[From A. M. Anile, Relativistic fluids and magneto-fluids, Cambridge Univ Press (1989)] (186)

12. Quasi-linear hyperbolic systems in conservation form

The equations of a perfect fluid are

$$\frac{\partial \rho}{\partial t} + v^i \frac{\partial \rho}{\partial x^i} + \rho \frac{\partial v^i}{\partial x^i} = 0$$

$$\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} + \frac{1}{\rho} \frac{\partial p}{\partial x^i} = 0$$

$$\frac{\partial S}{\partial t} + v^i \frac{\partial S}{\partial x^i} = 0$$

where ρ, p, S, v^i are respectively the ~~equations of~~ mass density, pressure, specific entropy, and velocity. The pressure p is assumed to be given by a state equation of the form

$$p = p(\rho, S)$$

Introduce the column vector

$$\mathbf{V} = \begin{pmatrix} \rho \\ v^i \\ S \end{pmatrix}$$

so that the equations can be written as

$$\frac{\partial \underline{V}}{\partial t} + A^i(\underline{V}) \frac{\partial \underline{V}}{\partial x^i} = 0$$

where the matrices A^i are

$$A^1 = \begin{bmatrix} v^1 & \rho & 0 & 0 & 0 \\ \frac{p_\rho}{\rho} & v^1 & 0 & 0 & \frac{p_s}{\rho} \\ 0 & 0 & v^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & v^1 \end{bmatrix},$$

$$A^2 = \begin{bmatrix} v^2 & 0 & \rho & 0 & 0 \\ 0 & v^2 & 0 & 0 & 0 \\ \frac{p_\rho}{\rho} & 0 & v^2 & 0 & \frac{p_s}{\rho} \\ 0 & 0 & 0 & v^2 & 0 \\ 0 & 0 & 0 & 0 & v^2 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} v^3 & 0 & 0 & \rho & 0 \\ 0 & v^3 & 0 & 0 & 0 \\ 0 & 0 & v^3 & 0 & 0 \\ \frac{p_\rho}{\rho} & 0 & 0 & v^3 & \frac{p_s}{\rho} \\ 0 & 0 & 0 & 0 & v^3 \end{bmatrix}$$

A quasi-linear system is said to be <u>hyperbolic</u> in the time direction if the eigenvalue problem

$$(-\lambda I + A^i(\underline{V}) n_i)\, \underline{d} = 0$$

where $\underline{n}$ is an arbitrary unit vector and I is the identity matrix, has N real eigenvalues $\lambda^{(i)}$ (where N is the number of components of $\underline{V}$), and the corresponding set of eigenvectors $\underline{d}^{(i)}$ spans the N dimensional Euclidean space $\mathbb{R}^N$.

The definition of a quasi-linear hyperbolic system can be formulated in a covariant framework as follows.

Let $\mathcal{M}$ be a space-time endowed with a Lorentzian metric g (As usual the speed of light $= 1$). Denote the covariant derivatives by ∇.

In this case the field $\underline{U}$ representing physical quantities will consist of a set of (piecewise) differentiable tensor fields. In local coordinates x^α (lower case Greek indices from 0 to 3 and Latin ones from 1 to 3 except when stated otherwise), the field $\underline{U}$ will have components $U^A(x^\alpha)$, $A = 1, \ldots, N$, and its covariant derivative ∇U by definition will have components $\nabla_\alpha U^A$.

In M we consider a quasi-linear system of N first order partial differential equations for the unknown field $\underline{U}$ which in local coordinates (x^μ) is is written as

$$A_B^{\alpha A}(\Psi^C)\nabla_\alpha \Psi^B = f^A(\Psi^C)$$

where $A_B^{\alpha A}(\Psi^C)$ and $f^A(\Psi^C)$ are differentiable functions of Ψ^C is some open domain $D \subseteq \mathbb{R}^N$.

Following Friedrichs (CPAM, 27, 749-808 (1974)) introduce the following definitions.

<u>Definition</u> Let ξ^α be a differentiable time-like unit vector field in W, W an open connected subset of M. The above system of partial differential equations is called <u>hyperbolic</u> in the time direction defined by ξ^α if the following two conditions hold in W:

(i) $\det(A^\alpha \xi_\alpha) \neq 0$, where A is the matrix whose components in a local chart $A^{\alpha A}_{\ \ B}$;

(ii) For any spacelike vector field ζ_α on W, the eigenvalue problem

$$A^\alpha(\zeta_\alpha - \mu \xi_\alpha)\underline{d} = 0$$

has only real eigenvalues μ and N linearly independent eigenvectors $\underline{d}$.

Note "spacelike" just roughly means

$$\zeta_\alpha \xi^\alpha = 0 \qquad \zeta_\alpha \zeta^\alpha = 1$$

i.e. space-like is a unit vector orthogonal to the time like vector.

More generally, ~~the~~ in fact any space-like ζ_α can be decomposed ~~into~~ as

$$\zeta_\alpha = b(\nu_\alpha - a\xi_\alpha)$$

where $\nu^\alpha \nu_\alpha = 1$, $\nu^\alpha \xi_\alpha = 0$, a and b real numbers, $b > 0$.

When the roots μ are all distinct the system will be said to be <u>strictly</u> <u>hyperbolic</u>.

The vectors $\zeta_\alpha - \mu\xi_\alpha$ are called <u>characteristic</u> and ξ_α is called <u>subcharacteristic</u>.

<u>Definition</u> The system of PDEs is said to be in conservation form if there is $F^\alpha(\underline{\Psi})$ such that

$$A^{\alpha A}_{B}(\Psi^C) = \frac{\partial F^{\alpha A}}{\partial \Psi^B}$$

$F^{\alpha A}(\Psi^C)$ being differentiable functions of $\underline{\Psi} \in \underline{D} \subseteq \mathbb{R}^N$.

In what follows we consider hyperbolic systems in conservation form. Also we are interested in cases where one can derive a "supplementary conservation law"

$$\nabla_\alpha h^\alpha(\Psi^c) = g(\Psi^c)$$

with h^α, g differentiable functions of $\underline{\Psi}^a \in D \subseteq \mathbb{R}^N$.

This is the case of the equations of compressible fluid dynamics given earlier since those equations imply the additional conservation law for conservation of energy:

$$\frac{\partial}{\partial t}\left(\rho\left(\varepsilon + |\underline{v}|^2/2\right)\right) + \frac{\partial}{\partial x^i}\left[\rho v^i\left(|\underline{v}|^2/2 + \varepsilon + p\right)\right] = 0$$

where ε is the specific internal energy. Here the internal energy ε must obey the equations

$$p(\rho, S) = \rho^2 \varepsilon_\rho(\rho, S), \quad \theta(\rho, S) = \varepsilon_S(\rho, S)$$

$\theta > 0$ is the absolute temperature. For example the polytropic gas satisfies

$$\varepsilon = c\rho^{\gamma-1} e^{S/c}, \quad p = R\rho^\gamma e^{S/c}, \quad \theta = \rho^{\gamma-1} e^{S/c}$$

From the first relation $\varepsilon = \varepsilon(\rho, S)$ we could solve for S as a function of ρ, ε and hence $p(\rho, S)$ becomes $p(\rho, \varepsilon)$. Then we would use the energy equation as our fundamental balance law and ~~keep~~ use the entropy equation as a supplementary balance law.

The existence of a supplementary conservation law is fundamental to the understanding of existence - uniqueness theory to systems of hyperbolic balance equations.

We remark that when $\mathcal{M}$ is Minkowski space time the we have

$$ds^2 = -dt^2 + g_{ij}\, dx^i dx^j$$

and the choice $\xi^\alpha = (1,0,0,0)$ in the previous definitions reduce to the usual ones in Cartesian coordinates which we gave earlier, with $-\mu$ as eigenvalues.

Next let us do the same discussion regarding conservation laws and the supplementary conservation law for a relativistic fluid. Recall that we had ~~Einstein's equations gave~~

$$\nabla_\alpha (\rho U^\alpha) = 0$$

for mass conservation and

$$\nabla_\alpha T^{\alpha\beta} = 0$$

where $T^{\alpha\beta}$ was the energy momentum tensor

$$T^{\alpha\beta} = (E+p)\, U^\alpha U^\beta + p g^{\alpha\beta}$$

Before we can do our computation of the conservation laws recall that in the Einstein equation g is an unknown quantity that must be solved for. The simpler case is when the gravitational effects are neglected and we take

$$T^{\alpha\beta} = (E+p)\, U^\alpha U^\beta + p \eta^{\alpha\beta}$$

Then

$\nabla_\alpha T^{\alpha\beta} = 0$ implies

$$\nabla_\alpha (E+p)\, U^\alpha U^\beta + (E+p) \nabla_\alpha U^\alpha U^\beta + (E+p)\, U^\alpha \nabla_\alpha U^\beta$$

$$+ \nabla_\alpha p\, \eta^{\alpha\beta} = 0$$

If we raise and lower subscripts and superscripts according to the Lorentzian metric $\eta^{\alpha\beta}$

we see first

$$U_\beta U^\beta = U_\alpha \eta^{\alpha\beta} U^\beta = -1$$

and contracting our conservation laws with U_β gives

$$U_\beta \nabla_\alpha E\, U^\alpha U^\beta + \underline{U_\beta \nabla_\alpha p\, U^\alpha U^\beta}$$
$$+ (E+p) \nabla_\alpha U^\alpha U_\beta U^\beta + \underline{\underline{(E+p) U^\alpha U_\beta \nabla_\alpha U^\beta}}$$
$$+ \underline{\nabla_\alpha p\, U_\beta \eta^{\alpha\beta}} = 0$$

$$\boxed{- \nabla_\alpha E\, U^\alpha - (E+p) \nabla_\alpha U^\alpha = 0}$$ Energy equation

(since $\nabla_\alpha U_\beta U^\beta = 0$).

Next contract with

$$\eta_{\mu\nu} + U_\mu U_\nu$$

Next contract the conservation laws with

$\eta_{\mu\beta} + U_\mu U_\beta$:

$$(\eta_{\mu\beta} + U_\mu U_\beta)\{\nabla_\alpha (E+p) U^\alpha U^\beta$$

$$+ (E+p)\nabla_\alpha U^\alpha U^\beta + (E+p) U^\alpha \nabla_\alpha U^\beta$$

$$+ \nabla_\alpha p\, \eta^{\alpha\beta}\} = 0$$

$$\underline{\nabla_\alpha (E+p) U^\alpha} U_\mu - \underline{\nabla_\alpha (E+p)} U_\mu U^\alpha$$

$$+ \underline{(E+p)\nabla_\alpha U^\alpha U_\mu} - \underline{(E+p)\nabla_\alpha U^\alpha} U_\mu$$

$$+ (E+p) U^\alpha \nabla_\alpha U_\mu - (E+p) U^\alpha U_\mu \underbrace{U_\beta \nabla_\alpha U^\beta}_{0}$$

$$+ \nabla_\alpha p\, \eta_{\mu\beta}\eta^{\alpha\beta} + U_\mu U_\beta \nabla_\alpha p\, \eta^{\alpha\beta} = 0$$

$$\eta_{\mu\beta}\{(E+p) U^\alpha \nabla_\alpha U^\beta + \nabla_\alpha p\, \eta^{\alpha\beta}\} + U_\mu U_\beta \nabla_\alpha p\, \eta^{\alpha\beta} = 0$$

$$(E+p) U^\alpha \nabla_\alpha U_\mu + \nabla_\alpha p\, \eta_{\mu\beta}\eta^{\alpha\beta} + U_\mu U_\beta \nabla_\alpha p\, \eta^{\alpha\beta} = 0$$

$$(E+p) U^\alpha \nabla_\alpha U_\mu + \eta^{\alpha\beta}\{(\eta_{\mu\beta} + U_\mu U_\beta)\nabla_\alpha p\} = 0$$

$$\eta^{\mu\beta}\ \{(E+p) U^\alpha \nabla_\alpha U_\mu + (\eta^{\alpha\beta}\eta_{\mu\beta} + U_\mu U^\alpha)\nabla_\alpha p\} = 0$$

$$(E+p) U^\alpha \nabla_\alpha U^\beta + (\eta^{\mu\beta} \underbrace{\eta^{\alpha\beta}\eta_{\mu\beta}}_{\delta^\alpha_\mu \eta^{\alpha\beta}} + U^\beta U^\alpha)\nabla_\alpha p = 0$$

since

$$\eta^{\beta\gamma}\eta_{\alpha\gamma} = \delta_\alpha^\beta \equiv \begin{cases} 1 & \alpha=\beta \\ 0 & \alpha\neq\beta \end{cases}.$$

Hence

$$\boxed{(E+p)\,U^\alpha \nabla_\alpha U^\beta + (\eta^{\alpha\beta} + U^\beta U^\alpha)\nabla_\alpha p = 0}$$ Momentum equations

F~~inally~~

$\cancel{U_\alpha \nabla_\beta T^{\alpha\beta}}$

Next rewrite the energy equation

as

$$-\nabla_\alpha\left((E+p)U^\alpha\right) + U^\alpha \nabla_\alpha p = 0$$

$$-\nabla_\alpha\left(\left(\frac{E+p}{\rho}\right)(U^\alpha \rho)\right) + U^\alpha \nabla_\alpha p = 0$$

$$-\rho\,\nabla_\alpha\left(\frac{E+p}{\rho}\right)\cdot U^\alpha + U^\alpha \nabla_\alpha p = 0$$

where we have used

$$\nabla_\alpha(\rho U^\alpha) = 0 \quad \text{(conservation of ~~momentum~~ mass)}$$

Now rewrite the above equation as

$$-\rho U^a \left(p \nabla_a \left(\frac{1}{\rho}\right) + \nabla_a \left(\frac{E}{\rho}\right)\right) = 0$$

The specific entropy S satisfies

$$\theta\, dS = dE + p\, d\left(\frac{1}{\rho}\right)$$

$$\theta \nabla_a S = \nabla_a E + p \nabla_a \left(\frac{1}{\rho}\right) \qquad (\theta \text{ absolute temperature})$$

Hence

$$-\rho\theta U^a \nabla_a S = 0$$

or

$$\boxed{U^a \nabla_a S = 0} \qquad \underline{\text{Entropy equation}}$$

The equations are as usual are closed with an equation of state or constitutive relation

$$p = p(E, S)$$

In summary we write equations of momentum, energy, entropy as

$$(E+p)\,U^{\alpha}\nabla_{\alpha}U^{\beta} + (\eta^{\alpha\beta} + U^{\alpha}U^{\beta}\nabla_{\alpha}p) = 0 \qquad \text{(momentum)}$$

$$(E+p)\quad \nabla_{\alpha}U^{\alpha} + U^{\alpha}\nabla_{\alpha}E \qquad = 0 \qquad \text{(energy)}$$

$$U^{\alpha}\nabla_{\alpha}S = 0 \qquad \text{(entropy)}$$

Hence with field vector

$$\Psi = \begin{pmatrix} U^{\alpha} \\ E \\ S \end{pmatrix}$$

the equations become

$$A^{\alpha A}_{\ \ B}\,\nabla_{\alpha}\Psi^{B} = 0$$

where $h_{\mu\nu} \doteq \eta_{\mu\nu} + U_{\mu}U_{\nu}$ and

$$A^{\alpha} = \begin{bmatrix} (E+p)\,U^{\alpha}\delta^{\mu}_{\nu} & h^{\alpha\mu}\,p_E & h^{\alpha\mu}\,p_S \\ (E+p)\,\delta^{\alpha}_{\nu} & U^{\alpha} & 0^{\alpha} \\ 0^{\alpha}_{\nu} & 0^{\alpha} & U^{\alpha} \end{bmatrix}$$

<u>Theorem</u> The above system for (U^0, E, S) with the addition restriction

$$0 < \frac{\partial p}{\partial E} \leq 1$$

is hyperbolic

This is proven with a long and rather unpleasant calculation. This issue will motivate us to recall the (by now) classic theory of Lax & Friedrichs and Godunov on symmetrization via the use supplementary a supplementary conservation law.

For example in the above set of equations the conservation of mass

$$\nabla_a(\rho U^a) = 0$$

did not enter our system of equations

and can be view as a supplementary conservation law. Similarly if we included the conservation of mass but deleted the conservation of ~~momenta~~ entropy then conservation of entropy would be the supplementary conservation law.

The theory of T. Ruggeri and A Strumia

(Main field and convex covariant density
for quasi-linear hyperbolic systems,

Annales de l'Institut Henri Poincaré,
Physique théoretique 34, 65–84, (1981))

Let us consider a quasi-linear system of
conservation laws

$$\nabla_\alpha F^{\alpha A} = f^A \quad , \quad F^{\alpha A} = F^{\alpha A}(\Psi^B),$$

We do not as of yet know that the relation

$$A_B^{\alpha A} \equiv \frac{\partial F^{\alpha A}}{\partial \Psi^B}$$

will yield the system to be hyperbolic
but we do assume the first condition of
hyperbolicity

$$\det\left(A_B^{\alpha A}\, \xi_\alpha\right) \neq 0$$

so as to make the system evolutionary in the time like ξ_α direction

As advertised earlier we also assume the existence of a supplementary conservation law of the form

$$\nabla_\alpha h^\alpha = g \qquad h^\alpha = h^\alpha(\Psi^B)$$

which holds as a consequence of the given system of conservation laws. More precisely we assume that the supplementary conservation law is obtained from the original system of conservation laws by linear combinations of the form

$$\Psi'^A A^{\alpha A}_B = \frac{\partial h^\alpha}{\partial \Psi^B},$$

$$\Psi'^A f^A = g.$$

where Ψ'^A are functions of the original column of unknowns Ψ^B, i.e

$$\bar{v}'^{A}\left(A^{\alpha A}_{B}\nabla_\alpha \bar{v}^{B} = f^{A}\right),$$

$$\underbrace{\bar{v}'^{A} A^{\alpha A}_{B}}_{\frac{\partial h^\alpha}{\partial \bar{v}^B}} \nabla_\alpha \bar{v}^{B} = \underbrace{\bar{v}'^{A} f^{A}}_{g},$$

$$\nabla_\alpha h^\alpha = g .$$

Let's rewrite

$$\bar{v}'^{A} A^{\alpha A}_{B} = \frac{\partial h^\alpha}{\partial \bar{v}^B}$$

as

$$\bar{v}'^{A} \frac{\partial F^{\alpha A}}{\partial \bar{v}^B} = \frac{\partial h^\alpha}{\partial \bar{v}^B}$$

or

$$\bar{v}'^{A}\, dF^{\alpha A} = dh^\alpha$$

[This shows that multipliers $\underline{\bar{v}'}$ are determined only by the structure of the system of conservation laws and the supplementary conservation law and does not depend on the particular choice of $\underline{V}$ chosen in order to write the conservation laws as a quasi-linear system. For this reason, in order to obtain explicit expressions for $\underline{V}'$ from the relations

$$V'^A \, dF^{\alpha A} = dh^\alpha$$

(also called compatibility conditions),

it is possible to start with a field $\underline{V}$ which is more suitable.

Because of the condition $\det(A^{\alpha A}_B \xi_\alpha) \neq 0$

we can take as a field

$$V^A = F^{\alpha A} \xi_\alpha ,$$

which implies

$$A^{\alpha A}_B \xi_\alpha = \delta^A_B .$$

Now multiply by ξ_α:

$$\cancel{\xi_\alpha\left(\Psi'^{A}\ f^{*}(F(g))\right)}$$

$$\xi_\alpha\left(\Psi'^{A}\ A^{\alpha A}_{B} = \frac{\partial h^\alpha}{\partial \Psi^B}\right)$$

$$\Psi'^{A}\ \underbrace{\xi_\alpha A^{\alpha A}_{B}}_{\delta^A_B} = \xi_\alpha \frac{\partial h^\alpha}{\partial \Psi^B}$$

$$\Psi'^{B} = \frac{\partial h}{\partial \Psi^B}$$

where

$$h \equiv h^\alpha \xi_\alpha$$

The equation

$$\Psi'^{B} = \frac{\partial h}{\partial \Psi^B}$$

gives a simple relation for the multipliers Ψ' in terms of the ~~unknown~~ field

$$\Psi^A = F^{\alpha A} \xi_\alpha$$

Now we assume the crucial property of of h:

$\underline{V}$ is defined on an open convex subset $\underline{D}$ of $\mathbb{R}^N$

and the h is convex in $\underline{V}$.

Systems of this form are called by Ruggeri and Strumia convex covariant density systems.

For such system we compute using first the known relation

$$V'^A = \frac{\partial h}{\partial V^A}$$

~~that~~

$$\frac{\partial V'^A}{\partial V^B} = \frac{\partial^2 h}{\partial V^A \partial V^B}.$$

Since the right hand side is symmetric and positive definite in the convex domain $\underline{D}$ of $\mathbb{R}^N$ so is the the Jacobian matrix on the ~~left~~ hand side. Therefore a theorem of analysis (M. Berger and M. Berger (1968), Perspectives in Nonlinearity, New York: Benjamin) shows the mapping $\underline{v} \Leftrightarrow \underline{v}'$ is globally invertible in D. Hence $\underline{v}'$ can be taken as the new field of unknowns in $\underline{D} \subseteq \mathbb{R}^N$.

This remarkable property is the motivation for calling $\underline{v}'$ the <u>main field associated (uniquely) with the convex covariant density system</u>

$$\left\{ \begin{array}{l} \nabla_\alpha F^{\alpha A} = f^A, \\ \nabla_\alpha \lambda^\alpha = g, \end{array} \right\}.$$

We now state the fundamental theorem.

<u>Theorem</u> A convex covariant density system is a conservative symmetric hyperbolic system in the field $\underline{v}'$.

<u>Proof</u> Consider the Legendre transformation

$$h'^{\alpha} \equiv v'^{A} F^{\alpha A} - h^{\alpha}$$

so that

$$h'^{\alpha} = h'^{\alpha}(v'^{B})$$

Then we compute

$$\cancel{\frac{\partial h'^{\alpha}}{\partial v'^{B}} = F^{\alpha B}}$$

$$\frac{\partial h'^{\alpha}}{\partial v'^{B}} = \underbrace{\frac{\partial v'^{A}}{\partial v'^{B}}}_{\delta^{AB}} F^{\alpha A} + \underbrace{\underbrace{v'^{A} \frac{\partial F^{\alpha A}}{\partial v'^{B}}}_{\frac{\partial h^{\alpha}}{\partial v'^{B}}} - \frac{\partial h^{\alpha}}{\partial v'^{B}}}_{=0}$$

(Recall

$$v'^{A}\, dF^{\alpha A} = dh^{\alpha}$$

so that $v'^{A} \dfrac{\partial F^{\alpha A}}{\partial v'^{B}} = \dfrac{\partial h^{\alpha}}{\partial v'^{B}}$)

So

$$\frac{\partial h'^{\alpha}}{\partial v'^{B}} = F^{\alpha B}$$

or rewriting this as

$$\frac{\partial h'^{\alpha}}{\partial v'^{A}} = F^{\alpha A}.$$

Now differentiate

$$\underbrace{\frac{\partial^2 h'^{\alpha}}{\partial v'^{A}\,\partial v'^{B}}}_{\text{symmetric}} = \frac{\partial F^{\alpha A}}{\partial v'^{B}}$$

So $\dfrac{\partial F^{\alpha A}}{\partial v'^{B}}$ is symmetric and we define

$$A'^{\alpha A}_{B} \equiv \frac{\partial F^{\alpha A}}{\partial v'^{B}}.$$

We can verify

$$\det\left(A'^{\alpha A}_{B}\,\xi_{\alpha}\right) \neq 0$$

(the first condition of hyperbolicity)

and furthermore

since

$$h'^{\alpha} \equiv v'^{A} F^{\alpha A} - h^{\alpha}$$

$$\underbrace{h'^{\alpha} \xi_{\alpha}}_{\parallel} = v'^{A} \underbrace{\xi_{\alpha} F^{\alpha A}}_{v^{A}} - \underbrace{\xi_{\alpha} h^{\alpha}}_{h}$$

$$h' = v'^{A} v^{A} - h$$ (see p. 206)

So h' is the Legendre transform of h.

But the Legendre transform of a convex function of $\underline{v}$ (i.e. h) is a convex function of $\underline{v}'$. Thus the second condition (ii) of hyperbolicity holds as well.

In summary we started with a system

$$\left\{ \begin{array}{l} \nabla_{\alpha} F^{\alpha A} = f^{A} \\ \nabla_{\alpha} h^{\alpha} = g \end{array} \right\}$$

for $\underline{v}$, we make the change of variables

$$V'^A = \frac{\partial h}{\partial V^A}.$$

to rewrite

$$\nabla_\alpha F^{\alpha A}(\underline{V}') = h^\alpha$$

$$\underbrace{\frac{\partial F^{\alpha A}}{\partial V'^B}}_{A'^{\alpha A}_B} \nabla_\alpha V'^B = h^\alpha$$

$$\boxed{\underbrace{A'^{\alpha A}_B}_{} \nabla_\alpha V'^B = h^\alpha}$$

~~system~~

(symmetric $A'^{\alpha A}_B = A'^{B}_A$) and

$(\det A'^{\alpha A}_B \xi_\alpha) \neq 0$

This is symmetric hyperbolic system.

Symmetric hyperbolic systems are well posed as reflected in the following useful theorem of A. Fisher and J.E. Marsden (The Einstein evolution equations as a first order quasi-linear symmetric hyperbolic system I. Comm. in Math. Physics 28, 1-38 (1972))

Theorem Let $H^s(\mathbb{R}^n, \mathbb{R}^m)$ denote the Sobolev spaces, $s \geq 0$, with norms

$$\|u\|_s^2 = \sum_{0 \leq |\alpha| \leq s} \int |D^\alpha u(x)|^2 \, dx$$

and let $\mathcal{O}^s \subseteq H^s(\mathbb{R}^n, \mathbb{R}^m)$ be an open set.

Let $\delta > 0$ and for $(t, x, u) \in (-\delta, \delta) \times \mathbb{R}^n \times \mathcal{O}^s$, let $A^i(t, x, u)$ be symmetric $m \times m$ matrices and $B(t, x, u)$ and m component vector, assumed to be H^s - functions of (t, x) and rational functions of u with non-zero denominators. Given $u_0 \in \mathcal{O}^s$, $s > \frac{n}{2} + 2$, there exists $0 < \varepsilon < \delta$ and a unique $u(t, x)$, $|t| < \varepsilon$, $x \in \mathbb{R}^n$, which is H^s in (t, x) and satisfies the following initial value problem

$$\frac{\partial u}{\partial t} = A^i(t,x,u)\frac{\partial u}{\partial x^i} + B(t,x,u),$$

$$u(0,x) = u_0(x)$$

Furthermore $u(t,x)$ depends continuously on u_0 in the H^s topology

Thus <u>convex covariant density systems</u>

$$\left\{\begin{array}{l} \nabla_\alpha F^{\alpha A} = f^A, \\ \nabla_\alpha h^\alpha = g \end{array}\right\}$$

<u>are locally well posed in H^s, $s > \frac{n}{2} + 2$</u>