Hence if we write a vector field

$$\overline{X} = (\overline{X}^1, \ldots, \overline{X}^m) = \sum X^i e_i$$

and the components of a differential

$$d\varphi = \sum \frac{\partial \varphi}{\partial u_i} du^i$$

To emphasize the distinction superscripts are used for the components of vector like or contravariant tensors

and subscripts are used for components of differential-like or covariant tensors.

Thus a vector field X has components X^j. Its covariant derivative has components $X^i_{;j}$. The semi-colon ";"

distinguishes the covariant derivative from usual partial derivative. $X^i_{,j} = \dfrac{\partial X^i}{\partial u_j}$.

The important formulas from Riemannian geometry are

$$X^i_{;j} = X^i_{,j} + \sum_k \Gamma^i_{jk} X^k \qquad (1)$$

and

$$\Gamma^i_{jk} = \frac{1}{2} \sum_l g^{il} \left(g_{lj,k} + g_{lk,j} - g_{jk,l} \right)$$

where

g^{ij} is the inverse of the metric g_{ij}

Γ^i_{jk} are the Christoffel symbols.

Note in formula (1) the usual partial derivative in the main term and the additional terms arise because the basis vectors are themselves turning.

In addition

$$\Gamma^{i}_{jk} = \Gamma^{i}_{kj} \qquad (3)$$

$$R_{jkl}^{\;\;\;\;i} = -\Gamma^{i}_{jk,l} + \Gamma^{i}_{jl,k} - \Gamma^{h}_{jk}\Gamma^{i}_{hl} + \Gamma^{h}_{jl}\Gamma^{i}_{hk} \qquad (4)$$

$R_{jkl}^{\;\;\;\;i}$ is the Riemann curvature tensor

related to its old version by

$$R_{ijkl} = g_{ih} R_{jkl}^{\;\;\;\;h} .$$

Also

$$R_{jlk}^{\;\;\;\;i} = - R_{jkl}^{\;\;\;\;i} \qquad (5)$$

$$R_{jkl}^{\;\;\;\;i} + R_{klj}^{\;\;\;\;i} + R_{ljk}^{\;\;\;\;i} = 0 \qquad (6)$$

(Bianchi identity).

The Ricci curvature is

$$R_{jl} = R^i_{jil} \qquad (7)$$

and scalar curvature is

$$R = g^{jl} R_{jl}.$$

Ricci's lemma says the covariant derivative of the metric is 0

$$g_{ij;k} = g^{ij}_{\ ;k} = 0 \qquad (8)$$

and Ricci's identity gives a nice formula for the difference of mixed partials

$$X^i_{;j;k} - X^i_{;k;j} = -R^i_{hjk} X^h \qquad (9)$$

So the Riemann curvature tensor provides the error in failure of mixed partials to equate.

One last formula is crucial to our program, namely covariant differentiation of tensors h_{ij}:

$$h_{ij;k} = h_{ij,k} - \Gamma_{ki}^{p} h_{pj} - \Gamma_{kj}^{p} h_{ip}. \qquad (10)$$

Recall Gauss's classical notation

$$\Pi_{11} = L \quad , \quad \Pi_{12} = M \quad , \quad \Pi_{22} = N$$

so that the Codazzi equations of Section 2 can be written as

$$\text{(a)} \qquad \frac{\partial \Pi_{11}}{\partial u_2} - \Gamma_{12}^{1}\,\Pi_{11} - \Gamma_{12}^{2}\,\Pi_{12} = \frac{\partial \Pi_{12}}{\partial u_1} - \Gamma_{11}^{1}\,\Pi_{12} - \Gamma_{11}^{2}\,\Pi_{22}$$

or adding

$$-\Gamma_{12}^{1}\,\Pi_{11} - \Gamma_{12}^{2}\,\Pi_{12}$$

to both sides of (a) gives

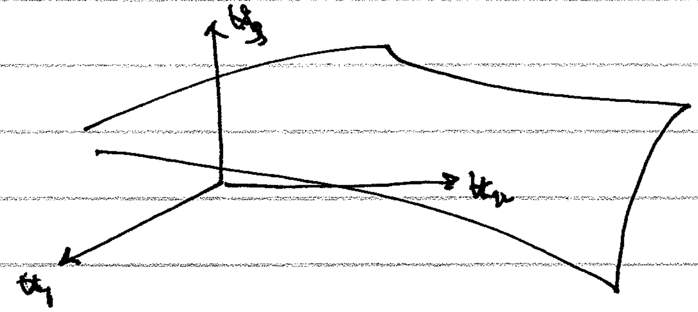
$$\frac{\partial \Pi_{11}}{\partial u_2} - \Gamma_{12}^{1}\,\Pi_{11} - \Gamma_{12}^{2}\,\Pi_{12} - \Gamma_{12}^{1}\,\Pi_{11} - \Gamma_{12}^{2}\,\Pi_{12} =$$

$$\frac{\partial \Pi_{12}}{\partial u_1} - \Gamma_{11}^{1}\,\Pi_{12} - \Gamma_{11}^{2}\,\Pi_{22} - \Gamma_{12}^{1}\,\Pi_{11} - \Gamma_{12}^{2}\,\Pi_{12}$$

and hence

$$\Pi_{11;2} = \Pi_{12;1}$$

If we apply formula (10) to the second fundamental II in the case of 2-dimensional surface in $\mathbb{R}^3$



We have

$$\mathrm{II}_{11;2} = \frac{\partial \mathrm{II}_{11}}{\partial x_2} - \Gamma^1_{21}\mathrm{II}_{11} - \Gamma^2_{21}\mathrm{II}_{21} - \Gamma^1_{21}\mathrm{II}_{11} - \Gamma^2_{21}\mathrm{II}_{21}$$

$$\mathrm{II}_{12;2} = \frac{\partial \mathrm{II}_{12}}{\partial x_2} - \Gamma^1_{21}\mathrm{II}_{12} - \Gamma^2_{21}\mathrm{II}_{22} - \Gamma^1_{22}\mathrm{II}_{11} - \Gamma^2_{22}\mathrm{II}_{12}$$

$$\mathrm{II}_{12;1} = \frac{\partial \mathrm{II}_{12}}{\partial x_1} - \Gamma^1_{11}\mathrm{II}_{12} - \Gamma^2_{11}\mathrm{II}_{22} - \Gamma^1_{12}\mathrm{II}_{11} - \Gamma^2_{12}\mathrm{II}_{12}$$

$$\mathrm{II}_{22;1} = \frac{\partial \mathrm{II}_{22}}{\partial x_1} - \Gamma^1_{12}\mathrm{II}_{12} - \Gamma^2_{12}\mathrm{II}_{22} - \Gamma^1_{12}\mathrm{II}_{21} - \Gamma^2_{12}\mathrm{II}_{22}$$

Similarly the second Codazzi equation can be written as

$$\text{(b)}\qquad \frac{\partial \Pi_{12}}{\partial u_2} - \Gamma^1_{22}\Pi_{11} - \Gamma^2_{22}\Pi_{12} =$$

$$\frac{\partial \Pi_{22}}{\partial u_1} - \Gamma^1_{12}\Pi_{12} - \Gamma^2_{12}\Pi_{22}$$

and adding

$$-\Gamma^1_{21}\Pi_{12} - \Gamma^2_{21}\Pi_{22}$$

to both sides of (b) gives

$$\frac{\partial \Pi_{12}}{\partial u_2} - \Gamma^1_{22}\Pi_{11} - \Gamma^2_{22}\Pi_{12} - \Gamma^1_{21}\Pi_{12} - \Gamma^2_{21}\Pi_{22} =$$

$$\frac{\partial \Pi_{22}}{\partial u_1} - \Gamma^1_{12}\Pi_{12} - \Gamma^2_{12}\Pi_{22} - \Gamma^1_{21}\Pi_{12} - \Gamma^2_{21}\Pi_{22}$$

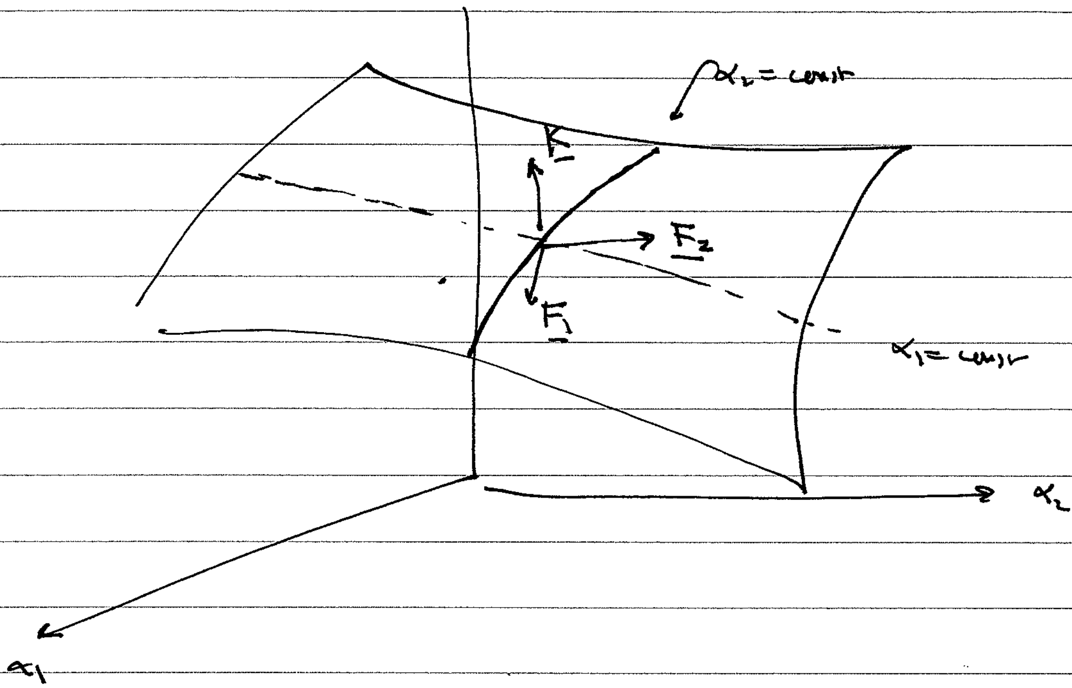
and hence

$$\Pi_{12;2} = \Pi_{22;1}$$

Thus the Codazzi equations in this case are just

$$\left\{ \begin{array}{l} \Pi_{11;2} = \Pi_{12;1}, \\ \Pi_{12;2} = \Pi_{22;1} \end{array} \right\}.$$

6. Elastic surfaces. (from "Confined developable elastic surfaces: cylinders, cones and the Elasticas", E Cerda and L Mahadevan, Proc of the Royal Society A (2005), 461, 671–700.



$\underline{F_2}$ is force on line $\alpha_2 = \text{const}$

Decompose

$$\underline{F_2} = N_{21}\,\underline{t_1} + N_{22}\,\underline{t_2} + Q_2\,\underline{n} \qquad (1)$$

Similarly

$\underline{F_1}$ is force on line $\alpha_1 = \text{const}$ and

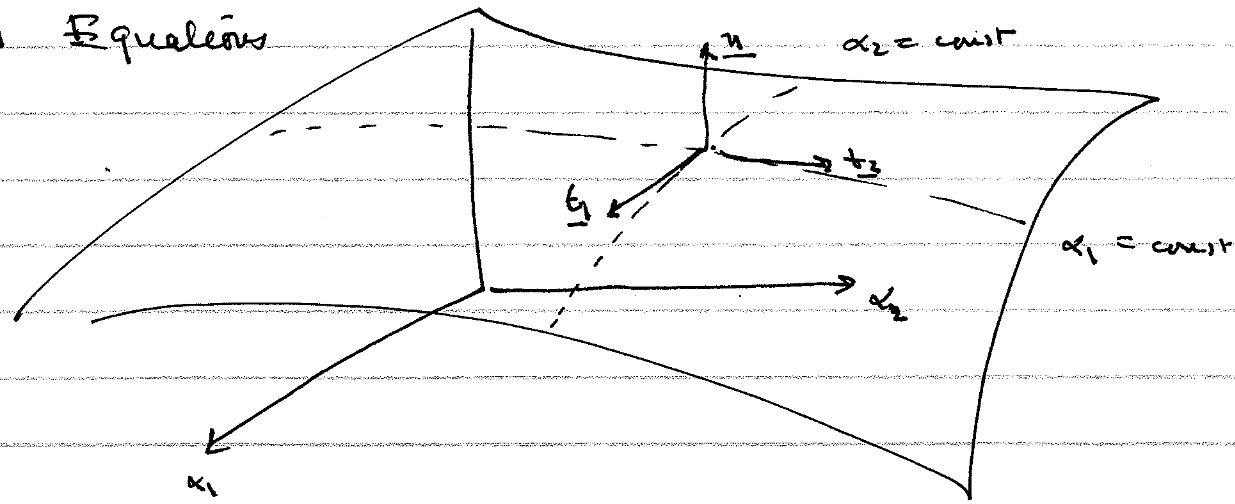
$$\underline{F_1} = N_{11}\,\underline{t_1} + N_{12}\,\underline{t_2} + Q_1\,\underline{n} \qquad (2)$$

Set

$A_1 = \sqrt{g_{11}}$, $A_2 = \sqrt{g_{22}}$

$\underline{K}$ force / unit area (body force)

6.1 Equations



$$\underline{t}_1 = \frac{\dfrac{\partial \underline{r}}{\partial \alpha_1}}{\sqrt{g_{11}}} \qquad \underline{t}_2 = \frac{\dfrac{\partial \underline{r}}{\partial \alpha_2}}{\sqrt{g_{22}}}$$

$$\underline{n} = \underline{t}_1 \times \underline{t}_2$$

$\underline{t}_1$ is tangent to line $\alpha_2 = \text{const}$,

$\underline{t}_2$ is tangent to line $\alpha_1 = \text{const}$,

$\underline{n}$ normal to surface.

i.e. Gauss frame as before

Balance of linear momentum

$$\frac{\partial}{\partial \alpha_1}(A_2 \underline{F}_1) + \frac{\partial}{\partial \alpha_2}(A_1 \underline{F}_2) + A_1 A_2 \underline{K} = \underline{0} \qquad (3)$$

(3 eqns)

Torques occur in

$\underline{n} \times \underline{t}_1$, $\underline{n} \times \underline{t}_2$ directions

So if we set

$\underline{M}_1$ as the couple / unit length on $\alpha_1 = \text{const}$

$\underline{M}_2$ as the couple / unit length on $\alpha_2 = \text{const}$

we can write via the decomposition

$$\underline{M}_1 = \underline{n} \times (m_{11} \underline{t}_1 + m_{12} \underline{t}_2),$$

$$\underline{M}_2 = \underline{n} \times (m_{21} \underline{t}_1 + m_{22} \underline{t}_2),$$

and balance of angular momentum (torque equilibrium) gives

$$\frac{\partial}{\partial \alpha_1}(A_2 \underline{M}_1) + \frac{\partial}{\partial \alpha_2}(A_1 \underline{M}_2) + A_1 A_2 (\underline{t}_1 \times \underline{F}_1 + \underline{t}_2 \times \underline{F}_2) = \underline{0} \quad (4)$$

(3 eqns)

So eqns (3), (4) give 6 eqns. in terms of

6 force components : $N_{11}, N_{12}, N_{21}, N_{22}, Q_1, Q_2$,

~~4~~ 4 torque components : $M_{11}, M_{12}, M_{21}, M_{22}$.

Hence 10 unknowns

As usual in continuum mechanics we need

a closure relation, i.e constitutive relations

which define the nature of our sheets.

Let as usual Π_{ij} denotes the

components of 2nd fundamental form.

Define

$$\kappa_1 = \frac{\Pi_{11}}{g_{11}}, \quad \kappa_2 = \frac{\Pi_{22}}{g_{22}}, \quad \tau = \frac{\Pi_{12}}{\sqrt{g_{11} g_{22}}}$$

(using Cerda and Mahadevan's notation).

κ_1 is normal curvature along $\underline{t}_1$,
κ_2 $\underline{t}_2$,
τ is geodesic torsion or twist of surface.

When the surface is deformed inextensibly (bending but no stretching) means the metric is preserved but the second fundamental form varies. By Gauss's Theorem Egregium the Gauss curvature is unchanged. (which for the case ~~an~~ originally flat sheet is of course zero)

At the lowest order the constitutive relations relating the couples and curvatures for a naturally flat sheet (A.E.H Love, A treatise on the mathematical theory of elasticity (1944), Dover, New York; C.L. Dym, Theory of shells; Hemisphere, New York)

$$M_{11} = B(\kappa_1 + \sigma \kappa_2), \quad M_{22} = B(\kappa_2 + \sigma \kappa_1),$$

$$M_{12} = M_{21} = B(1-\sigma)\tau \qquad (5)$$

σ is Poisson's ratio of the material of the sheet

$B = Eh^3 / 12(1-\sigma^2)$ bending stiffness

E Young's modulus

Now recall the Codazzi equations (§2.8 of these notes)

$$\partial_{\alpha_2} \mathrm{II}_{11} - \partial_{\alpha_1} \mathrm{II}_{12} = \Gamma^1_{12}\, \mathrm{II}_{11} + (\Gamma^2_{22} - \Gamma^1_{11})\, \mathrm{II}_{12} - \Gamma^2_{11}\, \mathrm{II}_{22},$$

$$\partial_{\alpha_2} \mathrm{II}_{12} - \partial_{\alpha_1} \mathrm{II}_{22} = \Gamma^1_{22}\, \mathrm{II}_{11} + (\Gamma^2_{22} - \Gamma^1_{12})\, \mathrm{II}_{12} - \Gamma^2_{12}\, \mathrm{II}_{22} \qquad (6)$$

and via the definition of the Gauss curvature

and Gauss's theorem Egregium ~~and~~ we have

$$\det \mathrm{II} = (\text{Gauss curv.}) \det g \qquad \left(\begin{matrix}\text{defn of} \\ \text{Gauss} \\ \text{curvature}\end{matrix}\right) \quad (7)$$

$$\text{Gauss curvature} = \frac{R_{1212}}{\det g}, \qquad (\text{thm Egregium})$$

$$R_{ijkl} = g_{lm}\left(\partial_k \Gamma^m_{ij} - \partial_j \Gamma^m_{ik} + \Gamma^n_{ij} \Gamma^m_{nk} - \Gamma^n_{ik} \Gamma^m_{nj}\right),$$

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right),$$

(definitions of Riemann curvature tensor and Christoffel symbols).

The constitutive relations (5) allow us to use the components of the second fundamental form as dependent state variables (3 variables), and the six force components ($N_{11}, N_{12}, N_{21}, N_{22}, Q_1, Q_2$) (6 more state variables) for a total of

9 state variables

The six equations (3), (4) combined with the 2 Codazzi equations (6) and Gauss equation (7) yield

9 equations

Of course the Gauss equation (7) is an algebraic relation which would allow elimination of one state variable to give us a system of

$$\left\{ \begin{array}{l} 8 \text{ balance laws} \quad \text{in} \\ 8 \text{ unknowns} \end{array} \right\}$$

One might be tempted to use the term "conservation laws" but the non-vanishing right hand sides of the Codazzi equations make "balance law" a better description.

Of course we readily see that the Gauss-Codazzi equations are decoupled from the force balance equations. Hence knowledge of a solution of the Gauss-Codazzi equations gives $\Pi_{11}, \Pi_{12}, \Pi_{22}$ and hence the couples $m_{11}, m_{12}, m_{21}, m_{22}$. We may now go back to the definitions of $\underline{t}_1$, $\underline{t}_2$, $\underline{n}$ to write

$$\underline{t}_1 = \frac{\frac{\partial \underline{r}}{\partial a_1}}{\sqrt{g_{11}}}, \qquad \underline{t}_2 = \frac{\frac{\partial \underline{r}}{\partial a_2}}{\sqrt{g_{22}}}, \qquad \underline{n} = \underline{t}_1 \times \underline{t}_2$$

to see that

$$\underline{M}_1 = \underline{n} \times (m_{11}\underline{t}_1 + m_{12}\underline{t}_2)$$

$$\underline{M}_2 = \underline{n} \times (m_{21}\underline{t}_1 + m_{22}\underline{t}_2)$$

are known functions of $\left\{ \frac{\partial \underline{r}}{\partial a_1}, \frac{\partial \underline{r}}{\partial a_2} \right\}$.

Hence the force and torque balance equations (3), (4) give us 6 equations in terms of

$N_{11}, N_{12}, N_{21}, N_{22}, Q_1, Q_2$ and $\underline{r}$ (9 unknowns)

But recall from §2 of these notes the second derivatives of $\underline{r}$ are determined by the second fundamental norm.

In our notation this is (using the more general definitions)

$$\Pi_{ij} = -\underline{n} \cdot \partial_{\alpha_i} \partial_{\alpha_j} \underline{r} \quad , \qquad (8)$$

$$\underline{n} = \frac{\frac{\partial r}{\partial \alpha_1} \times \frac{\partial r}{\partial \alpha_2}}{\sqrt{g_{11} g_{22}}}$$

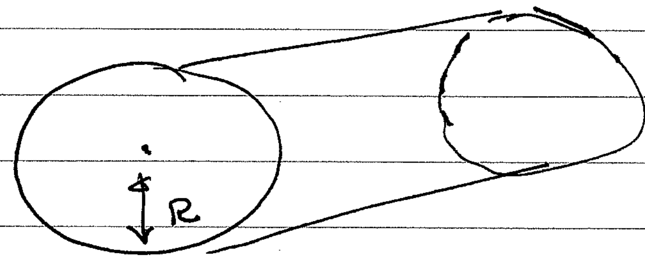
And hence the force and torque balance equations (3), (4) combined with definition of second fundamental form (8) yields

(3), (4), (8) 9 equations

$N_{11}, N_{12}, N_{21}, N_{22}, Q_1, Q_2, \underline{r}$ 9 unknowns

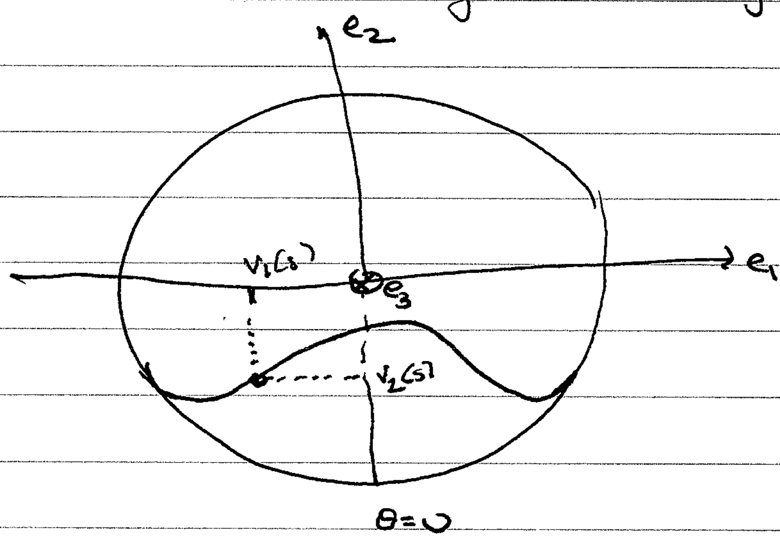
And finally we ask the question:
How do we know (8) can be solved to recover a class of possible deformations $\underline{r}$? The answer of course is that the solvability of (8) was guaranteed by the associated integrability conditions, i.e. the Codazzi equations which is where we began!

6.2 Example : the puckered cylinder



Consider a very long sheet rolled into a circular cylinder of radius R.

The above cylinder is then introduced into a second cylinder of radius $b < R$



The excess length of the sheet causes it to pucker with a natural dimensionless control parameter to describe the packing of the system

$$\varepsilon^2 = \left(\frac{R-b}{b}\right) \tag{9}$$

Assume that the deformation of the sheet is cylindrical, then the position vector describing the sheet is can be written as

$$\underline{r}(s,z) = \underline{v}(s) + z\,\underline{e_3},$$

$$\underline{v}(s) = v_1(s)\underline{e_1} + v_2(s)\,e_2,$$

s is the arc length of the 2 dimensional curve defined by $\underline{v}(s)$,

z is the length along the axis of the cylinder

The metric for arc length along the surface is just the classical Euclidean metric in $\mathbb{R}^3$

$$\underbrace{dx^2 + dy^2}_{ds^2} + dz^2 = ds^2 + dz^2$$

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = 1$$

where s, z are the local coordinates

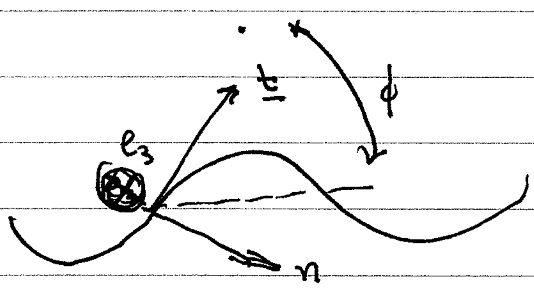
$$\frac{\partial}{\partial s} = \partial_1, \qquad \frac{\partial}{\partial z} = \partial_2$$

The tangent along the surface is

$$\underline{t} = \frac{\partial \underline{x}}{\partial s} = \underline{t}_1$$

$$\underline{t}_2 = \underline{e}_3$$

$$\underline{n} = \underline{t} \times \underline{e}_3$$



Hence

$$\underline{t} = \cos\phi \, \underline{e}_1 + \sin\phi \, \underline{e}_2$$
$$\underline{n} = \sin\phi \, \underline{e}_1 - \cos\phi \, \underline{e}_2$$

Recall the second fundamental form is given in general by

since normal has zero e_3 component

$$\Pi_{ij} = -(\underline{n},0)\cdot \partial_{\alpha_i}\partial_{\alpha_j}\underline{r}$$

So $$\Pi_{ij} = -(\underline{n},0)\underline{r}_{,ij}$$

$$\Pi_{11} = -(\underline{n},0).(v_1''(s), v_2''(s), 0)$$

$$\Pi_{12} = -(\underline{n},0)(0, 0, 0)$$

$$\Pi_{22} = -(\underline{n},0)(0, 0, 0)$$

$\Pi_{12} = \Pi_{22} = 0$ and

$$\Pi_{11} = -(\sin\phi, -\cos\phi, 0)(v_1''(s), v_2''(s), 0)$$

$$\Pi_{11} = -\sin\phi\, v_1''(s) + \cos\phi\, v_2''(s)$$

Also since

$$\underline{t} = \frac{\partial x}{\partial s} = v_1'(s)\,\underline{e_1} + v_2'(s)\,\underline{e_2}$$

$$= \cos\phi\,\underline{e_1} + \sin\phi\,\underline{e_2}$$

$\cos\phi = v_1'(s)$, $\quad \sin\phi = v_2'(s)$

$-\sin\phi\;\phi'(s) = v_1''(s)$, $\quad \cos\phi\;\phi'(s) = v_2''(s)$

$$\Pi_{11} = -\sin\phi(-\sin\phi)\,\phi'(s) + \cos\phi\,\cos\phi\;\phi'(s)$$

i.e. $\quad \Pi_{11} = \phi'(s)$

$k(s) = \phi'(s)$ is the curvature of the planar curve, which of course is consistent with the usual definition of curvature of a planar curve (i.e rate of change of angle ϕ with respect to arc length).

Since

$$\tau = \frac{g_{12}}{\sqrt{g_{11}\, g_{22}}}$$

In our problem $\tau = 0$ and

$m_{12} = m_{21} = 0$ (by our constitutive relation) and

also

$$m_{11} = B\,\kappa(s), \qquad m_{22} = B\sigma\,\kappa(s),$$

$$\underline{M}_1 = \underline{n} \times (B\,\kappa(s)\,\underline{t}_1)$$

$$\underline{M}_2 = \underline{n} \times (B\sigma\,\kappa(s)\,\underline{t}_2)$$

so since $\underline{M}_1, \underline{M}_2$ have zero component in the $\underline{n}$ direction.

Hence the ~~$\underline{n}$ component~~ balance of angular momentum (torque balance) is

$$\partial_s \left(\overset{=1}{A_2} \, \overbrace{\underline{n} \times \underline{t}_1}^{\underline{e}_3} \, B\,\kappa(s) \right)$$

$$+ \partial_z \left(\overbrace{A_1 \, \underline{n} \times \underline{t}_2 \, B\,\sigma\,\kappa(s)}^{=0} \right) \qquad \text{assume } z \text{ dependence.}$$

$$+ \underset{=1}{A_1} \underset{=1}{A_2} \left(\underline{t}_1 \times \underline{F}_1 + \underline{t}_2 \times \underline{F}_2 \right) = \underline{0}$$

$$\partial_s \left(\underline{e}_3 B\,\kappa(s) \right) + \underline{t}_1 \times \underline{F}_1 + \underline{t}_2 \times \underline{F}_2 = 0$$

$$\underline{F}_1 = N_{11}\,\underline{t}_1 + N_{12}\,\overset{=e_3}{\underline{t}_2} + Q_1\,\underline{n}$$

$$\underline{F}_2 = N_{21}\,\underline{t}_1 + N_{22}\,\underset{e_3}{\underline{t}_2} + Q_2\,\underline{n}$$

$$\left\{ \begin{array}{l} \underline{t}_1 = \underline{t} \\ \underline{t}_2 = \underline{e}_3 \\ \underline{n} = \underline{t} \times \underline{e}_3 \end{array} \right\}$$

$$\partial_s \left(\underline{e}_3 B\,\kappa(s) \right) + N_{12}\,\underline{n} + Q_1\,\overset{=e_3}{\underline{t}_1 \times \underline{n}}$$

$$+ N_{21}\,\underbrace{\underline{t}_2 \times \underline{t}_1}_{-\underline{n}} + Q_2\,\underbrace{\underline{t}_2 \times \underline{n}}_{\underline{t}}$$

In $\underline{n}$ direction we see

$$\boxed{N_{12} = N_{21}} \qquad (10)$$

Hence $\boxed{B\,\kappa'(s) = Q}$ (11)

from torque balance

Also $\boxed{Q_2 = 0}$

Now we use balance of linear momentum to see

$$\partial_s (\underbrace{A_2}_{=1} \underline{F}_1) + \underbrace{\partial_z (\underbrace{A_1}_{=1} \underline{F}_2)}_{=\underline{0}} + \underbrace{A_1 A_2}_{=1} \underline{K} = \underline{0}$$

$$\partial_s \underline{F}_1 + \underline{K} = \underline{0}$$

$$\underline{F}_1 = N_{\phi\phi} \underline{t}_1 + N_{\phi z} \underline{t}_2 + Q_\phi \underline{n}$$

$$= N_{\phi\phi} \underline{t} + N_{\phi z} \underline{e}_3 + Q_\phi \underline{n} , \quad \underline{n} = \underline{t} \times \underline{e}_3 ,$$

Any external force must be along the normal so

$$\underline{K} = - \tilde{F} \underline{n} , \quad \text{so}$$

$$\partial_s (N_{\phi\phi} \underline{t} + N_{\phi z} \underline{e}_3 + Q_\phi \underline{n}) - \tilde{F} \underline{n} = 0.$$

Also since

$$\underline{t} = \cos\phi \, \underline{e}_1 + \sin\phi \, \underline{e}_2$$

$$\underline{n} = \sin\phi \, \underline{e}_1 - \cos\phi \, \underline{e}_2$$

$$\frac{\partial \underline{t}}{\partial s} = (-\sin\phi\, \underline{e}_1 + \cos\phi\, \underline{e}_2)\, \phi'(s)$$

$$= -\underline{n}\, \phi'(s) = -\underline{n}\, \kappa$$

$$\frac{\partial \underline{n}}{\partial s} = (\cos\phi\, \underline{e}_1 + \sin\phi\, \underline{e}_2)\, \phi'(s)$$

$$= \underline{t}\, \kappa$$

we see

$$\left(\frac{\partial}{\partial s} N_{\phi\phi}\right) \underline{t} + N_{\phi\phi}(-\underline{n}\,\kappa) + \left(\frac{\partial}{\partial s} N_{\phi z}\right) \underline{e}_3$$

$$+ \frac{\partial Q_\phi}{\partial s}\, \underline{n} + Q_\phi\, \underline{t}\, \kappa - \tilde{k}\, \underline{n} = 0$$

$$\frac{\partial}{\partial s} N_{\phi\phi} + \kappa\, Q_\phi = 0 \qquad (12) \quad [\underline{t} \text{ direction}]$$

$$-\kappa\, N_{\phi\phi} + \frac{\partial Q_\phi}{\partial s} - \tilde{k} = 0 \qquad (13) \quad (\underline{n} \text{ direction})$$

$$\frac{\partial N_{\phi z}}{\partial s} = 0 \qquad (14) \quad [\underline{e}_3 \text{ direction}]$$

Insert (11) $B\kappa'(s) = Q_1$

into (12), (13) yields

$$\frac{\partial}{\partial s} N_{11} + \kappa B \kappa'(s) = 0 \qquad (15)$$

$$-\kappa N_{11} + B\kappa''(s) - \tilde{\kappa} = 0 \qquad (16)$$

Integrate (15)

$$N_{11}(s) + \frac{B\kappa^2(s)}{2} = -\frac{Ba^2}{2}$$

($-\frac{Ba^2}{2}$ is constant of integration).

Now substitute into (16) yields

$$\boxed{B\kappa''(s) + \kappa\left(a^2 + \frac{1}{2}\kappa^2\right) = \tilde{\kappa}(s) \qquad (17)}$$

which is the equilibrium shape of

Euler's elastica.

Of course in the region where the confined sheet is in contact with the external cylinder the curvature is equal to the (radius)$^{-1}$ i.e

$$k = \frac{1}{b}$$

which from (17) implies

$$\tilde{K}(s) = \text{const} = \tilde{K}_c$$

in this region. Furthermore from in the ~~contact~~ contact region (11) implies

$$Q_1 = 0 \quad (\text{in the contact region})$$

and hence (13) implies

$$\underset{\displaystyle \left(-\frac{1}{b}\right)}{\overset{\parallel}{(-k)}} N_{11} = \tilde{K}_c \quad (\text{in contact region})$$

$$N_{11} = -b\tilde{K}_c \quad (\text{in contact region})$$

A solution in the free region for the elastica equation (17) is known when

$$\tilde{\kappa}(s) \equiv 0,$$

i.e when

$$\kappa''(s) + (a^2 + \tfrac{1}{2}\kappa^2)\,\kappa(s) = 0 \qquad (18)$$

in terms of Jacobi elliptic function.

To do this just multiply (18) by $\kappa'(s)$:

$$\kappa'(s)\,\kappa''(s) + (a^2 + \tfrac{1}{2}\kappa^2(s))\,\kappa(s)\,\kappa'(s) = 0$$

so that

$$\frac{d}{ds}\left\{\frac{\kappa'(s)^2}{2} + \left(a^2\frac{\kappa^2(s)}{2} + \frac{1}{8}\kappa^4(s)\right)\right\} = 0,$$

$$\frac{\kappa'(s)^2}{2} + a^2\frac{\kappa^2(s)}{2} + \frac{1}{8}\kappa^4(s) = c_1^2 \ (\text{const}),$$

$$\kappa'(s)^2 = c_1^2 - a^2\kappa^2(s) - \frac{\kappa^4(s)}{4},$$

$$\kappa'(s) = \left(c_1^2 - a^2\kappa^2(s) - \frac{\kappa^4(s)}{4}\right)^{1/2},$$

$$\frac{\kappa'(s)}{\left(c_1^2 - \partial^2 \kappa^2(s) - \frac{\kappa^4(s)}{4}\right)^{1/2}} = 1$$

$$\frac{d}{ds} \int^{\kappa} \frac{\partial \kappa}{\left(c_1^2 - \partial^2 \kappa^2 - \frac{\kappa^4}{4}\right)^{1/2}} = 1$$

$$\int^{\kappa} \frac{\partial \kappa}{\left(c_1^2 - \partial^2 \kappa^2 - \frac{\kappa^4}{4}\right)^{1/2}} = s \, ,$$

which defines $\kappa(s)$ implicitly.

We also of course need to satisfy the Gauss–Codazzi equations. We know from our earlier discussion

$$\Pi_{11} = \alpha'(s) = \kappa(s)$$

$$\Pi_{12} = 0$$

$$\Pi_{22} = 0$$

so

$$\det \Pi = 0$$

and the Gauss curvature is zero.

Also $g_{11} = 1$, $g_{12} = 0$, $g_{22} = 1$

the components of the metric are constant and the Christoffel symbols are zero.

Hence the Codazzi equations are

$$\partial_z \Pi_{11} - \partial_s \Pi_{12} = 0$$

$$\partial_z \Pi_{12} - \partial_s \Pi_{22} = 0$$

and the Gauss equation is just

$$0 = 0$$