The scalar quantity

$$\left(0, 0, \frac{\partial^2 f_3}{\partial x_1^2}\right) \cdot \left(0, 0, \frac{\partial^2 f_3}{\partial x_2^2}\right) - \left(0, 0, \frac{\partial^2 f_3}{\partial x_1 \partial x_2}\right) \left(0, 0, \frac{\partial^2 f_3}{\partial x_1 \partial x_2}\right)$$

= Gauss curvature

In general we still take

trace $\mathrm{II} = \underline{H}$ = mean curvature vector

$a_{11} \cdot a_{22} - a_{12} \cdot a_{12} =$ Gauss curvature $= G$

Neither $\underline{H}$ nor G depends on the choice of orthonormal coordinates.

<u>Remark</u> $T_pS^\perp$ and P change from point to point.

If $T_pS^\perp$ is just the $x_3, \dots, x_n$ - plane as shown in the picture

$$P(a_1, a_2, a_3, a_4 \dots) = (0, 0, a_3, a_4 \dots)$$

More generally,

By the usual Gram-Schmidt process if

$\underline{x}_1, \underline{x}_2$ given an orthogonal basis for T_pS

then

$$P(\underline{w}) = w - \frac{\underline{w} \cdot \underline{x}_1}{\underline{x}_1 \cdot \underline{x}} x_1 - \frac{\underline{w} \cdot \underline{x}_2}{\underline{x}_2 \cdot \underline{x}_2} x_2$$

If $\underline{x}_1, \underline{x}_2$ are not orthogonal compute P

by replacing $\underline{x}_2$ by

$$\underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{x}_1}{\underline{x}_1 \cdot \underline{x}_1} x_1$$

(this is Gram-Schmidt)

An example is crucial to understanding

Consider the surface

$$\{ (w,z) \in \mathbb{C}^2 ;\ w = e^z \}$$

so that w, z are each complex valued.

Write $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, so

$$w = e^{x+iy} = e^x(\cos y + i \sin y)$$

and

$$\underline{x} = (x, y, e^x \cos y, e^x \sin y)$$

for this example

$$\underline{x}_1 = (1, 0, e^x \cos y, e^x \sin y)$$

$$\underline{x}_2 = (0, 1, -e^x \sin y, e^x \cos y)$$

$$\underline{x}_{11} = (0, 0, e^x \cos y, e^x \sin y)$$

$$\underline{x}_{12} = (0, 0, -e^x \sin y, e^x \cos y)$$

$$\underline{x}_{22} = (0, 0, -e^x \cos y, -e^x \sin y)$$

Proposition 3.1 For any local coordinates u_1, u_2 and a point p in a C^2 surface S in $\mathbb{R}^n$, the second fundamental tensor Π at p is similar to

$$g^{-1}\,\underline{P}(D^2\underline{x}) = g^{-1}\begin{bmatrix} \underline{P}(\underline{x}_{11}) & \underline{P}(\underline{x}_{12}) \\ \underline{P}(\underline{x}_{12}) & \underline{P}(\underline{x}_{22}) \end{bmatrix}$$

where $\underline{P}$ denotes the projection onto $T_pS^{\perp}$

and $\underline{x}_{ij} = \dfrac{\partial^2 \underline{x}}{\partial u_i\, \partial u_j}$.

Consequently

$$\underline{H} = \operatorname{trace} g^{-1} P(D^2\underline{x})$$

$$= \underline{P}\ \frac{(\underline{x}_2^2\, x_{11} - 2(\underline{x}_1\cdot\underline{x}_2)\,\underline{x}_{12} + \underline{x}_1^2\,\underline{x}_{22})}{\underline{x}_1^2\, x_2^2 - (\underline{x}_1\cdot\underline{x}_2)}$$

$$G = \det\,(g^{-1}\,\underline{P}(D^2\underline{x}))$$

$$= \frac{(\underline{P}\,\underline{x}_{11})\cdot(\underline{P}\,\underline{x}_{22}) - (\underline{P}\,\underline{x}_{12})^2}{\underline{x}_1^2\,\underline{x}_2^2 - (\underline{x}_1\cdot\underline{x}_2)^2}$$

Now $\underline{x_1}^2 = |\underline{x_1}|^2 = 1 + e^{2x}$ $\left(= \underline{x_2}^2 = |\underline{x_2}|^2 \right)$

$\underline{x_1} \cdot \underline{x_2} = 0$ and $\underline{x_{11}} + \underline{x_{22}} = (0, 0, 0, 0)$.

$$\underline{H} = \mathbb{P} \frac{(1+e^{2x})(\underline{x_{11}} + \underline{x_{22}})}{(1+e^{2x})^2},$$

$$\underline{H} = \mathbb{P}(0,0,0,0) = (0,0,0,0)$$

where

$\mathbb{P}(a_1, a_2, a_3, a_4) = (0, 0, a_3, a_4)$.

To compute G:

Since $\underline{x_1}, \underline{x_2}$ are orthogonal are in tangent space T_pS, at p.

$\underline{x_1}, \underline{x_2}$ given an orthogonal basis for T_pS.

$$\mathbb{P}(\underline{x_{11}}) = \underline{x_{11}} - \frac{\underline{x_{11}} \cdot \underline{x_1}}{\underline{x_1} \cdot \underline{x_1}} \underline{x_1} - \frac{\underline{x_{11}} \cdot \underline{x_2}}{\underline{x_2} \cdot \underline{x_2}} \underline{x_2}$$

$$= (0, 0, e^x \cos y, e^x \sin y)$$

$$- \frac{(0 + 0 + e^{2x}(\cos^2 y + \sin^2 y))}{1+e^{2x}} \underline{x_1} - \frac{(0+0+0)}{1+e^{2x}} \underline{x_2}$$

$$\mathbb{I}(\underline{x}_{11}) = (0,0,e^x\cos y, e^x\sin y) - \frac{e^{2x}}{1+e^{2x}}\,\underline{x}_1$$

$$= (0,0,e^x\cos y, e^x\sin y) - \frac{e^{2x}}{1+e^{2x}}(1,0,e^x\cos y, e^x\sin y)$$

$$= \frac{(0,0,e^x\cos y + e^{3x}\cos y,\ e^x\sin y + e^{3x}\sin y)}{1+e^{2x}}$$

$$- \frac{(e^{2x},0,e^{3x}\cos y,\ e^{3x}\sin y)}{1+e^{2x}}$$

$$\mathbb{I}(\underline{x}_{11}) = \frac{(-e^{2x},0,e^x\cos y, e^x\sin y)}{1+e^{2x}}$$

Similarly

$$\mathbb{I}(\underline{x}_{12}) = \frac{(0,-e^{2x},-e^x\sin y, e^x\cos y)}{1+e^{2x}}$$

$$\mathbb{I}(\underline{x}_{22}) = \frac{(e^{2x},0,-e^x\cos y, -e^x\sin y)}{1+e^{2x}}$$

Hence

$$G = \frac{(e^{4x} - e^{2x}) - (e^{4x} + e^{2x})}{(1+e^{2x})^2\left[(1+e^{2x})^2 - 0\right]}$$

$$= \frac{-2e^{2x}}{(1+e^{2x})^3}$$

Gauss's Theorem Egregium still holds:

If G is given in local coordinates in which $g = I$ to first order then

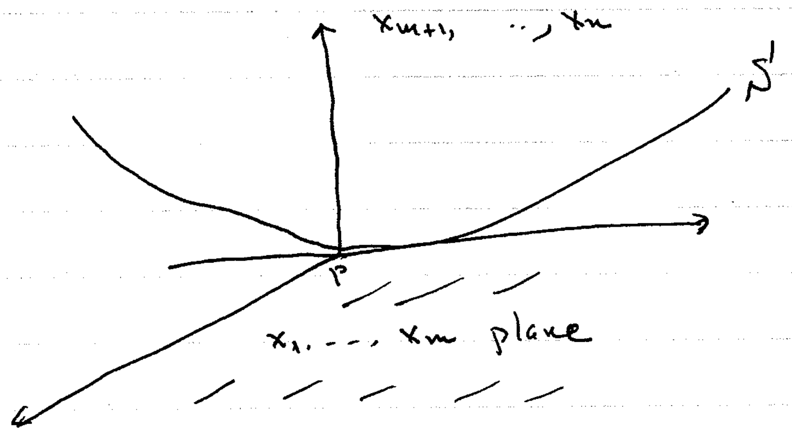
$$G = \frac{\partial^2 g_{12}}{\partial u_1 \partial u_2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2^2}$$

and G is "INTRINSIC."

4 m - Dimensional surfaces in $\mathbb{R}^n$

We now extend the theory to C^2 m-dimensional surfaces S' in $\mathbb{R}^n$. As before origin at p and S tangent to the $x_1, \dots, x_m$ plane at p. Locally S is the graph of the a function

$$f : T_p S' \to (T_p S)^{\perp}$$



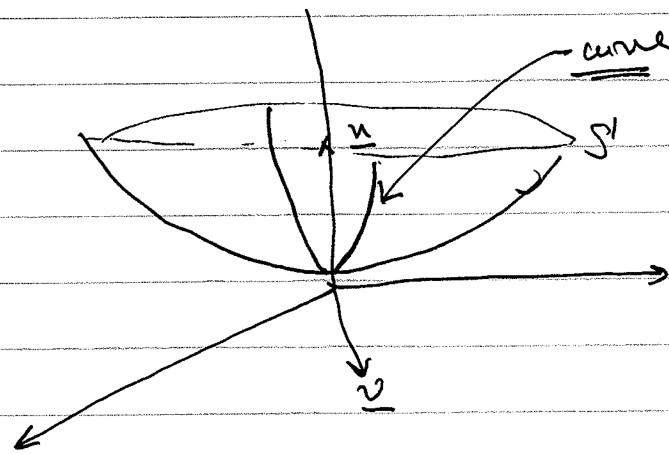
$$\left.\begin{cases} x_{m+1} = f_{m+1}(x_1, \dots, x_m) \\ x_{m+2} = f_{m+2}(x_1, \dots, x_m) \\ \\ x_n = f_n(x_1, \dots, x_m) \end{cases}\right\} \text{ \underline{describes} } S'$$

Again a unit vector $\underline{v}$ tangent to S' at p

together with vectors normal to S' at p

spans a plane, which intersect S'

in a curve



The curvature vector $\underline{K}$ of the curve

which we call the curvature in direction

$\underline{v}$ is just the second derivative

$$\underline{K} = (D^2 f)_p (\underline{v}, \underline{v}) .$$

$$\left(= \underline{v}^T \; (D^2 f)_p \; \underline{v} \right),$$

and $f = (\underbrace{0, 0, \ldots 0}_{m}, f_{m+1}, \ldots f_n)$

The bilinear form $(D^2 f)_p$ on $T_p S$ with values in $(T_p S)^\perp$ is called the second fundamental tensor $\underline{II}$ ~~of~~ S at p, given in coordinates as a symmetric $m \times m$ matrix with entries in $(T_p S)^\perp$:

$$\underline{II} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & & \dfrac{\partial^2 f}{\partial x_1 \partial x_m} \\ & & \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_m} & & \dfrac{\partial^2 f}{\partial x_m} \end{bmatrix}$$

The trace of $\underline{\mathrm{II}}$ is called the mean curvature vector $\underline{H}$ (Sometimes people use a scalar factor and write $\underline{H} = \frac{\operatorname{trace} \underline{\mathrm{II}}}{n}$)

<u>Hypersurfaces</u> In the special case of hypersurfaces when $n = m+1$, $\underline{\mathrm{II}}$ is just the unit normal $\underline{n}$ times a scalar matrix, called the second fundamental form and also denote by II.

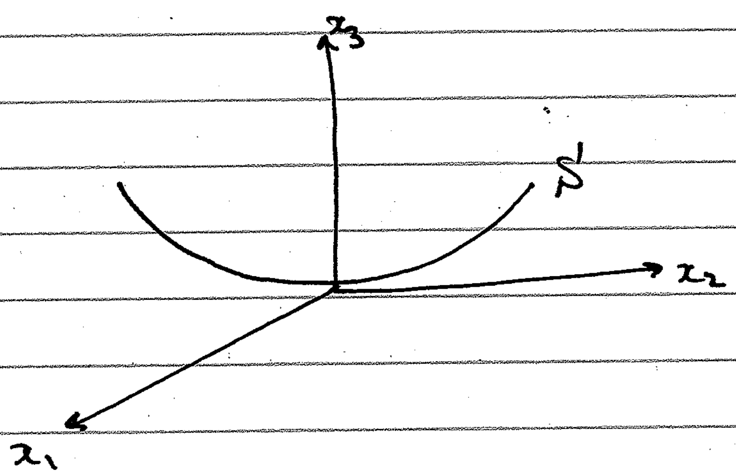
$\underline{H} = H \underline{n}$ where H is the scalar mean curvature. If we choose coordinates to make the second fundamental form diagonal

$$\Pi = \begin{bmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_M \end{bmatrix}$$

then

$$H = k_1 + \cdots + k_m$$

Consider for the moment $m=2$



$$S' : \quad x_3 - f_3(x_1, x_2) = 0$$

$$\underline{n} = \frac{\left(-\frac{\partial f_3}{\partial x_1}, -\frac{\partial f_3}{\partial x_2}, 1\right)}{\left(1 + \left(\frac{\partial f_3}{\partial x_1}\right)^2 + \left(\frac{\partial f_3}{\partial x_2}\right)^2\right)^{1/2}} = (n_1, n_2, n_3)$$

$$\frac{\partial n_3}{\partial x_3} = 0$$

and

$$\frac{\partial n_1}{\partial x_1} = -k_1, \quad \frac{\partial n_2}{\partial x_2} = -k_2$$

so

$$H = -\sum_{i=1}^{3} \frac{\partial n_i}{\partial x_i} = -\operatorname{div} \underline{n}$$

Formula true for general hypersurfaces

$$H = -\operatorname{div} \underline{n}$$

as well.

Useful theorem

Theorem Let S be a C^2 m-dimensional surface in $\mathbb{R}^n$. The first variation of area S with respect to a compactly supported C^2 vector field $\underline{V}$ on S is given by integrating $\underline{V}$ against the mean curvature vector:

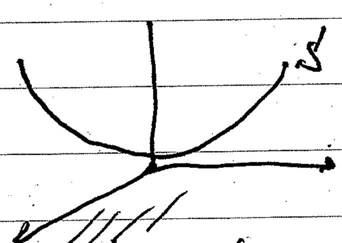
$$\delta'(S) = -\int_S \underline{V} \cdot \underline{H} \, d\sigma$$

4.2 Sectional and Riemann curvature

The <u>sectional curvature</u> K_{sec} of S at p assign to every 2-plane $P \subset T_p S$ the Gauss curvature of the 2-dimensional surface

$$S \cap (P \oplus T_p S^{\perp})$$

This is easier to state than to sketch since I can only draw a sketch where the tangent space $T_p S'$ is already a two dimensional surface



$T_p S'$ (already 2 dim surface)

If v, w are 2 vectors that give an orthonormal basis ~~for~~ for the 2-plane P we can find the components of Π by just remembering that components of a matrix are defined by how the linear operator acts on basis vectors ~~vector~~, i.e

$$\Pi = \begin{bmatrix} \Pi(v,v) & \Pi(v,w) \\ \Pi(v,w) & \Pi(w,w) \end{bmatrix}$$

and hence taking determinants

$$K_{sect}(P) = \Pi(v,v)\,\Pi(w,w) - \Pi(v,w)\cdot\Pi(v,w) \qquad [1]$$

For example if

$$\Pi = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and P is spanned by unit vectors $e_1 = [1,0]$, $e_2 = [0,1]$ so that P is just the x_1, x_2 plane

$$K_{sect}(P) = \Pi(e_1,e_1)\,\Pi(e_2,e_2) - \Pi(e_1,e_2)\,\Pi(e_1,e_2)$$

$$= a_{11}\cdot a_{22} - a_{12}\cdot a_{12}$$

Lets compute formula in (1) explicitly.

$\underline{v} = (v_1, \dots, v_m)$, $\underline{w} = (w_1, \dots, w_m)$, $\Pi = [a_{ij}]$

$$(v_1, \dots, v_m)^t \overbrace{\begin{bmatrix} a_{11} & \dots & a_{1m} \\ & \vdots & \\ a_{m1} & & a_{mm} \end{bmatrix}}^{\Pi} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} =$$

$$(v_1 \dots, v_m) \begin{bmatrix} \sum a_{1j} v_j \\ \sum a_{2j} v_j \\ \vdots \\ \sum a_{mj} v_j \end{bmatrix} =$$

$$\sum a_{1j} v_1 v_j + \sum a_{2j} v_2 v_j \dots$$

$$= \sum \sum a_{ik} v_i v_k ,$$

etc.

So

$$v^T \Pi v = \sum \sum a_{ik} v_i v_k$$

$$w^T \Pi w = \sum \sum a_{jl} w_j w_l$$

$$v^T \Pi w = \sum \sum a_{jk} v_k w_j .$$

so formula (1) becomes

$$K(P) = \sum\sum a_{ik} v_i v_k \sum\sum a_{jl} w_j w_l$$
$$- \sum\sum a_{jk} v_k w_j \sum\sum a_{il} v_i w_l$$

$$= \sum\sum\sum\sum R_{ijkl} v_i w_j v_k w_l$$

or using summation convention for repeated indices

$$K_{sect}(P) = R_{ijkl} v_i w_j v_k w_l \qquad (2)$$

where

$$R_{ijkl} = a_{ik} \cdot a_{jl} - a_{jk} \cdot a_{il} \qquad (3)$$

$\underline{\underline{R}}$ is called the <u>Riemann</u> <u>curvature</u> <u>tensor</u>

If we write

$$\Pi = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{12} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{13} & a_{23} & a_{33} & a_{34} & \cdots \\ & & & & \end{bmatrix}$$

we see

$$R_{1234} = a_{13} \cdot a_{24} - a_{14} \cdot a_{23}$$

i.e a minor shown in $\square$

$$R_{1212} = a_{11} \cdot a_{22} - a_{12} \cdot a_{12},$$

i.e a minor shown in $\bigcirc$

In fact R_{1212} is just the sectional curvature of x_1, x_2 plane

We also see that the Riemann curvature tensor is just the 2×2 minors of the second fundamental tensor II.

Immediately we see

$$R_{jikl} = R_{ijlk} = -R_{ijkl} \qquad (4)$$

(Interchanging 2 columns or rows changes the sign of the minor),

and

$$R_{klij} = R_{ijkl} \qquad (5)$$

(because II is symmetric)

In addition we can check

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \qquad (6)$$

on permutation of last 3 indices. This is Bianchi's identity.

The <u>Ricci</u> <u>curvature</u> is

$$R_{jl} = R_{ijil} \qquad (7)$$

The <u>scalar</u> <u>curvature</u> is

$$R = R_{ii} \qquad (8)$$

(Again using summation convention)

5. Covariant derivatives

Let's start with a simple example.

Consider the unit tangent vector

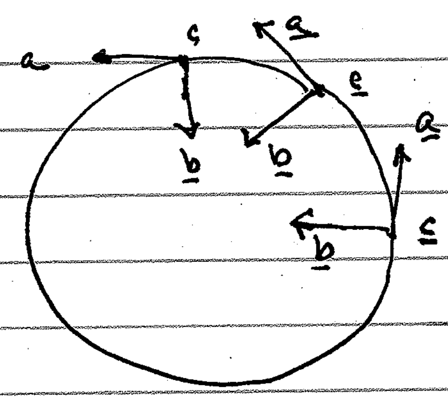
$\underline{T}(t) = (\sin t, \cos t)$ to the unit circle

As we move around the circle the unit tangent changes with derivative

$$\frac{d\underline{T}}{dt}(t) = (\cos t, -\sin t)$$

which of course is no longer in the tangent plane to the circle.

The <u>covariant derivative</u> is just the projection of back into the tangent plane.

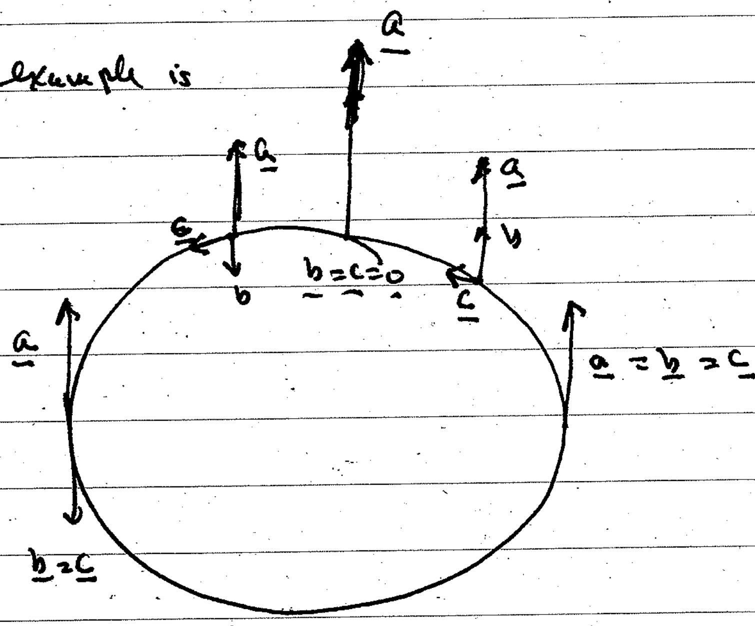


$\underline{a}$ is unit tangent to unit circle

$\underline{b}$ is its derivative

$\underline{c}$ is its covariant derivative $\underline{D}$.

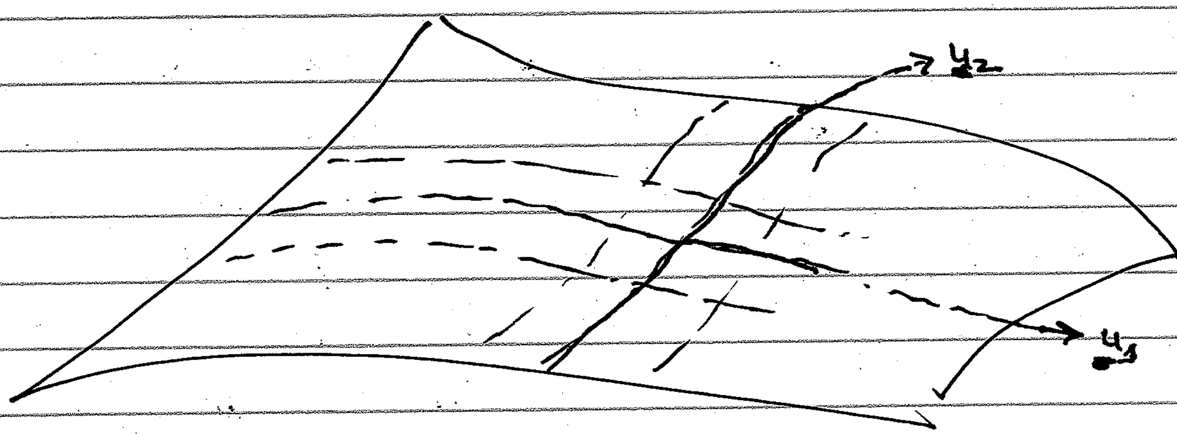
A second example is



Notice the vector field $\underline{a}$ on the circle always has same direction. It increases slowly in magnitude as we move from $\theta = 0$ counter-clockwise. Hence its derivative $\underline{b}$ is as shown at $\theta = \pi/4$. If we project back onto tangent place we see $\underline{c}$ at $\theta = \frac{\pi}{4}$ is covariant derivative.

Of course the "intrinsic" notion is the covariant derivative since we then are differentiating only along the surface, in this case the unit circle.

Another way to think about this is in the following simple picture



Notice y_1 - axis is not orthogonal to to the level set $\{y_1 = 0\}$. Infinitesimally

$e_1 = \frac{\partial}{\partial y_1}$ is not perpendicular to $\{dy_1 = 0\}$