i.e

$$\partial_z \phi'(s) = 0$$

which is course true. So the Gauss and Codazzi equations are satisfied and we know the shape of puckered cylinder has curvature given by a Jacobi elliptic function (~~[illegible]~~ for the case $\bar{\kappa} = 0$).

7. The Ricci curvature revisited

7.1 Recall our previous definitions:

g_{ij} is a given metric (with inverse g^{ij})

the Christoffel symbols (with summation convention)

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(g_{lj,k} + g_{lk,j} - g_{jk,l} \right) \tag{1}$$

$$R^i_{jkl} = -\Gamma^i_{jk,l} + \Gamma^i_{jl,k} + \left(-\Gamma^h_{jk} \Gamma^i_{hl} + \Gamma^h_{jl} \Gamma^i_{hk} \right) \tag{2}$$

(the Riemann curvature tensor) and

$$R_{ijkl} = g_{ih} R^h_{jkl} .$$

Ricci curvature is

$$R_{jl} = R^i_{jil} , \tag{3}$$

the scalar curvature is

$$R = g^{jl} R_{jl}$$

In terms of the Christoffel symbols (1) we see the Ricci curvature is just (from (2))

$$R_{jl} = -\Gamma^{i}_{ji,l} + \Gamma^{i}_{jl,i} + \left(-\Gamma^{k}_{ji}\Gamma^{i}_{kl} + \Gamma^{k}_{jl}\Gamma^{i}_{ki}\right) \qquad (4)$$

If we rewrite (4), (1) in usual PDE notation we see they are weakly coupled system which will determined the metric g_{ij} given the Ricci curvature R_{jl}, i.e.

$$\frac{\partial \Gamma^{i}_{ji}}{\partial x_l} - \frac{\partial \Gamma^{i}_{jl}}{\partial x_i} = -\Gamma^{k}_{ji}\Gamma^{i}_{kl} + \Gamma^{k}_{jl}\Gamma^{i}_{ki} - R_{jl} \qquad (5)$$

$$\frac{\partial g_{lj}}{\partial x_k} + \frac{\partial g_{lk}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_l} = 2\, g_{il}\Gamma^{i}_{jk} \qquad (6)$$

In this section will now assume that g_{ij} need not be positive definite but allow g_{ij} to be a pseudo-metric, specifically a Lorentzian metric

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2$$

in 4 dimensional space-time. In spherical coordinates we have

$$d\tau^2 = -dr^2 - r^2 d\varphi^2 - r^2 \sin^2\varphi \, d\theta^2 + dt^2$$

where τ has replaced s.

7.2. General relativity and the Schwarzschild metric

Assume the Lorentzian metric has the special form

$$d\tau^2 = -e^{\lambda(r)}dr^2 - r^2(d\varphi^2 + \sin^2\varphi\ d\theta^2) + e^{\nu(r)}dt^2 \qquad (7)$$

where $\lambda(r)$, $\nu(r)$ are functions yet to be determined

To make sense of equations (5), (6) for the pseudo-metric g_{ij} we have to know something about R_{jk} (the Ricci curvature).

For his theory of general relativity Einstein assumed that the Einstein tensor

$$G^i_k \equiv g^{ij} R_{jk} - \frac{1}{2} R\ \delta^i_k \qquad (8)$$

vanishes. (δ^i_k is the Kronecker delta), i.e

$$g^{ij} R_{jk} - \frac{1}{2} R\ \delta^i_k = 0. \qquad (9)$$

If we label our variables r, φ, θ, t. to note the 1,2,3,4 ordering of tensors. Then from (7)

$$g_{11} = -e^{\lambda}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\varphi, \quad g_{44} = e^{\nu} \qquad (10)$$

and the other g_{ij} vanish. Since g_{ij} is diagonal it is easy to invert and get

$$g^{11} = -e^{-\lambda}, \quad g^{22} = -r^{-2}, \quad g^{33} = -r^{-2}\sin^{-2}\varphi, \quad g^{44} = e^{-\nu}, \qquad (11)$$

and again the other components vanish.

Equation (6) gives us the Christoffel symbols.

$$\Gamma_{11}^{\ 1} = \frac{1}{2}\lambda', \quad \Gamma_{22}^{\ 1} = -re^{-\lambda}, \quad \Gamma_{33}^{\ 1} = -re^{-\lambda}\sin^2\varphi,$$

$$\Gamma_{44}^{\ 1} = \frac{1}{2}\nu' e^{\nu-\lambda}, \quad \Gamma_{12}^{\ 2} = \Gamma_{13}^{\ 3} = -r^{-1}, \quad \Gamma_{33}^{\ 2} = -\sin\varphi\cos\varphi,$$

$$\Gamma_{23}^{\ 3} = \cot\varphi, \quad \Gamma_{14}^{\ 4} = \frac{1}{2}\nu', \qquad (12)$$

and again the others vanish.

We can now substitute (12) into the formula for the Riemann curvature tensor (2) to see

$$R^2_{\ 121} = R^3_{\ 131} = \tfrac{1}{2} r^{-1} \lambda', \quad R^4_{\ 141} = -\tfrac{1}{2}\nu'' + \left(\tfrac{1}{2}\nu'\right)\left(\tfrac{1}{2}\lambda'\right) - \tfrac{1}{4}(\nu')^2$$

$$R^{\ 1}_{2\ 12} = \tfrac{1}{2} r \lambda' e^{-\lambda}, \quad R^{\ 3}_{2\ 32} = 1 - e^{-\lambda}, \quad R^{\ 4}_{2\ 42} = \tfrac{1}{2}\nu'(-r e^{-\lambda}),$$

$$R^{\ 1}_{3\ 13} = \tfrac{1}{2} r \lambda' \sin^2\varphi \, e^{-\lambda}, \quad R^{\ 2}_{3\ 23} = \sin^2\varphi\,(1 - e^{-\lambda}),$$

$$R^{\ 4}_{3\ 43} = \tfrac{1}{2}\nu'(-r)\, e^{-\lambda} \sin^2\varphi,$$

$$R^{\ 1}_{4\ 14} = \tfrac{1}{2} e^{\nu-\lambda}\left(\nu'' + \tfrac{1}{2}(\nu')^2 - \tfrac{1}{2}\nu'\lambda'\right), \qquad (13)$$

$$R^{\ 2}_{4\ 24} = R^{\ 3}_{4\ 34} = \tfrac{1}{2} r^{-1} \nu' e^{\nu-\lambda}.$$

Then

$$R_{11} = \underbrace{R^1_{111}}_{\substack{\parallel \\ 0}} + R^2_{121} + R^3_{131} + R^4_{141} \tag{14}$$

$$= r^{-1}\lambda' - \frac{1}{2}\nu'' + \frac{1}{4}\nu'\lambda' - \frac{1}{4}\nu'^2$$

$$R_{22} = R^1_{212} + \underbrace{R^2_{222}}_{\substack{\parallel \\ 0}} \quad R^3_{232} + R^4_{242}$$

$$= \frac{1}{2}r\lambda' e^{-\lambda} + 1 - e^{-\lambda} + \frac{1}{2}\nu'(-re^{-\lambda})$$

$$= 1 + \frac{1}{2}re^{-\lambda}(\lambda' - \nu') - e^{-\lambda}$$

$$R_{33} = R^1_{313} + R^2_{323} + \underbrace{R^3_{333}}_{\substack{\parallel \\ 0}} + R^4_{343}$$

$$= \frac{1}{2}r\lambda' \sin^2\varphi\, e^{-\lambda} + \sin^2\varphi(1 - e^{-\lambda}) + \frac{1}{2}\nu'(-r)e^{-\lambda}\sin^2\varphi$$

$$= \sin^2\varphi\left(1 + \frac{1}{2}re^{-\lambda}(\lambda' - \nu') - e^{-\lambda}\right)$$

$$R_{44} = R^1_{414} + R^2_{424} + R^3_{434} + \underbrace{R^4_{444}}_{\substack{\parallel \\ 0}}$$

$$= \frac{1}{2}e^{\nu-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\nu'\lambda'\right)$$
$$+ r^{-1}\nu' e^{\nu-\lambda}$$

$$= \frac{1}{2}e^{\nu-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\nu'\lambda' + 2r^{-1}\nu'\right)$$

R (the scalar curvature) =

$$\underbrace{-e^{-\lambda}}_{g^{11}} R_{11} - \underbrace{r^{-2}}_{g^{22}} R_{22} - \underbrace{r^{-2}\sin^{-2}\varphi}_{g^{33}} R_{33} + \underbrace{e^{-\nu}}_{g^{44}} R_{44} =$$

$$= -2r^{-2} + e^{-\nu}\left(\nu'' - 2r^{-1}\lambda' - \frac{1}{2}\nu'\lambda' + \frac{1}{2}\nu'^2 + 2r^{-1}\nu' + 2r^{-2}\right) \quad (15)$$

Recall the Einstein tensor is

$$G^i_k = g^{ij} R_{jk} - \frac{1}{2} R\, \delta^i_k$$

So

one

$$G^1_1 = g^{1j} R_{j1} - \frac{1}{2} R$$

$$= g^{11} R_{11} + \overset{=0}{g^{12}} R_{21} + \overset{=0}{g^{13}} R_{31} + \overset{=0}{g^{14}} R_{41} - \frac{1}{2} R$$

$$= -e^{-\lambda} R_{11} - \frac{1}{2} R$$

$$= e^{-\lambda}\left(r^{-1}\lambda' - \frac{1}{2}\nu'' + \frac{1}{4}\nu'\lambda' - \frac{1}{4}\nu'^2\right)$$

$$-\frac{1}{2}\left(-2r^{-2} + e^{-\lambda}\left(\nu'' - 2r^{-1}\lambda' - \frac{1}{2}\nu'\lambda' + \frac{1}{2}\nu'^2\right.\right.$$

$$\left.\left. + 2r^{-1}\nu' + 2r^{-2}\right)\right)$$

and so

$$G_1^1 = r^{-2} + e^{-\lambda}\left(-r^{-1}\nu' - r^{-2}\right) \tag{16}$$

Similarly we find

$$G_2^2 = G_3^3 = e^{-\lambda}\left(-\tfrac{1}{2}\nu'' + \tfrac{1}{2} r^{-1}\lambda' - \tfrac{1}{2} r^{-1}\nu' + \tfrac{1}{4}\nu'\lambda' - \tfrac{1}{4}\nu'^2 \right)$$

$$G_4^4 = r^{-2} + e^{-\lambda}\left(r^{-1}\lambda' - r^{-2}\right).$$

So our guess (in fact Schwartzschild's guess) of the special form of the pseudo-metric reduces to solving the ordinary differential equations implied by Einstein's assumptions for the vanishing of the Einstein tensor.

First set $G^q_q = 0$, i.e

$$r^{-2} + e^{-\lambda}\left(r^{-1}\lambda' - r^{-2}\right) = 0$$

Set

$$y = e^{-\lambda}, \quad y' = -\lambda' e^{-\lambda},$$

so

$$r^{-2} - r^{-1} y' - r^{-2} y = 0$$

or

$$y' + \frac{y}{r} = \frac{1}{r}$$

Integrating factor is

$$e^{\int \frac{dr}{r}} = r,$$

so

$$(ry)' = 1$$

$$ry = r - \gamma \qquad (\gamma = \text{const})$$

$$\boxed{e^{-\lambda} = 1 - \frac{\gamma}{r}} \qquad (17)$$

We set the constant γ equal to $2GM$,

G the gravitational constant, M the central mass (of the sun) taken as a point mass

so

$$e^{-\lambda} = 1 - 2GM r^{-1}$$

Now since $G_1^1 - G_4^4 = 0$

$$r^{-1} e^{-\lambda} (\nu + \lambda)' = 0 \quad \text{and}$$

$$(\nu + \lambda)' = 0$$

i.e.

$\nu + \lambda$ is constant

and if we have boundary condition at $r = \infty$

$\nu + \lambda = 0$ (the metric should be flat at $r = \infty$ since we are far away from sun's influence)

then $\nu + \lambda = 0$ for all r.

So

$$e^{\nu} = e^{-\lambda} = 1 - 2GMr^{-1} \qquad (18)$$

If we substitute $\nu = -\lambda$ into $G^2_2 = G^3_3 = 0$

we see

$$0 = e^{-\lambda}\left(-\tfrac{1}{2}\nu'' - \tfrac{1}{2} r^{-1}\nu' - \tfrac{1}{2} r^{-1}\nu' - \tfrac{1}{4}(\nu')^2 - \tfrac{1}{4}(\nu')^2 \right)$$

$$0 = -\tfrac{1}{2}\nu'' - r^{-1}\nu' - \tfrac{1}{2}(\nu')^2$$

$$\tfrac{1}{2} z' + \tfrac{1}{2} z^2 + r^{-1} z = 0, \quad \text{where } z = \nu',$$

$$\tfrac{1}{2} \frac{z'}{z^2} + \tfrac{1}{2} + r^{-1}\frac{1}{z} = 0,$$

$$\frac{d}{dr}\left(-\frac{1}{z}\right) + 1 + 2r^{-1}\frac{1}{z} = 0,$$

$$\frac{d}{dr}\left(-\frac{1}{z}\right) + 2r^{-1}\left(\frac{1}{z}\right) = -1$$

Integrating factor is

$$e^{\int -\frac{2}{r}dr} = e^{(-2\ln r)} = \frac{1}{r^2}$$

$$\frac{d}{dr}\left(\frac{1}{r^2 z}\right) = \frac{1}{r^2}$$

$$\frac{1}{r^2 z} = -\frac{1}{r} + \text{const}$$

$$\frac{1}{z} = -r + \text{const}\, r^2$$

$$z = \frac{1}{(-r + \text{const } r^2)},$$

$$\nu' = \frac{1}{(-r + \text{const } r^2)},$$

From our formula (18)

$$e^{\nu} = 1 - 2GM\, r^{-1}$$

so

$$e^{\nu}\nu' = +2GM\, r^{-2}$$

$$\nu' = \frac{2GM\, r^{-2}}{1 - 2GM\, r^{-1}} = \frac{2GM}{-2GMr + r^2}$$

$$\nu' = \frac{1}{\left(-r + \frac{r^2}{2GM}\right)}$$

so $\text{const} = \frac{1}{2GM}$ and

hence $G^2_2 = G^3_3 = 0$ are satisfied identically.

In summary

$$d\tau^2 = -(1-2GM\,r^{-1})^{-1}\,dr^2 - r^2(d\varphi^2 + \sin^2\varphi\, d\theta^2) + (1-2GM\,r^{-1})\,dt^2 \qquad (19)$$

which is the classical Schwarzschild solution.

Notice the mass M introduces a singularity.

Also when r decreases to $2GM$ (the Schwarzschild radius) the metric becomes singular, a "black hole" of Schwarzschild radius $r = 2GM$.

[Also notice that when $M=0$ (no sun)

$$d\tau^2 = -dr^2 - r^2(d\varphi^2 + \sin^2\varphi\, d\theta^2) + dt^2$$

$(= -(dx)^2 - (dy)^2 - (dz)^2 + (dt)^2)$ which is the Lorentz metric $-(dx)^2 - (dy)^2 - (dz)^2 + c^2 dt^2$ with speed of light c set equal 1. So that motion along geodesics will be straight lines. See § 7.4]

7.3 Motion along a geodesic

Einstein postulated the motion in space-time occurs along geodesics. If parametrize the location of the geodesic by arc length τ the we have four functions $r(\tau)$, $\varphi(\tau)$, $\theta(\tau)$, $t(\tau)$ that satisfy the equation of a geodesic

$$0 = \frac{d^2u^i}{d\tau^2} + \Gamma^i_{jk}\frac{du^j}{d\tau}\frac{du^k}{d\tau} \qquad (20)$$

with $u^1(\tau) = r(\tau)$, $u^2(\tau) = \varphi(\tau)$, $u^3(\tau) = \theta(\tau)$, $u^4(\tau) = t(\tau)$.

(Notice of course when the Christoffel symbols are zero motion would be trivially computed)

So in principle if we want to know the influence of a central mass on motion all we have to do is compute the

Christoffel symbols from the known Schwarzschild metric g substitute into (20), and solve for $r(\tau), \varphi(\tau), \theta(\tau), t(\tau)$. However there is a more efficient way to do the computation. Recall

$$(d\tau)^2 = g_{11}(dr)^2 + g_{22}(d\varphi)^2 + g_{33}(d\theta)^2 + g_{44}(dt)^2$$

and so

$$-(1-2GMr^{-1})\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\varphi}{d\tau}\right)^2 \qquad (21)$$
$$- r^2\sin^2\varphi\left(\frac{d\theta}{d\tau}\right)^2 + (1-2GMr^{-1})\left(\frac{dt}{d\tau}\right)^2 = 1$$

Next from (18) we have

$$\lambda' = -\nu' = -2GM(r^2-2GMr)^{-1},$$
$$\lambda'' = -\nu'' = 2GM(r^2-2GMr)^{-2}(2r-2GM)$$

and from the formulas (12) for the Christoffel symbols we have

$\Gamma_{11}^{1} = -GM(r^2-2GMr)^{-1},$

$\Gamma_{22}^{1} = -r(1-2GMr^{-1}) = 2GM-r,$

$\Gamma_{33}^{1} = (2GM-r)\sin^2\varphi,$

$\Gamma_{44}^{1} = \frac{1}{2}(1-2GMr^{-1})(2GMr^{-2}),$

$\Gamma_{12}^{2} = \Gamma_{13}^{3} = r^{-1},$

$\Gamma_{33}^{2} = -\sin\varphi\cos\varphi,$

$\Gamma_{23}^{3} = \cot\varphi,$

$\Gamma_{14}^{4} = GM(r^2-2GMr)^{-1}.$

Now substitute in the geodesic equations (20) and we find

$$\frac{d^2\varphi}{d\tau^2} + 2r^{-1}\left(\frac{dr}{d\tau}\right)\frac{d\varphi}{d\tau} - \sin\varphi\cos\varphi\left(\frac{d\theta}{d\tau}\right)^2 = 0 \qquad (a)$$

$$\frac{d^2\theta}{d\tau^2} + 2r^{-1}\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right) + 2\cot\varphi\,\frac{d\varphi}{d\tau}\,\frac{d\theta}{d\tau} = 0 \qquad (b) \qquad (22)$$

$$\frac{d^2t}{d\tau^2} + \left(\frac{2GM}{r^2-2GMr}\right)\left(\frac{dr}{d\tau}\right)\left(\frac{dt}{d\tau}\right) \qquad (c)$$

where $\frac{dr}{d\tau}$ is then known from (21).

To simplify the computation assume initial conditions

$$\frac{d\varphi}{d\tau}=0, \quad \cos\varphi=0. \quad \left(\text{say } \varphi=\frac{\pi}{2}\right).$$

Then this is an equilibrium point for (22 a)

and φ remains equal to $\frac{\pi}{2}$. Thus even

relativistically the orbit is in a plane.

So (21), (22 b, c) become

$$-\left(1-2GMr^{-1}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2-r^2\left(\frac{d\theta}{d\tau}\right)^2 + \left(1-2GMr^{-1}\right)\left(\frac{dt}{d\tau}\right)^2=1, \qquad (a)$$

(23)

$$\frac{d^2\theta}{d\tau^2}+2r^{-1}\frac{dr}{d\tau}\frac{d\theta}{d\tau}=0 \qquad (b)$$

$$\frac{d^2t}{d\tau^2}+2GM\left(r^2-2GMr\right)^{-1}\left(\frac{dr}{d\tau}\right)\left(\frac{dt}{d\tau}\right)=0 \qquad (c).$$

Let us solve (23b)

$$\frac{\frac{d^2\theta}{d\tau^2}}{\left(\frac{d\theta}{d\tau}\right)} + 2r^{-1}\frac{dr}{d\tau} = 0$$

$$\frac{d}{d\tau}\ln\left(\frac{d\theta}{d\tau}\right) + 2\frac{d}{d\tau}\left(\ln r(\tau)\right) = 0$$

$$\ln\frac{d\theta}{d\tau} + 2\ln r(\tau) = \text{const}$$

$$\ln\left(\frac{d\theta}{d\tau}\, r(\tau)^2\right) = \text{const}$$

$$\frac{d\theta}{d\tau}\, r(\tau)^2 = \text{const} = h \qquad (24)$$

Let us solve (23c)

$$\frac{\frac{d^2 t}{d\tau^2}}{\left(\frac{dt}{d\tau}\right)} + 2GM\left(r^2 - 2GMr\right)^{-1}\frac{dr}{d\tau} = 0$$

and similarly as in (24)

$$\left(1 - 2GMr^{-1}\right)\frac{dt}{d\tau} = \beta \quad (\text{a const}) \qquad (25)$$

Substitute (24), (25) into (23a) and we see

$$-r^{-4}\left(\frac{dr}{d\theta}\right)^2 - r^{-2}\left(1-2GMr^{-1}\right) + \beta^2 h^{-2} = h^{-2}\left(1-2GMr^{-1}\right)$$

and set

$u = \frac{1}{r}$ $\left(r = \frac{1}{u}\right)$ we see

$-r^{-4}$ ~~$\left(\frac{dr}{d\theta}\right)^2$~~.

$$\left(\frac{du}{d\theta}\right)^2 = 2GM\left(u^3 - \frac{1}{2GM}u^2 + \beta_1 u + \beta_0\right) \qquad (26)$$

for constant β_1, β_0.

The maximum ("u_1") and minimum ("u_2") values of of u must be roots of r.h.s of (26) since $\frac{du}{d\theta} = 0$ at the max, min. So

$$\left(\frac{du}{d\theta}\right)^2 = 2GM(u-u_1)(u-u_2)(u-u_3)$$

$$= 2GM\left(u^3 - u^2(u_1+u_2+u_3) + \cdots\right)$$

we see

$$u_1 + u_2 + u_3 = \frac{1}{2GM}$$

$$u_3 = \frac{1}{2GM} - u_1 - u_2$$

and

$$\left(\frac{du}{d\theta}\right)^2 = 2GM\,(u-u_1)(u-u_2)\left(u - \frac{1}{2GM} + u_1 + u_2\right). \quad (27)$$

or

$$\left|\frac{d\theta}{du}\right| = \frac{1}{\sqrt{(u_1-u)(u-u_2)}}\left[1 - 2GM(u+u_1+u_2)\right]^{-1/2}$$

$$\approx \frac{1}{\sqrt{(u_1-u)(u-u_2)}}\left[1 + GM(u+u_1+u_2) + \cdots\right] \quad (28)$$

(expanding $[1+x]^{-1/2} = 1 - \frac{1}{2}x + \cdots$)

So to ~~third~~ first order in G we can integrate (28)

$$\left|\frac{d\theta}{du}\right| = \frac{1}{\sqrt{(u_1-u)(u-u_2)}} \quad (29)$$

$$d\theta = \pm \frac{du}{\sqrt{(u_1-u)(u-u_2)}}$$

$$\theta = \pm 2\sin^{-1}\sqrt{\frac{u-u_2}{u_1-u_2}} + \text{const}$$

$$\frac{\theta}{2} - \text{const} = \pm \sin^{-1}\sqrt{\frac{u-u_2}{u_1-u_2}}$$

$$\sin\left(\frac{\theta}{2}-\text{const}\right) = \pm\sqrt{\frac{u-u_2}{u_1-u_2}}$$

$$\sin^2\left(\frac{\theta}{2}-\text{const}\right) = \frac{u-u_2}{u_1-u_2}$$

But

$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$$

So

$$\frac{1}{2}\left(1-\cos\left(\theta-2(\text{const})\right)\right) = \frac{u-u_2}{u_1-u_2}$$

$$\frac{u_1-u_2}{2}\left(1-\cos\left(\theta-2(\text{const})\right)\right) = u-u_1$$

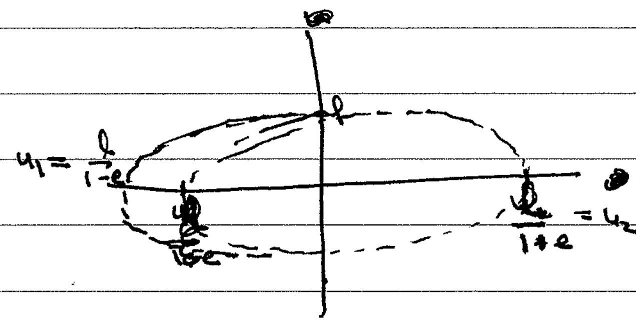
$$u = u_2 + \left(\frac{u_1-u_2}{2}\right) - \frac{(u_1-u_2)}{2}\cos\left(\theta-2(\text{const})\right)$$

$$u = \left(\frac{u_1+u_2}{2}\right)\left[1 - \frac{(u_1-u_2)}{(u_1+u_2)} \cos(\theta - 2\,(\text{const}))\right]$$

Set $\quad l^{-1} = \left(\frac{u_1+u_2}{2}\right)$

$$e = \left(\frac{u_2-u_1}{u_1+u_2}\right)$$

$$\text{const} = 0$$



$$u = l^{-1}(1 + e\cos\theta)$$

which is the classical ellipse: $r = \dfrac{l}{1+e\cos\theta} \left(= \frac{1}{u}\right)$.

Now we use both the first and second terms on

r.h.s. of (28):

Our ODE is

$$\frac{du}{d\theta} = \sqrt{(u_2-u)(u-u_1)}\,\left(1-2GM(u+u_1+u_2)\right)^{1/2}$$

$$= \sqrt{(u_2-u)(u-u_1)}\,\left(1-GM(u+u_1+u_2)+\cdots\right)$$

and we look for solution of the form

$$u = U^{(0)}(\theta) + GM\,U^{(1)}(\theta) + \cdots$$

where $U^{(0)}(\theta)$ is a solution of

$$\frac{du}{d\theta} = \sqrt{(u_2-u)(u-u_1)}.$$

Then we see

$$\frac{dU^{(0)}(\theta)}{d\theta} + GM\frac{dU^{(1)}(\theta)}{d\theta} + \cdots =$$

$$\sqrt{\left(u_2 - U^{(0)}_{\theta}(\theta) - GM\,U^{(1)}_{\theta}(\theta)\right)\left(U^{(0)}(\theta) + GM\,U^{(1)}(\theta) - u_1\right)}$$

$$\cdot\left(1 - GM\left(U^{(0)}_{\theta}(\theta) + GM\,U^{(1)}(\theta) + u_1 + u_2\right) + \cdots\right)$$