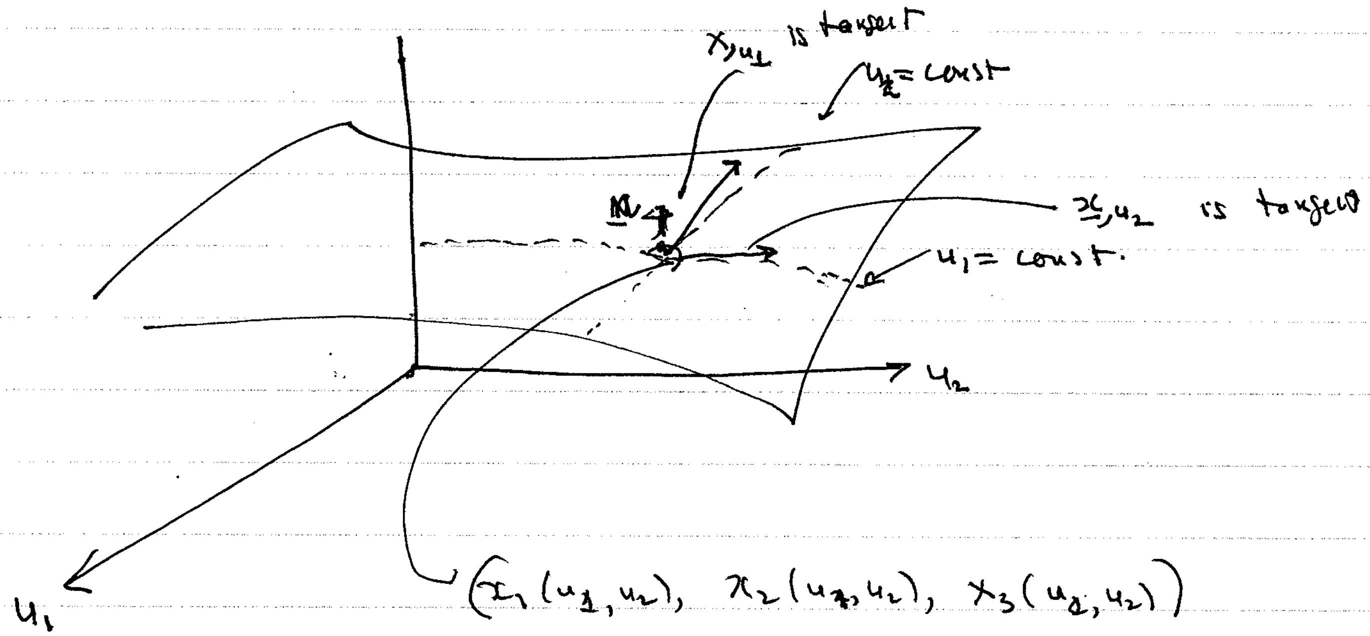
$$\frac{d\underline{T}}{ds} = \kappa \underline{N},$$

$$\frac{d\underline{N}}{ds} = -\kappa \underline{T} + \tau \underline{B},$$

$$\frac{d\underline{B}}{ds} = -\tau \underline{N},$$

κ is the <u>curvature</u> of curve C,
τ is the <u>torsion</u>

There is a similar result for surfaces in $\mathbb{R}^3$



$$\underline{x}(u,v) = (x_1(u,v),\ x_2(u,v),\ x_3(u,v))$$

So as before space curves

$\dfrac{\partial \underline{x}}{\partial u_1}$, $\dfrac{\partial \underline{x}}{\partial u_2}$ are tangent to surface

and

$$\frac{\dfrac{\partial \underline{x}}{\partial u_1} \times \dfrac{\partial \underline{x}}{\partial u_2}}{\left|\dfrac{\partial \underline{x}}{\partial u_1} \times \dfrac{\partial \underline{x}}{\partial u_2}\right|} = \underline{N}$$

is unit normal.

The triple $\left(\frac{\partial \underline{x}}{\partial u_1}, \frac{\partial \underline{x}}{\partial u_2}, \underline{n}\right)$ is called the Gauss frame.

The generalization of the Frenet-Serret formulas are the Gauss-Weingarten equations:

$$\underline{x}_{,u_1u_1} = \Gamma_{11}^{1}\,\underline{x}_{,u} + \Gamma_{11}^{2}\,\underline{x}_{,v} + L\,\underline{n}$$

$$\underline{x}_{,u_1u_2} = \Gamma_{12}^{1}\,\underline{x}_{,u} + \Gamma_{12}^{2}\,\underline{x}_{,v} + M\,\underline{n}$$

$$\underline{x}_{,u_2u_2} = \Gamma_{22}^{1}\,\underline{x}_{,u} + \Gamma_{22}^{2}\,\underline{x}_{,v} + N\,\underline{n}$$

$$\underline{n}_{,u} = \alpha_1\,\underline{x}_{,u_1} + \alpha_2\,\underline{x}_{,u_2}$$

$$\underline{n}_{,v} = \beta_1\,\underline{x}_{,u_1} + \beta_2\,\underline{x}_{,u_2}$$

where the Christoffel symbols are

$$\Gamma_{11}^{1} = G E_{,u_1} + F E_{,u_2} - 2F F_{,u_2} / 2(EG-F^2)$$

$$\Gamma_{11}^{2} = - F E_{,u_1} + E E_{,u_2} - 2E F_{,u_1} / 2(EG-F^2)$$

$$\Gamma_{12}^{1} = G E_{,u_2} - F G_{u_1} / 2(EG-F^2)$$

$$\Gamma_{12}^{2} = E G_{,u_1} - F E_{,u_2} / 2(EG-F^2)$$

$$\Gamma_{22}^{1} = - F G_{,u_2} - G G_{,u_1} + 2 G F_{,u_2} / 2(EG-F^2)$$

$$\Gamma_{22}^{2} = E G_{,u_2} + F G_{,u_1} - 2F F_{,u_2} / 2(EG-F^2)$$

$$\alpha_1 = MF - LG / (EG-F^2)$$

$$\alpha_2 = LF - ME / (EG-F^2)$$

$$\beta_1 = NF - MG / (EG-F^2)$$

$$\beta_2 = MF - NE / (EG-F^2)$$

Here we have reverted to the classical notation for the first and second fundamental forms for surfaces in $\mathbb{R}^3$:

$$g_{11} = E \qquad g_{12} = F \qquad g_{22} = G,$$

$$\mathrm{II}_{11} = L, \quad \mathrm{II}_{12} = M, \quad , \mathrm{II}_{22} = N.$$

2.8 The Gauss- Codazzi Equations

In the case of space curves the Frenet- Serret equations say that given $\kappa(s), \tau(s)$ a space curve is determined uniquely (up to position and orientation) by the usual ODE existence, uniqueness theorem.

We can ask the same question for surfaces. Is there a set of functions the define a surface to within a position in space? Specifically do the first and second fundamental forms as functions of u_1, u_2 define a surface up to a position in space? Do the Gauss- Weingarten Equations themselves provide the appropriate generalization of the Frenet- Serret formulas?

The answer is no and the reason is obvious from one glance at the Gauss- Weingarten system, i.e.

we must satisfy equivalence of mixed

partial derivatives

$$(\bar{X}_{,u_1})_{,u_1 u_2} = (\bar{X}_{,u_1})_{,u_2, u_1}$$

$$(\bar{X}_{,u_2})_{u_1 u_2} = (\bar{X}_{,u_2})_{,u_2 u_1}$$

The mixed partials equality given above when applied to the Gauss- Weingarten system is satisfied if and only if the first and second fundamental forms satisfy the Gauss and Codazzi equations:

Codazzi equations

$$\frac{\partial L}{\partial u_2} - \frac{\partial M}{\partial u_1} = L\Gamma_{12}^1 + M\left(\Gamma_{12}^2 - \Gamma_{11}^1\right) - N\Gamma_{11}^2$$

$$\frac{\partial M}{\partial u_2} - \frac{\partial N}{\partial u_1} = L\Gamma_{22}^1 + M\left(\Gamma_{22}^2 - \Gamma_{12}^1\right) - N\Gamma_{12}^2$$

Gauss equation:

$$LN - M^2 = F\left((\Gamma_{22}^{2})_{,u_1} - (\Gamma_{12}^{2})_{,u_2} + \Gamma_{22}^{1}\Gamma_{11}^{2} - \Gamma_{12}^{1}\Gamma_{12}^{2} \right)$$

But recall the Christoffel symbols are defined in terms of first derivatives of the metric i.e. the first fundamental form. And Gauss's Theorem Egregium relates the 2nd derivatives of the first fundamental form to the Gauss curvature K. So it is no surprise that right hand side of Gauss equation above which is 2nd derivatives of the first fundamental form is nothing more than $K(EF - G^2)$

Hence the Gauss equation is

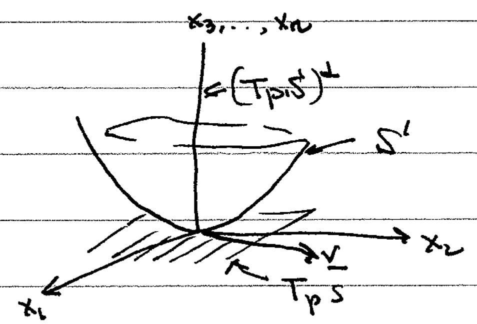
$$LN - M^2 = K(EF - G^2).$$

<u>Fundamental Theorem of Surfaces</u> If first and second fund. forms are sufficiently diff in u_1, u_2, satisfy Gauss-Codazzi eqns and $EG - F^2 > 0, E > 0, G > 0$ there exists a surface uniquely determined up to its position in space which has respectively the given first and second fundamental forms.

3. Surfaces in $\mathbb{R}^n$

We extend the previous section from surfaces in $\mathbb{R}^3$ to surfaces in $\mathbb{R}^n$. As before choose orthonormal coordinates on $\mathbb{R}^n$ with the origin at p and S tangent to x_1, x_2 plane at p. Locally S is the graph of a function

$$f : T_pS \to T_pS^{\perp}$$



$x_3 = f_3(x_1, x_2), \ldots, x_n = f_n(x_1, x_2)$

~~$f = (f_3, \ldots, f_n)$~~

$f = (0, 0, f_3, \ldots, f_n)$

Any unit vector $\underline{v}$ tangent to S at p, together with the vectors normal to S at p, spans a hyperplane, which intersects S in a curve. The curvature vector $\underline{k}$ of this curve, which we call the curvature in the direction $\underline{v}$, is just the second derivative

$$\underline{k} = (D^2 f)_p (\underline{v}, \underline{v})$$

$$= \underline{v}^t \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(p) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(p) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(p) & \frac{\partial^2 f}{\partial x_2^2}(p) \end{bmatrix} \underline{v}$$

Notice now the entries of the 2×2 matrix are not scalar but have values in $(T_p S)^\perp$.

$$\mathrm{II} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \qquad (*)$$

is again the second fundamental tensor of S at p in local coordinates x_1, x_2. More generally

$$\mathrm{II} = (D^2 f)_p .$$

(Again this formula $(*)$ is good only at the point where the surface S is tangent to the x_1, x_2)

Notice when $n=3$

$$\mathrm{II} = \begin{bmatrix} \left(0, 0, \frac{\partial^2 f_3}{\partial x_1^2}\right) & \left(0, 0, \frac{\partial^2 f_3}{\partial x_1 \partial x_2}\right) \\ \left(0, 0, \frac{\partial^2 f_3}{\partial x_1 \partial x_2}\right) & \left(0, 0, \frac{\partial^2 f_3}{\partial x_2^2}\right) \end{bmatrix}$$

The trace of $\mathrm{II} =$

$$\left(0, 0, \frac{\partial^2 f_3}{\partial y_1^2} + \frac{\partial^2 f_3}{\partial y_2^2}\right) = \underbrace{\left(\frac{\partial^2 f_3}{\partial y_1^2} + \frac{\partial^2 f_3}{\partial y_2^2}\right)}_{H} \underline{n}$$

$\underline{H} = H \underline{n}$ where $\underline{H}$ is the mean curvature vector