Now write

$x(u_1,u_2)$, $y = y(u_1,y_2)$ so that

$\underline{x} = (x(u_1,u_2), y(u_1,u_2), f(x(u_1,u_2), y(u_1,u_2))$

which gives the surface S in terms of the new local coordinates u_1, u_2

$$z = f(x(u_1,u_2), y(u_1,u_2))$$

So

$$\frac{\partial z}{\partial u_1} = f_x^{\,0} \frac{\partial x}{\partial u_1} + f_y^{\,0} \frac{\partial y}{\partial u_1} = 0 \qquad \text{at} \quad x=0, y=0$$

$$\frac{\partial z}{\partial u_2} = f_x^{\,0} \frac{\partial x}{\partial u_2} + f_y^{\,0} \frac{\partial y}{\partial u_2} = 0$$

and also

$$g = J^T J \quad \text{at} \quad x=0, y=0$$

and

$$\frac{\partial \underline{x}}{\partial x} \cdot \underline{n} = 0 \quad , \quad \frac{\partial \underline{x}}{\partial y} \cdot \underline{n} = 0 \qquad \text{as before}$$

Now we do the computations:

$$\frac{\partial \underline{x}}{\partial u_1} = \left(\frac{\partial x}{\partial u_1}, \frac{\partial y}{\partial u_1}, f_x \frac{\partial x}{\partial u_1} + f_y \frac{\partial y}{\partial u_1} \right)$$

$$\frac{\partial^2 \underline{x}}{\partial u_1^2} = \left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 y}{\partial u_1^2}, f_{xx} \left(\frac{\partial x}{\partial u_1}\right)^2 + 2 f_{xy} \frac{\partial x}{\partial u_1} \frac{\partial y}{\partial u_1} + f_{yy} \left(\frac{\partial y}{\partial u_1}\right)^2 \right)$$

$$\frac{\partial^2 \underline{x}}{\partial u_1 \partial u_2} = \left(\frac{\partial^2 x}{\partial u_1 \partial u_2}, \frac{\partial^2 y}{\partial u_1 \partial u_2}, f_{xx} \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} + f_{xy} \left(\frac{\partial x}{\partial u_1} \frac{\partial y}{\partial u_2} + \frac{\partial y}{\partial u_1} \frac{\partial x}{\partial u_2} \right) + f_{yy} \frac{\partial y}{\partial u_1} \frac{\partial y}{\partial u_2} \right)$$

$$\frac{\partial^2 \underline{x}}{\partial u_2^2} = \left(\frac{\partial^2 x}{\partial u_2}, \frac{\partial^2 y}{\partial u_2^2}, f_{xx} \left(\frac{\partial x}{\partial u_2}\right)^2 + 2 f_{xy} \frac{\partial x}{\partial u_2} \frac{\partial y}{\partial u_2} + f_{yy} \left(\frac{\partial y}{\partial u_2}\right)^2 \right)$$

Since $\underline{u} = (0,0,1)$

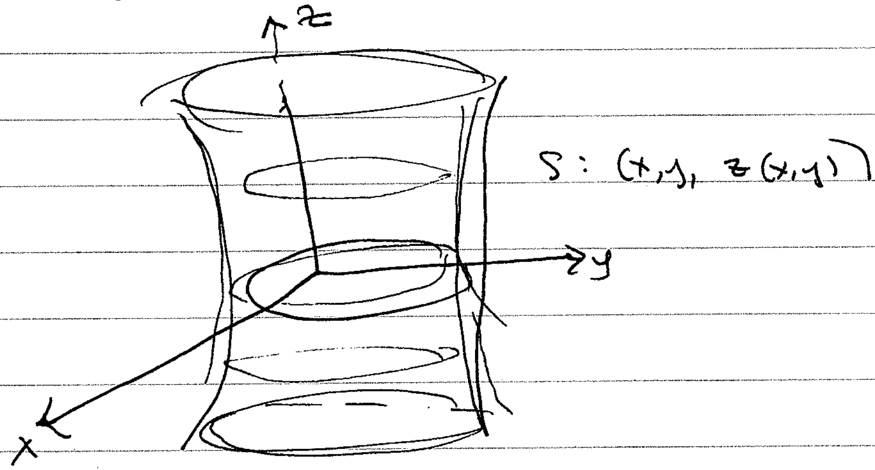
$$\begin{bmatrix} \underline{x}_{11} \cdot \underline{u} & \underline{x}_{12} \cdot \underline{u} \\ \underline{x}_{12} \cdot \underline{u} & \underline{x}_{22} \cdot \underline{u} \end{bmatrix} = J^T \Pi J$$

But $J^T J = g$ so $J^T = g J^{-1}$ and

$$\begin{bmatrix} \underline{x}_{11} \cdot \underline{u} & \underline{x}_{12} \cdot \underline{u} \\ \underline{x}_{12} \cdot \underline{u} & \underline{x}_{22} \cdot \underline{u} \end{bmatrix} = g J^{-1} \Pi J$$

And this proves the proposition.

Example The catenoid



$$\sqrt{x^2+y^2} = \cosh z$$

Instead of using (x,y) as local coordinates we use polar angle θ and z as local coordinates

$x = r\cos\theta$, $y = r\sin\theta$ implies the surface is given by

$$r = \cosh z$$

and hence

$$(x,y,z) = (\cosh z\cos\theta, \cosh z\sin\theta, z)$$

With $u_1 = z$, $u_2 = \theta$ [our local coordinates]

$$\underline{x}_1 = (\sinh z \cos\theta, \sinh z \sin\theta, 1)$$

$$\underline{x}_2 = (-\cosh z \sin\theta, +\cosh z \cos\theta, 0)$$

$$\underline{x}_{11} = (\cosh z \cos\theta, \cosh z \sin\theta, 0)$$

$$\underline{x}_{12} = (-\sinh z \sin\theta, \sinh z \cos\theta, 0)$$

$$\underline{x}_{22} = (-\cosh z \cos\theta, -\cosh z \sin\theta, 0)$$

$$\underline{n} = \frac{(\cos\theta, \sin\theta, \sinh z)}{\cosh z} \quad \left(= \frac{\underline{x}_1 \times \underline{x}_2}{|\underline{x}_1 \times \underline{x}_2|} \right)$$

By Proposition 2.5 (1):

$$H = \frac{(\underline{x}_2^2 \, \underline{x}_{11} - 2(\underline{x}_1 \cdot \underline{x}_2)\underline{x}_{12} + \underline{x}_1^2 \cdot \underline{x}_{22})}{\text{something}} \cdot \underline{n}$$

$$= \frac{(\cosh^2 z \, \underline{x}_{11} - 0 \, \underline{x}_{12} + \cosh^2 z \, \underline{x}_{22})}{\text{something}} \cdot \underline{n}$$

$$= \frac{\cosh^2 z \, \overbrace{(\underline{x}_{11} + \underline{x}_{22})}^{= \underline{0}}}{\text{something}} \cdot \underline{n}$$

$$= 0,$$

and so $K_1 + K_2 = 0$.

By Proposition 2.5 (2):

$$K = \frac{(\underline{x}_{11}\cdot\underline{u})(\underline{x}_{22}\cdot\underline{n}) - (\underline{x}_{12}\cdot\underline{u})^2}{\underline{x}_1^2\,\underline{x}_2^2 - (\underline{x}_1\cdot\underline{x}_2)^2}$$

$$= \frac{1\cdot(-1) - 0}{\cosh^2 z\,\cosh^2 z - 0} = -\cosh^{-4} z$$

Hence $\kappa_1 \kappa_2 = -\cosh^{-4} z$ and since $\kappa_1 = -\kappa_2$

$$\kappa_1^2 = \cosh^{-4} z, \quad \text{and so.}$$

$$\left\{ \begin{array}{l} \kappa_1 = \cosh^{-2} z, \\ \kappa_2 = -\cosh^{-2} z. \end{array} \right\}$$

Notice the catenoid has negative Gauss curvature everywhere but the Gauss curvature goes to zero exponentially fast as $|z| \to \infty$.

2.6 Gauss's Theorem Egregium.

<u>Theorem.</u> The Gauss curvature $\mathbb{K}$ is intrinsic.
Specifically there are local coordinates u_1, u_2 about any point p in a C^2 surface $\mathcal{S}$ in $\mathbb{R}^3$ such that the first fundamental form g at p is I to first order.
In any such coordinate system, the Gauss curvature is

$$\mathbb{K} = \frac{\partial^2 g_{12}}{\partial u_1 \partial u_2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2^2}$$

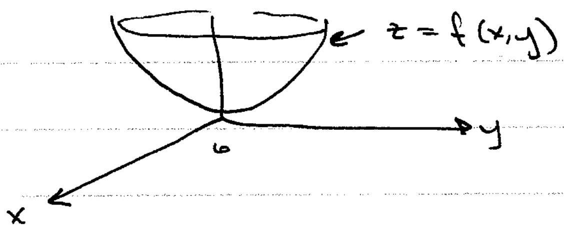
<u>Remark</u> To say that $g = I$ to first order means that

$$g_{11}(p) = g_{22}(p) = 1, \quad g_{12}(p) = 0 \text{ and each}$$

$$g_{ij,k}(p) \equiv \frac{\partial g_{ij}}{\partial u_k}(p) = 0.$$

<u>Proof</u> Locally S' is the graph of a function f over its tangent plane.

Choose orthonormal coordinates x, y



Since $f_x(0) = f_y(0) = 0$,

$$g = \begin{bmatrix} 1 + f_x^2 & f_{xy} \\ f_{xy} & 1 + f_y^2 \end{bmatrix} = I + \text{quadratic terms}$$

So g equals I to to first order ~~[illegible]~~.
in neighborhood of $(0,0)$

Now a direct to computation shows

$$\frac{\partial^2 g_{12}}{\partial x \partial y} = \frac{\partial}{\partial x \partial y}(f_{xy}) = f_{xx} f_{yy} + \text{quadratic terms}$$

$$\frac{\partial g_{22}}{\partial x^2} = \frac{\partial}{\partial x^2}(f_y^2) = f_{xy}^2 + \cdots$$

$$\frac{\partial g_{11}}{\partial y^2} = \frac{\partial}{\partial y^2}(f_x^2) = f_{xy}^2 + \cdots$$

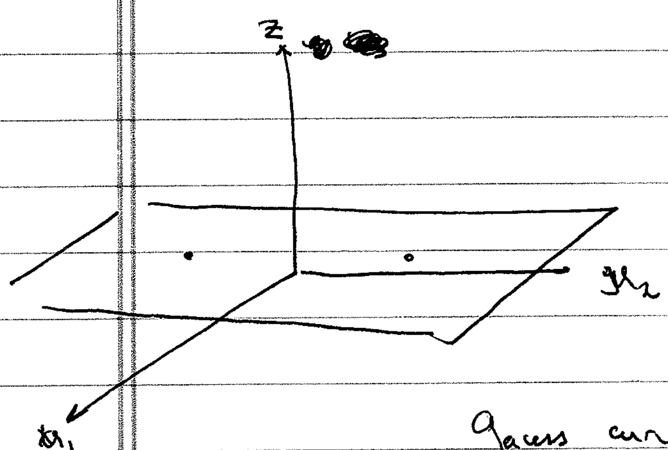
and so

$$\frac{\partial^2 g_{12}}{\partial x \partial y} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial y^2} =$$

$$f_{xx} f_{yy} - f_{xy}^2 = \det D^2 f = \det II = G \quad \text{at } x=0, y=0$$

But any coordinates u_1, u_2 for which the metric at p is I agree with the special x, y coordinates on the tangent plane to first order and will not change the result.

<u>Example</u>



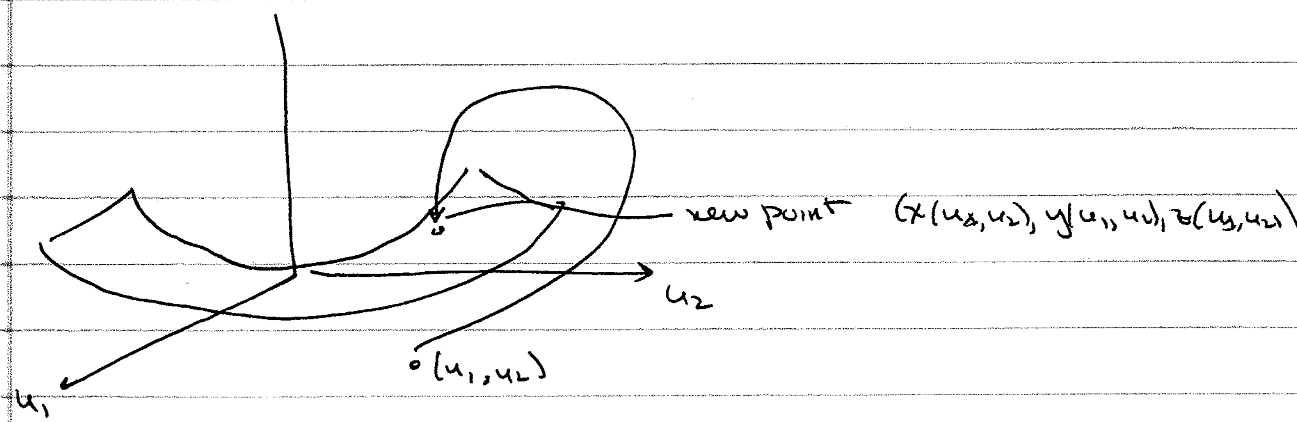
(flat piece of paper)

Gauss curvature $= 0$

k_1, k_2 = Principle curvatures $= (0, 0)$

Assume the paper is bent but not stretched so that a point (u_1, u_2) in the flat paper $z=0$ moves to a new point

$(x(u_1,u_2), \; y(u_1,u_2), \; z(u_1,u_2))$



We know the metric at the new point

$$g_{ij} = \frac{\partial \mathbf{x}}{\partial u_i} \cdot \frac{\partial \mathbf{x}}{\partial u_j}$$

i.e.,

$$g_{11} = \left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2 ,$$

$$g_{12} = \frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2} + \frac{\partial y}{\partial u_1}\frac{\partial y}{\partial u_2} + \frac{\partial z}{\partial u_1}\frac{\partial z}{\partial u_2} ,$$

$$g_{22} = \left(\frac{\partial x}{\partial u_2}\right)^2 + \left(\frac{\partial y}{\partial u_2}\right)^2 + \left(\frac{\partial z}{\partial u_2}\right)^2 .$$

But what if we assume more i.e. that new metric is exactly the same as the old metric on the flat piece of paper, i.e distances are preserved in our bending process so that paper is deformed <u>isometrically</u>. In our example this means

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = 1$$

since on the flat paper

$$ds^2 = (du_1)^2 + (du_2)^2 .$$

Since Gauss's Theorem Egregium states the Gauss curvature depends only on the metric and metric is preserved the Gauss curvature is unchanged under isometric deformation. In our example this means $K = 0$ always.

For example bending into a cylinder yields principal curvatures (κ_1, κ_2) going from $(0,0)$ to $(\frac{1}{r}, 0)$ and mean curvature going from 0 to $\frac{1}{r}$.

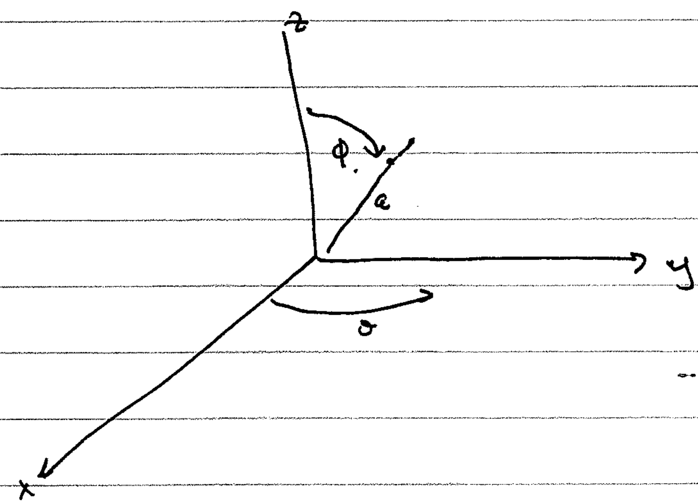
Hence the Gauss curvature is "intrinsic", once we know the metric we know the Gauss curvature. The curvatures are "extrinsic" and are not determined by metric alone.

Notice also that trying to solve for allowable x, y, z as functions of u_1, u_2 leads to systems of <u>fully</u> <u>non-linear</u> <u>partial</u> <u>differential</u> <u>equations</u>.

A calculus exercise.

The surface of a sphere may defined by local coordinate ϕ, θ

$$x = a \sin\phi \cos\theta, \quad y = a \sin\phi \sin\theta, \quad z = a \cos\phi$$



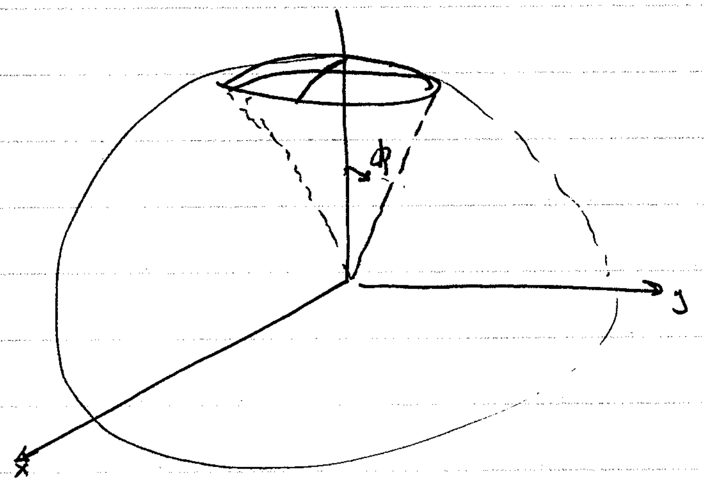
by using spherical coordinates and radius of the sphere $= a$.

The Gauss curvature from our formula $= \frac{1}{a^2}$.

But increment of surface area on a sphere is

$a^2 \sin\phi \, d\phi \, d\theta$ so that area of a polar cap

with angle ϕ_1 can easily be computed



$A =$ area of polar cap $= \int_0^{2\pi}\int_0^{\phi_1} a^2 \sin\phi \, d\phi \, d\theta$

$$A = 2\pi a^2 (1 - \cos\phi_1)$$

We define the intrinsic radius for the ~~sphere~~ polar cap as $r = a\phi_1$.

$$A = 2\pi r^2 \frac{(1-\cos\phi_1)}{\phi_1^2} = 2\pi r^2 \left(\frac{1}{2!} - \frac{\phi_1^2}{4!} + \dots \right)$$

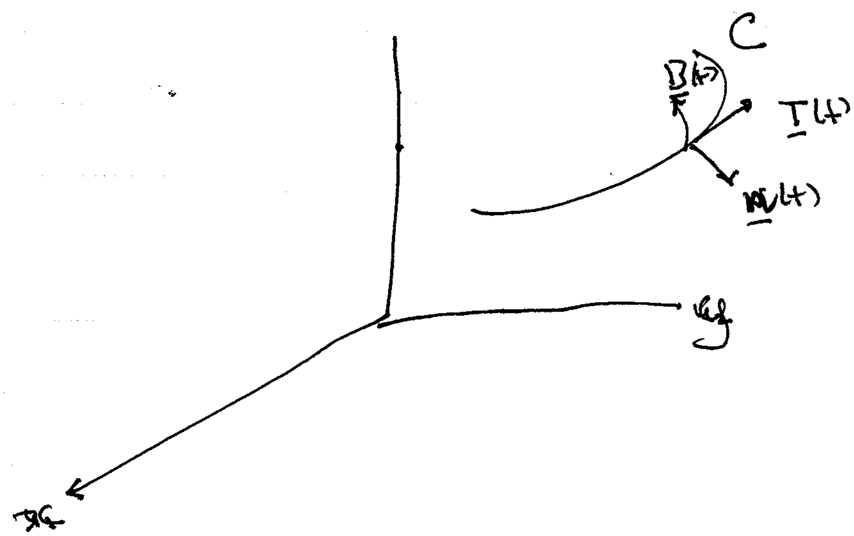
$$= 2\pi r^2 \left(\frac{1}{2!} - \frac{r^2}{a^2 4!} + \dots \right)$$

$$= \pi r^2 \left(1 - \frac{r^2}{12 a^2} + \dots \right)$$

$$A = \pi r^2 \left(1 - K \frac{r^2}{12} + \dots \right)$$

2.7 The Gauss-Weingarten equations

Recall for a space curve C



we can set up a local orthonormal system of coordinates

$\underline{T}(t)$ unit tangent, $\underline{N}(t)$ unit normal,
$\underline{B}(t)$ unit binormal.

If s denotes arc length along C,

$\underline{T}(t)$, $\underline{N}(t)$, $\underline{B}(t)$ are related by the

Frenet-Serret formulas