

$$\left\{ r\frac{B_r}{B} - \frac{1-A}{A} \right\} = \frac{\kappa}{A^2} r^2 T^{11}$$

(1F2)

$$- \left\{ \left(\frac{1}{A}\right)_{tt} - B_{rr} + \Phi \right\} = 2\frac{\kappa B}{A} r^2 T^{22}$$

where

$$\Phi = \frac{B_t A_t}{2A^2 B} - \frac{1}{2A}\left(\frac{A_t}{A}\right)^2 - \frac{B_r}{r} - \frac{B A_r}{rA}$$

$$+ \frac{B}{2}\left(\frac{B_r}{B}\right)^2 - \frac{B B_r A_r}{2BA}$$

where we have the letters r, t for convenience instead of $\bar{r}, \bar{t}$.

Notice the scaling of the components of the momentum - energy tensor in our special case was ~~is~~ tuned appropriately., ~~i.e.~~ ~~e.g.~~ ~~[illegible] special solution~~

$$T^{00} \sim \frac{1}{t^2} \approx \frac{1}{t^2}\left(1 \pm (1-\xi^2)^{1/2}\right)^{-2},$$

~~and hence~~

$$\bar{r}^2 T^{00} \sim \frac{\bar{r}^2}{\bar{t}^2}\left(1 \pm (1-\xi^2)^{1/2}\right)^{-2}$$

$$\sim \xi^2\left(1 \pm (1-\xi^2)^{-1/2}\right)^{-2}$$

are thus we assume $r^2 T^{00}$, $r^{2} T^{01}$, etc

all scale to be functions of ξ so that

$v(\xi)$ is now an unknown. Hence

there are 3 unknowns: A, B, v.

If we set

$$G = \frac{\xi}{\sqrt{AB}}$$

we can eliminate B in favor of G. The

three ordinary differential equations for A, B, v.

Also the four velocity that

enters the energy-momentum tensor

now has the form

$(U^0, U^1, 0, 0)$ where

$$\frac{U^1}{U^0} = v\sqrt{AB}$$

to allow for desired self similarity.

The ODEs are

$$\xi A_\xi = -\left[\frac{4(1-A)\nu}{(3+\nu^2)G - 4\nu}\right],$$

$$\xi G_\xi = -G\left\{\left(\frac{1-A}{A}\right)\frac{2(1+\nu^2)G - 4\nu}{(3+\nu^2)G-4\nu} - 1\right\},$$

$$\xi \nu_\xi = -\left(\frac{1-\nu^2}{2\{\cdot\}_D}\right)\left\{(3+\nu^2)G - 4\nu + \frac{4\left(\frac{1-A}{A}\right)\{\cdot\}_N}{(3+\nu^2)G-4\nu}\right\},$$

where

$$\{\cdot\}_N = \left\{-2\nu^2 + 2(3-\nu^2)\nu G - (3-\nu^4)G^2\right\},$$

$$\{\cdot\}_D = \left\{(3\nu^2-1) - 4\nu G + (3-\nu^2)G^2\right\}.$$

with energy E satisfying

$$\kappa E = \frac{3(1-\nu^2)(1-A)G}{(3+\nu^2)G-4\nu}\frac{1}{r^2}.$$

(Notice the momentum-energy tensor will be of the form ~ (function of ξ) t^{-2} as we desired.)

Notice that ξ does not enter the right hand sides of our system of ODEs.

Hence the change of independent variable

$$\xi = e^s$$

changes the non-autonomous system into an autonomous one, i.e.

$$\xi A_\xi = A_s, \text{ etc.}$$

Also ~~note that the special~~ solution we ~~derived initially via our change~~ of ~~independent~~ variables is now

~~with~~