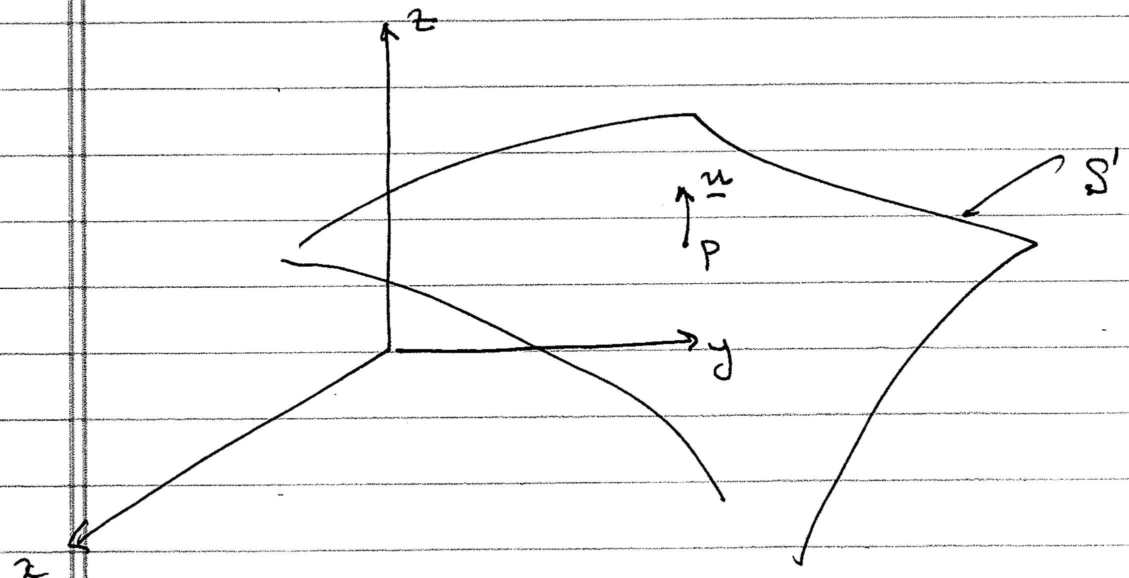
2. Surfaces in $\mathbb{R}^3$

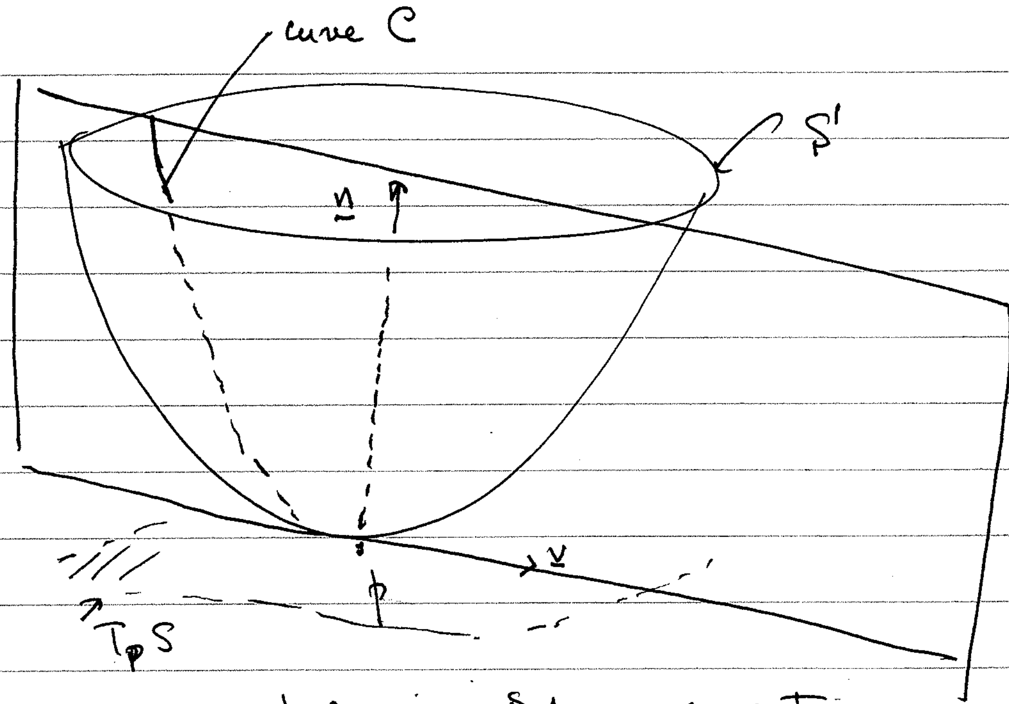
S C^2 surface in $\mathbb{R}^3$



Assume S is graph $(x, y, f(x,y))$

T_pS : <u>tangent space</u> of vectors tangent to S at p

To study curvature of S, we slice S by planes containing $\underline{n}$ and consider the curvature $\underline{k}$ of the resulting curves



Since as noted in §1 curvature vector is always orthogonal to $\underline{v}$ which lies in T_pS, $\underline{\kappa}$ lies in direction of $\underline{n}$, so

$$\underline{\kappa} = \kappa \underline{n}.$$

The largest and smallest curvatures lie in 2 orthogonal directions. These are κ_1, κ_2 (the <u>principal curvatures</u>).

Choose orthonormal coordinates on $\mathbb{R}^3$ with origin at p, S tangent to the x, y plane, $\underline{n}$ pointing in positive z direction as shown above.

The vectors $\underline{v}, \underline{n}$ define a plane as shown which intersects S in a curve. C.

The curvature κ of this curve at p which is called the curvature in the direction $\underline{v}$, is the second derivative

$$\kappa = (D^2 f)_p (\underline{v}, \underline{v}) \equiv \underline{v}^t \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(p) & \frac{\partial^2 f}{\partial x \partial y}(p) \\ \frac{\partial^2 f}{\partial x \partial y}(p) & \frac{\partial^2 f}{\partial y^2}(p) \end{bmatrix} \underline{v}$$

e.g. if $\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\kappa = \frac{\partial^2 f}{\partial x^2}(p)$.

The bilinear form $(D^2 f)_p$ on $T_p S$ is called the <u>second fundamental form</u> II of S at p. In our local coordinates

$$\mathrm{II} = D^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

(Of course the formula is good only a point where the surface is taking to the x, y plane)

For the second fundamental form we will always use orthonormal coordinates.

Since II is symmetric, we may choose coordinates x, y such that II is diagonal:

$$\mathrm{II} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

Then the curvature k is direction

$$\underline{v} = (\cos\theta, \sin\theta)$$

is given by Euler's formula (1760)

$$k = \mathrm{II}(\underline{v}, \underline{v}) = \underline{v}^t\, \mathrm{II}\, \underline{v}$$
$$= k_1 \cos^2\theta + k_2 \sin^2\theta$$

2.1 Definitions

At a point p in a surface $S \subset \mathbb{R}^3$ the eigenvalues $k_1 \geq k_2$ of the second fundamental form II are called the principal curvatures and the corresponding eigenvectors (uniquely determined unless $k_1 = k_2$) are called the principal directions or directions of curvature.

The trace of II $= k_1 + k_2$ is called the mean curvature H

The determinant of II $= k_1 k_2$ is called the Gauss curvature K

Note the signs of II and H but not K depend on choice of unit normal $\underline{n}$.

Also sometimes (but not here) $H = \frac{k_1 + k_2}{2}$.

2.2 Coordinates, length, metric

Local coordinates or parameters u_1, u_2 on a C^2 surface $S \subset \mathbb{R}^3$ are provided by a C^2 diffeomorphism (or parametrization) between a domain in the u_1, u_2 plane a portion of S.

Example The standard spherical coordinates φ, θ provide local coordinates on all of the sphere of radius a except for the poles where longitude θ is undefined and φ is not differentiable. The position vector is determined by the coordinates

$$\underline{x} = (x, y, z) = (a \sin\varphi \cos\theta,\ a \sin\varphi \sin\theta,\ a \cos\varphi).$$

In general the position is some function of the coordinates u_i. Along a curve these coordinates are function of a single parameter t.

Notation: Subscript i $= \frac{\partial}{\partial u_i}$

$$\underline{x}_i = \frac{\partial \underline{x}}{\partial u_i} = \left(\frac{\partial x}{\partial u_i}, \frac{\partial y}{\partial u_i}, \frac{\partial z}{\partial u_i}\right)$$

Dot $\cdot$ denote differentiation w.r.t t

$$\dot{\underline{x}} = \frac{d\underline{x}}{dt} = \sum \underline{x}_i \dot{u}_i \quad \text{(chain rule)}$$

The arc length of a curve in the surface with coordinate $u(t)$ is given by

$$L = \int_{t_0}^{t_1} |\dot{\underline{x}}(t)|\, dt = \int_{t_0}^{t_1} |\underline{x}_1 \dot{u}_1 + \underline{x}_2 \dot{u}_2|\, dt$$

$$= \int_{t_0}^{t_1} \left((\underline{x}_1 \cdot \underline{x}_1)\, \dot{u}_1^2 + 2\, (\underline{x}_1 \cdot \underline{x}_2)\, \dot{u}_1 \dot{u}_2 + (\underline{x}_2 \cdot \underline{x}_2)\, \dot{u}_2^2\right)^{1/2} dt$$

$$= \int_{t_0}^{t_1} \left(\sum g_{ij}\, \dot{u}_i \dot{u}_j\right)^{1/2} dt$$

where

$$g_{ij} = \underline{x}_i \cdot \underline{x}_j \equiv \frac{\partial \underline{x}}{\partial u_i} \cdot \frac{\partial \underline{x}}{\partial u_j}$$

In other words $L = \int ds$ where

$$ds^2 = \sum g_{ij}\, du_i\, du_j$$

Example Sphere of radius a

$$ds^2 = a^2 d\varphi^2 + a^2 \sin^2\varphi \, d\theta^2$$

$$= (a^2 \dot\varphi^2 + a^2 \sin^2\varphi \, \dot\theta^2) \, dt^2$$

so

$$g_{11} = a^2, \quad g_{22} = a^2 \sin^2\varphi, \quad g_{12} = g_{21} = 0.$$

The matrix $g = [g_{ij}]$ is called the first fundamental form or metric.

It is an "intrinsic" form i.e. only depends on information on the surface, since the metric relates to measurements inside the surface.

Notice that in the formula for length

$$\sum g_{ij} \dot u_i \dot u_j = g_{11} \dot u_1^2 + 2 g_{12} \dot u_1 \dot u_2 + g_{22} \dot u_2^2$$

For many surfaces in $\mathbb{R}^3$ it is convenient to use x, y as local coordinates. Then

$$\underline{x}_1 = (1, 0, z_x), \quad \underline{x}_2 = (0, 1, z_y)$$

Proposition 2.5 For any local coordinates u_1, u_2 about a point p in a C^2 surface in $\mathbb{R}^3$, the second fundamental form II at p is similar to

$$g^{-1}(D^2\underline{x})\cdot\underline{n} = g^{-1}\begin{bmatrix} \underline{x}_{11}\cdot\underline{n} & \underline{x}_{12}\cdot\underline{n} \\ \underline{x}_{12}\cdot\underline{n} & \underline{x}_{22}\cdot\underline{n} \end{bmatrix}$$

where

$$\underline{x}_{ij} = \frac{\partial^2 \vec{x}}{\partial u_i \partial u_j} \quad \text{and}$$

$$\underline{n} = \frac{\underline{x}_1 \times \underline{x}_2}{|\underline{x}_1 \times \underline{x}_2|}$$

Consequently

$$H = \operatorname{trace} g^{-1}(D^2\underline{x})\cdot\underline{n} = \frac{\underline{x}_2^2\underline{x}_{11} - 2(\underline{x}_1\cdot\underline{x}_2)\underline{x}_{12} + \underline{x}_1^2\underline{x}_{22}}{\underline{x}_1^2\underline{x}_2^2 - (\underline{x}_1\cdot\underline{x}_2)^2}\cdot\underline{n} \qquad (1)$$

$$(\underline{x}_1^2 \equiv |\underline{x}_1|^2,\ \underline{x}_2^2 \equiv |\underline{x}_2|^2)$$

$$K = \det\left(g^{-1}(D^2\underline{x})\cdot\underline{n}\right) \qquad (2)$$

$$= \frac{(\underline{x}_{11}\cdot\underline{n})(\underline{x}_{22}\cdot\underline{n}) - (\underline{x}_{12}\cdot\underline{n})^2}{\underline{x}_1^2\underline{x}_2^2 - (\underline{x}_1\cdot\underline{x}_2)^2}$$

Note (A) If we solve for the eigenvalues of N and eliminate in favor H, K we have principal curvatures:

$$\kappa = \frac{H \pm \sqrt{H^2 - 4G}}{2}$$

(B) If the surface is a graph

$$\underline{x} = (x, y, f(x,y))$$

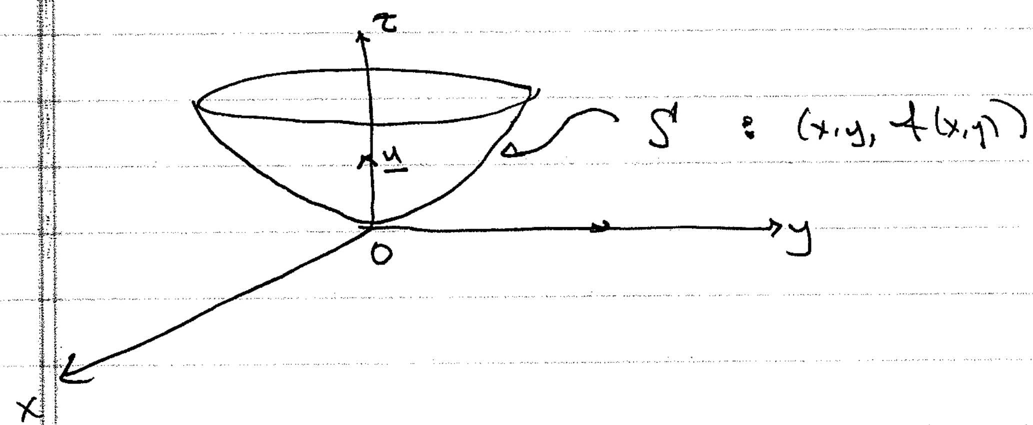
Then

$$H = \frac{(1+f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1+f_x^2) f_{yy}}{(1+f_x^2+f_y^2)^{3/2}}, \qquad (3)$$

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2}. \qquad (4)$$

Proof of Proposition:

Assume S is tangent to the x,y plane at $p = 0$ so S is locally a graph $z = f(x,y)$ with $f_x(0) = f_y(0) = 0$, $\underline{n}(0) = (0,0,1)$.



For these particular local coordinates

$$\underline{x} = (x, y, f(x,y))$$

$$g_{ij} = \underline{x}_i \cdot \underline{x}_j$$

$$g_{11} = (1, 0, \overset{=0}{f_x(0,0)}) \cdot (1, 0, \overset{=0}{f_x(0,0)}) = 1$$

$$g_{22} = (0, 1, f_y(0,0)) \cdot (0, 1, \overset{=0}{f_y(0,0)}) = 1$$

$$g_{12} = (1, 0, \overset{=0}{f_x(0,0)}) \cdot (0, 1, \overset{=0}{f_y(0,0)}) = 0$$

so $\underline{g}(0) = \underline{I}$ (identity matrix)

So the proposition says that Π is similar to

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{\text{at } x=0,\, y=0}$$

which is correct since this is in fact Π.

Now let u_1, u_2 be <u>any</u> local coordinates, and J denote the Jacobian at 0:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} \end{bmatrix}$$