From Frank Morgan, Riemannian Geometry a beginner's guide, Jones and Bartlett 1993

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Lectures on

Riemann Geometry and Nonlinear Conservation Laws

1. Curves in $\mathbb{R}^n$

Let $\underline{x}(t)$ be a parametrized curve in $\mathbb{R}^n$

Velocity $\dot{\underline{x}}(t) = \underline{v}$

Unit tangent $\frac{\underline{v}}{|\underline{v}|} = \underline{T}$

Curvature $\underline{\kappa}$ is rate of change of $\underline{T}$ w.r.t to arc length

$$\underline{\kappa} = \frac{d\underline{T}}{ds} = \frac{\frac{d\underline{T}}{dt}}{\frac{ds}{dt}} = \frac{1}{|\underline{v}|}\dot{\underline{T}}$$

Since $\underline{T}\cdot\underline{T} = 1$

$$\dot{\underline{T}}\cdot\underline{T} = 0$$

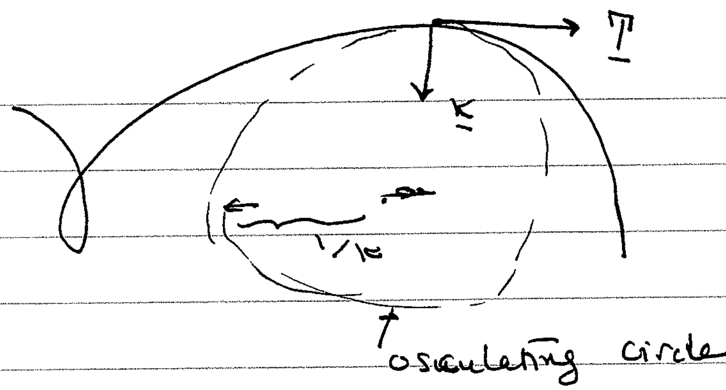
$$\Rightarrow \quad \underline{\kappa}\cdot\underline{T} = 0$$

and curvature vector is orthogonal to $\underline{T}$.

The scalar curvature is

$$\kappa = |\underline{\kappa}|.$$

(For a planar curve with unit normal $\kappa = \left|\frac{d\underline{n}}{ds}\right|$)



Of course since

$$\frac{ds}{dt} = |\underline{v}|,$$

$$\frac{d\underline{x}}{ds} = \frac{d\underline{x}}{dt}\frac{dt}{ds} = \frac{\underline{v}}{|\underline{v}|} = \underline{T}$$

we have

$$\underline{k} = \frac{d^2\underline{x}}{ds^2}.$$

As usual in the plane if we take

$$\underline{x}(t) = (t, f(t))$$

$$\frac{d\underline{x}}{ds} = \left(\frac{dt}{ds}, f'(t)\frac{dt}{ds}\right) = \frac{(1, f'(t))}{\left(\frac{ds}{dt}\right)}$$

$$= \frac{(1, f'(t))}{\sqrt{1+f'^2(t)}}$$

$$\frac{d^2\underline{x}}{ds^2} = \frac{d}{ds}\frac{(1, f'(t))}{\sqrt{1+f'(t)^2}}$$

$$= \frac{d}{dt}\left(\frac{(1, f'(t))}{\sqrt{1+f'^2(t)}} \right) \frac{dt}{ds}$$

$$= \left(\left(-\tfrac{1}{\cancel{2}}\right)(1+f'^2(t))^{-3/2}\, \cancel{2} f'(t)\, f''(t),\right.$$

$$\left. \frac{f''(t)}{\sqrt{1+f'^2(t)}} + f'(t)\left(-\tfrac{1}{\cancel{2}}(1+f'^2(t))^{-3/2}\, \cancel{2} f'(t)\, f''(t) \right) \right) \frac{dt}{ds}$$

$$= \frac{dt}{ds}\, (1+f'^2(t))^{-3/2}\, \big(-f'(t) f''(t),$$

$$f''(t)(1+f'^2(t)) - f'^2(t) f''(t) \big)$$

$$= \frac{dt}{ds}\, (-f'(t) f''(t), f''(t))\, (1+f'^2(t))^{-3/2}$$

$$= (-f'(t) f''(t), f''(t)) \quad (1+f'^2(t))^{-1/2}(1+f'^2(t))^{-3/2}$$

$$= f''(t)\,(-f'(t), 1)\,(1+f'^2(t))^{-2}$$

$$\frac{d^2 \vec{x}}{ds^2} = f''(t)\,(1+f'^2(t))^{-2}\,(-f'(t), 1)$$

$$\kappa = \left| \frac{d^2\vec{x}}{ds^2} \right| = \frac{|f''(t)|}{(1+f'^2(t))^{3/2}}$$

(The usual calculus formula for curve in the plane written as a graph)