Expand in powers of GM:

$$\frac{dU^{0}(\theta)}{d\theta} + GM\frac{dU^{(1)}}{d\theta} + \cdots =$$

$$\sqrt{u_2 - U_0^{(0)}(\theta)}\sqrt{U^{(0)}(\theta) - u_1}\left(1 - \frac{GM\,U^{(1)}(\theta)}{u_2 - U_0^{(0)}(\theta)} + \cdots\right)^{1/2}\left(1 + \frac{GM\,U^{(1)}(\theta)}{U^{(0)}(\theta) - u_1}\cdots\right)^{1/2}$$

$$\cdot\left(1 - GM\,U_0^{(0)} + \cdots\right)$$

$$= \sqrt{u_2 - U_0^{(0)}}\sqrt{U^{(0)}(\theta) - u_1}\left(1 - \frac{1}{2}\frac{GM\,U^{(1)}(\theta)}{u_2 - U^{(0)}(\theta)} \cdots\right)\left(1 + \frac{1}{2}\frac{GM\,U^{(1)}(\theta)}{U^{(0)}(\theta) - u_1}\right)$$

$$\cdot\left(1 - GM\,U_0^{(0)}(\theta)\right)$$

$$= \sqrt{u_2 - U^{(0)}}\sqrt{U^{(0)}(\theta) - u_1} - GM\,U^{(0)}(\theta)\sqrt{u_2 - U_0^{(0)}(\theta)}\sqrt{U^{(0)}(\theta) - u_1}$$

$$+ \sqrt{u_2 - U_0^{(0)}(\theta)}\sqrt{U^{(0)}(\theta) - u_1}\left(-\frac{1}{2}GM\frac{U^{(1)}(\theta)}{u_2 - U^{(0)}(\theta)}\right.$$

$$\left. + \frac{1}{2}GM\frac{U^{(1)}(\theta)}{U^{(0)} - u_1}\right) + \cdots$$

Hence

$$\frac{dU^{(1)}(\theta)}{d\theta} = \frac{U^{(1)}(\theta)}{2}\left(-\frac{\sqrt{U^{(0)}(\theta) - u_1}}{\sqrt{u_2 - U^{(0)}(\theta)}} + \frac{\sqrt{u_2 - U^{(0)}(\theta)}}{\sqrt{U^{(0)} - u_1}}\right)$$

$$- U^{(0)}\sqrt{u_2 - U_0^{(0)}(\theta)}\sqrt{U^{(0)} - u_1}$$

Recall

$$U^{(0)}(\theta) = \frac{1}{2}(u_1+u_2) - \frac{(u_1-u_2)}{2}\cos\theta$$

$$u_2 - U^{(0)}(\theta) = \frac{1}{2}(u_2-u_1) + \frac{(u_1-u_2)}{2}\cos\theta = \frac{1}{2}(u_2-u_1)(1-\cos\theta) = e\ell^{-1}(1-\cos\theta)$$

$$U^{(0)}(\theta) - u_1 = \frac{1}{2}(u_2-u_1) - \frac{(u_1-u_2)}{2}\cos\theta = \frac{1}{2}(u_2-u_1)(1+\cos\theta) = e\ell^{-1}(1+\cos\theta)$$

since $\frac{u_2-u_1}{2} = e\frac{(u_1+u_2)}{2} = \frac{2e\ell^{-1}}{2} = e\ell^{-1}$.

So the ODE for $U^{(1)}(\theta)$ is

$$\frac{dU^{(1)}(\theta)}{d\theta} = \frac{U^{(1)}(\theta)}{2}\left(-\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} + \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)$$

$$- (1+e\cos\theta)\, e\sqrt{1-\cos\theta}\sqrt{1+\cos\theta}\;\ell^{-2}$$

$$-\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} + \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{-(1+\cos\theta)+(1-\cos\theta)}{\sqrt{(1-\cos\theta)(1+\cos\theta)}}$$

$$= \frac{-2\cos\theta}{\sqrt{1-\cos^2\theta}} = \frac{-2\cos\theta}{\sin\theta},$$

$$\frac{dU^{(1)}(\theta)}{d\theta} = \frac{-\cos\theta}{\sin\theta}U^{(1)}(\theta) - (1+e\cos\theta)\,e\sin\theta\;\ell^{-2}$$

$$\sin\theta \frac{dv^{(1)}(\theta)}{d\theta} + \cos\theta\, v^{(1)}(\theta) =$$

$$- (1+e\cos\theta) e \sin^2\theta\, l^{-1}$$

$$\frac{d}{d\theta}\left(v^{(1)}(\theta)\sin\theta\right) = -(1+e\cos\theta) e \sin^2\theta\, l^{-2}$$

$$v^{(1)}(\theta)\sin\theta = -\int_0^\theta (1+e\cos\theta) e\sin^2\theta\, d\theta\, l^{-2}$$

$$= \left(-\int_0^\theta e\sin^2\theta\, d\theta + e^2\int_0^\theta \sin^2\theta\cos\theta\, d\theta\right) l^{-2}$$

$$v^{(1)}(\theta)\sin\theta = e\left(-\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) l^{-2} - e^2\frac{\sin^3\theta}{3} l^{-2}$$

Alternatively

$$\frac{d\theta}{du} \approx \frac{1 + GM\,(u + u_1 + u_2)}{\sqrt{(u_1 - u)\,(u - u_2)}}$$

$$\theta(u) - \theta(u_1) \approx \int_{u_1}^{u} \frac{1 + GM\,(u + u_1 + u_2)}{\sqrt{(u_1 - u)(u - u_2)}}\, du$$

Make the change of variable

$$u = \ell^{-1}(1 + e\cos\tilde{\theta})$$

$$\theta(u) - \theta(u_1) \approx \int_{0}^{\theta} \frac{1 + GM\,(\ell^{-1} + e\ell^{-1}\cos\tilde{\theta} + u_1 + u_2)}{e\ell^{-1}\sqrt{1 - \cos^2\theta}}\, \ell^{-1} e \sin\tilde{\theta}\, d\theta$$

$$\theta(u) - \theta(u_1) \approx \int_0^{\theta} \frac{1 - GM\,(\ell^{-1} + e\cos\tilde{\theta}\,\ell^{-1} + 2\ell^{-1})}{e\ell^{-1}\sin\theta}\, \ell^{-1} e \sin\theta\, d\tilde{\theta}$$

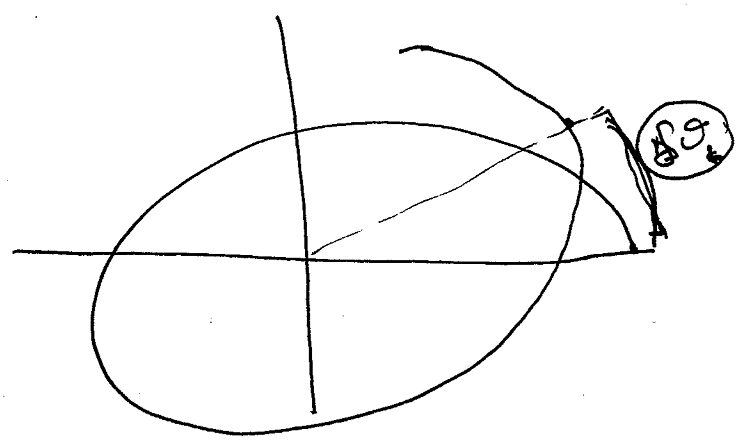
$$= \int_0^{\theta} 1 - GM\ell^{-1}(3 + e\cos\tilde{\theta})\, d\tilde{\theta}$$

So as u goes around one complete rotation

$\tilde{\theta}$ goes from 0 to 2π

$$\theta(\text{final}) - \theta(\text{original}) \approx \int_0^{2\pi} 1 - GM\ell^{-1}(3 + e\cos\tilde{\theta})\, d\tilde{\theta}$$

$$\Delta\theta = \theta(\text{final}) - \theta(\text{original}) \simeq 2\pi + \underbrace{6\pi M/\ell}_{\delta\theta_b}$$



So both calculations show the ellipse is precessing in θ but the second calculation actually gives amount of precession over a complete cycle (say from min dist to sun back to min distance to sun).

In fact that is the classical result computed by Einstein $\left(\delta\theta = \frac{6\pi M}{\ell} \right)$ for Mercury and shows the departure from Kepler's ~~theory~~ calculation based on Newtonian mechanics. ~~It agrees with~~ Einstein's calculation thus predicts the experimentally observed error from classical Newtonian mechanics

The numbers for Mercury are $\delta\theta = 43.1'' /$century

from our formula and this is the same as experiment

to one ~~one~~ decimal place (maybe more?)

7.4 Some casual remarks on general relativity.

First lets recall some of the well known observations that can motivate the search for a new metric g_{ij} to describe the appearance of gravitational force.

Consider ~~a~~ the motion of a particle in space with constant velocity (a,b,c)

$$x(t) = at + x_0, \quad y(t) = bt + y_0, \quad z(t) = ct + z_0.$$

If we consider any metric of the form

$$(ds)^2 = a_1 (dx)^2 + a_2 (dy)^2 + a_3 (dz)^2 + a_4 (dt)^2 \qquad (a_1, a_2, a_3 \text{ pos. const})$$

then minimizing the path in space-time

$$J = \int_{t_0}^{t_1} \left(\frac{ds}{dt}\right) dt = \int_{t_0}^{t} \sqrt{a_1 \left(\frac{dx}{dt}\right)^2 + a_2 \left(\frac{dy}{dt}\right)^2 + a_3 \left(\frac{dz}{dt}\right)^2 + a_4} \; dt$$

is done by considering the usual variation

$$x(t) = \bar{x}_0(t) + \epsilon x_1(t)$$

$$y(t) = \bar{y}(t) + \delta y_1(t)$$

$$z(t) = \bar{z}(t) + \gamma z_1(t)$$

So

$$J(\epsilon, \delta, \gamma) = \int_{t_0}^{t_1} \sqrt{a_1\left(\frac{d\bar{x}}{dt} + \epsilon \frac{dx_1}{dt}\right)^2 + a_2\left(\frac{d\bar{y}}{dt} + \delta \frac{dy_1}{dt}\right)^2 + a_3\left(\frac{d\bar{z}}{dt} + \gamma \frac{dz_1}{dt}\right)^2 + a_4}$$

If $\bar{x}, \bar{y}, \bar{z}$ is a minimizer the $J(0,0,0)$ is the minimum value and so

$$\frac{\partial J}{\partial \epsilon} = \frac{\partial J}{\partial \delta} = \frac{\partial J}{\partial \gamma} = 0 .$$

But

$$\left.\frac{\partial J}{\partial \epsilon}\right|_{\substack{\epsilon=0\\ \delta=0\\ \gamma=0}} = \int_{t_0}^{t_1} \frac{1}{\cancel{2}}\left(a_1\left(\frac{d\bar{x}}{dt}\right)^2 + a_2\left(\frac{d\bar{y}}{dt}\right)^2 + a_3\left(\frac{d\bar{z}}{dt}\right)^2 + a_4\right)^{1/2} \cancel{2}\, \frac{d\bar{x}}{dt}\frac{dx_1}{dt}\, dt$$

and so

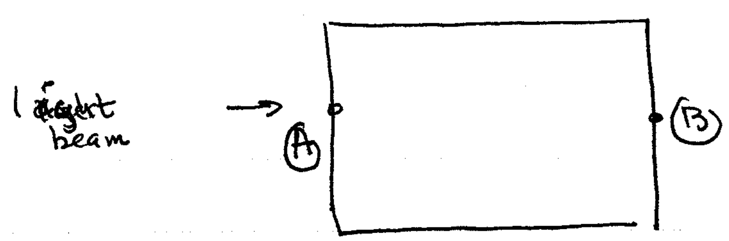
$$\frac{d}{dt}\left[\left(a_1\left(\frac{d\bar{x}}{dt}\right)^2 + a_2\left(\frac{d\bar{y}}{dt}\right)^2 + a_3\left(\frac{d\bar{z}}{dt}\right)^2 + a_4\right)^{1/2} \frac{d\bar{x}}{dt}\right] = 0$$

with similar expressions for $\bar{y}, \bar{z}$ from $\frac{\partial J}{\partial \delta} = \frac{\partial J}{\partial \gamma} = 0$ at $\epsilon = 0, \delta = 0, \gamma = 0$.

Clearly if $\frac{dx}{dt} = a$, $\frac{dy}{dt} = b$, $\frac{dz}{dt} = c$.

The equations become $\frac{d}{dt}(\text{const}) = 0$ which is of course satisfied. (The longer way is to actually solve the ODEs.) Notice also a critical pt (stationary pt) will suffice and we do not need a minimizer to derive our ODEs. Hence we know particle motion can be related to motion along a geodesic in space-time.

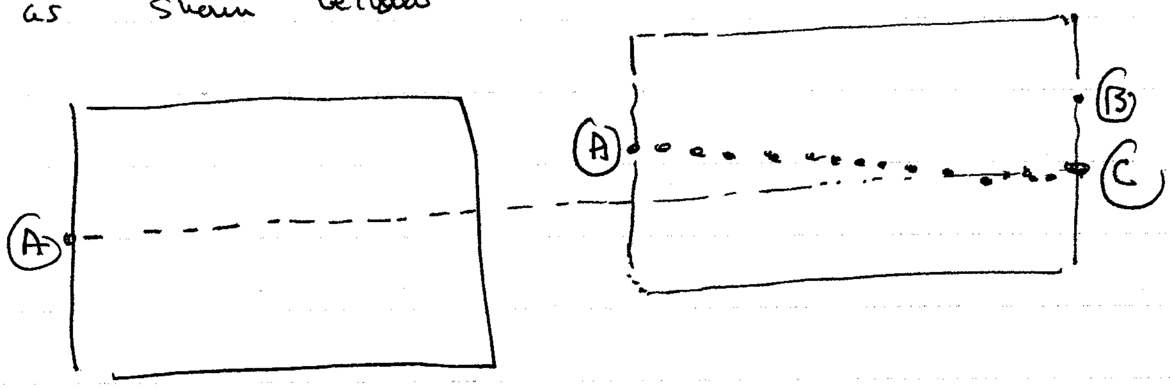
Next do the classical thought experiment which I sketch below



Light enters a hole on left of elevator compartment at A. If the compartment is not moving of course the light reaches point B and moves in straight line

But now if the elevator is accelerating upwards

as shown below



and in fact the light hits point C.

Thus to an observer in the elevator the light appears to bend and does not go in a straight line. Hence to the observer in elevator a metric with a_1, a_2, a_3 positive ~~pos~~ constants can be correct. Einstein postulated that the laws of physics should be the same whether we are in accelerating frame or not. So we use the Lorentzian metric where g_{ij} need not be positive definite. From our point of view the main feature is that Einstein has told us that to find motion we must search for the metric the solves the Einstein equations.

Finally I quote from Einstein's address in Kyoto (Dec. 1922) as is given in p2[illegible] of "Subtle is the Lord. The Science and Life of Albert Einstein" by Abraham Pais, Oxford Univ Press (1982).

" If all [accelerated] systems are equivalent, then Euclidean geometry cannot hold in all of them. To throw out geometry and keep [physical] laws is equivalent to describing thoughts without words. We must search for words before we can express thoughts. What must be search for at this point? This problem remained unsolvable to me until 1912, when I suddenly realized that Gauss's theory of surfaces holds the key for unlocking this mystery. I realized the Gauss's surface coordinates had a profound significance. However, I did not know at that time the Riemann had studied the foundations of geometry in an even more profound way. I suddenly remembered Gauss's theory was contained in the geometry course given by Geiser when I was a student ... I realized that the foundations of geometry have physical significance. My dear friend Grossman was there when I returned from Prague to Zürich. From him I learned for the first time about Ricci and later about Riemann. So I asked my friend whether any problem could be solved by Riemann's theory, namely, whether the invariants of the line element could completely determine the quantities I had been looking for".

(Of course what Einstein means by line element is just the metric g_{ij}).

8. Compensated compactness and the div-curl lemma of F. Murat and L. Tartar

In this section we review the compensated compactness theory of Murat–Tartar as presented in the book of C. M. Dafermos "Hyperbolic conservation laws in continuum physics".

<u>Theorem</u> Given an open subset Ω of $\mathbb{R}^m$, let $\{G_j\}$ and $\{H_j\}$ be sequences of vector fields in $L^2(\Omega;\mathbb{R}^m)$ converging weakly to respective limits $\overline{G}$ and $\overline{H}$ as $j\to\infty$. Assume both

$\{\operatorname{div} G_j\}$ and $\{\operatorname{curl} H_j\}$ lie in compact subsets of $W^{-1,2}(\Omega)$.

Then

$$G_j\cdot H_j \to \overline{G}\cdot\overline{H} \quad \text{as } j\to\infty \tag{1}$$

in the sense of distributions.

Proof. We establish (1) for Ω (since our convergence is only in sense of distributions with test functions of compact support). Also since

$$G_j \to \overline{G} \quad \text{as } j \to \infty$$

then $$G_j \cdot \overline{H} \to \overline{G} \cdot \overline{H} \quad \text{as } j \to \infty$$

And write

$$G_j \cdot H_j = G_j \cdot (H_j - \overline{H}) + G_j \cdot \overline{H}$$

Hence $$G_j \cdot H_j \to \overline{G} \cdot \overline{H}$$ (in sense of distribution) if and only if

$$G_j \cdot (H_j - \overline{H}) \to 0 \quad \text{(in sense of distributions)}$$

Hence we may define $\tilde{H}_j = H_j - \overline{H}$ and only need prove

$$G_j \cdot \tilde{H}_j \to 0 \quad \text{(in sense of distributions)}$$

. if

$$G_j \to \overline{G}, \quad \tilde{H}_j \to 0.$$

So without loss of generality we drop $\sim$ and assume $\overline{H} = 0$.

Now we can get on with the technicalities of the proof.

Let $\Phi_j \in W_0^{1,2}(\Omega; \mathbb{R}^m) \cap W^{2,2}_{loc}(\Omega; \mathbb{R}^m)$

denote the solution of the boundary value problem

$$\begin{cases} \Delta \Phi_j = H_j & \text{in } \Omega, \\ \Phi_j = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $H_j \to 0$ weakly in $L^2(\Omega)$,

$\Delta \Phi_j \to 0$ weak in $L^2(\Omega)$, $\Phi_j \to 0$ weakly in $W^{2,2}_{loc}(\Omega)$ and hence

$\{\text{div } \Phi_j\} \to 0$ ~~converges~~ weakly in $W^{1,2}_{loc}(\Omega)$.

On the other hand since

$$\Delta(\text{curl } \Phi_j) = \text{curl}(\Delta \Phi_j) = \text{curl } H_j$$

and curl H_j lies in a compact subset of $W^{-1,2}(\Omega)$

then $\Delta(\text{curl } \Phi_j)$ lies in a compact subset of $W^{-1,2}(\Omega)$

and so

$\{\text{curl } \Phi_j\}$ lies in a compact subset of $W^{1,2}_{loc}(\Omega)$.

and hence $\{\text{curl } \Phi_j\}$ has a strongly convergent subsequence in $W^{1,2}_{loc}(\Omega)$: what is the limit?

From $$\Delta(\text{curl } \Phi_j) = \text{curl } H_j$$

and $$H_j \to 0 \ (\text{weakly in } L^2(\Omega))$$

we see $\text{curl } \Phi_j \to 0$ weakly in $W^{1,2}_{loc}(\Omega)$

and hence the limit must be zero, i.e.

$\{\text{curl } \Phi_j\} \to 0$ strongly in $W^{1,2}_{loc}(\Omega)$

We now set

$$V_j = H_j - \text{grad div } \Phi_j \qquad (V_j \text{ a vector in } \mathbb{R}^m) \quad (2)$$

or component wise

$$V_{j\alpha} = H_{j\alpha} - \frac{\partial}{\partial x_\alpha} \text{div } \Phi_j \qquad \alpha = 1, \ldots, m$$

But

$$\Delta \Phi_j = H_j \quad (\text{by definition } \Phi_j)$$

or component wise

$$\Delta \Phi_{j\alpha} = H_{j\alpha} \qquad \alpha = 1, \ldots, m$$

and of course

$$\text{div } \Phi_j = \sum_{\beta=1}^{m} \frac{\partial \Phi_{j\beta}}{\partial x_\beta}$$

Hence

$$V_{j\alpha} = H_{j\alpha} - \frac{\partial}{\partial x_\alpha} \operatorname{div} \Phi_j$$

$$= \Delta \Phi_{j\alpha} - \frac{\partial}{\partial x_\alpha} \sum_{\beta=1}^{m} \frac{\partial \Phi_{j\beta}}{\partial x_\beta}$$

$$= \sum_{\beta=1}^{m} \frac{\partial^2}{\partial x_\beta^2} \Phi_{j\alpha} - \sum_{j=1}^{m} \frac{\partial}{\partial x_\beta} \frac{\partial \Phi_{j\beta}}{\partial x_\alpha}$$

$$= \sum_{\beta=1}^{m} \frac{\partial}{\partial x_\beta} \underbrace{\left\{ \frac{\partial \Phi_{j\alpha}}{\partial x_\beta} - \frac{\partial}{\partial x_\alpha} \Phi_{j\beta} \right\}}_{\operatorname{curl} \Phi_j}$$

Since $\operatorname{curl} \Phi_j \to 0$ strongly in $W^{1,2}_{loc}(\Omega)$

$$\sum_{\beta=1}^{m} \frac{\partial}{\partial x_\beta} \left(\operatorname{curl} \Phi_j \right) \to 0 \quad \text{strongly in } L^2_{loc}(\Omega) \qquad (3)$$

and so

$$V_j \to 0 \quad \text{strongly in } L^2_{loc}(\Omega). \qquad (4)$$

Now go back to the definition of V_j in (2).

$$V_j = H_j - \text{grad div } \Phi_j$$

$$G_j \cdot V_j = G_j \cdot H_j - \underbrace{G_j \cdot \text{grad div}}\, \Phi_j.$$

$$-\text{div}\left((\text{div } \Phi_j) G_j\right) + (\text{div } \Phi_j)(\text{div } G_j)$$

and

$$G_j \cdot H_j = \underset{\substack{\downarrow \\ 0 \text{ in } L^2_{loc}(\Omega)}}{G_j \cdot V_j} + \underset{\substack{\downarrow \\ 0 \text{ in } W^{-1,2}_{loc}(\Omega)}}{\text{div}\left((\text{div } \Phi_j) G_j\right)} - \underset{\substack{\downarrow \\ 0 \text{ in } W^{-1,2}_{loc}(\Omega)}}{(\text{div } \Phi_j)(\text{div } G_j)}$$

Multiply by a C^∞ test function and integrate by parts on the $\text{div}\left((\text{div } \Phi_j) G_j\right)$ term

and we see

$G_j \cdot H_j \to 0$ $j \to \infty$ in the sense of distributions. This completes our proof.

One remark that is important for later use:

For $\Phi \in \mathbb{R}^m$ we have defined

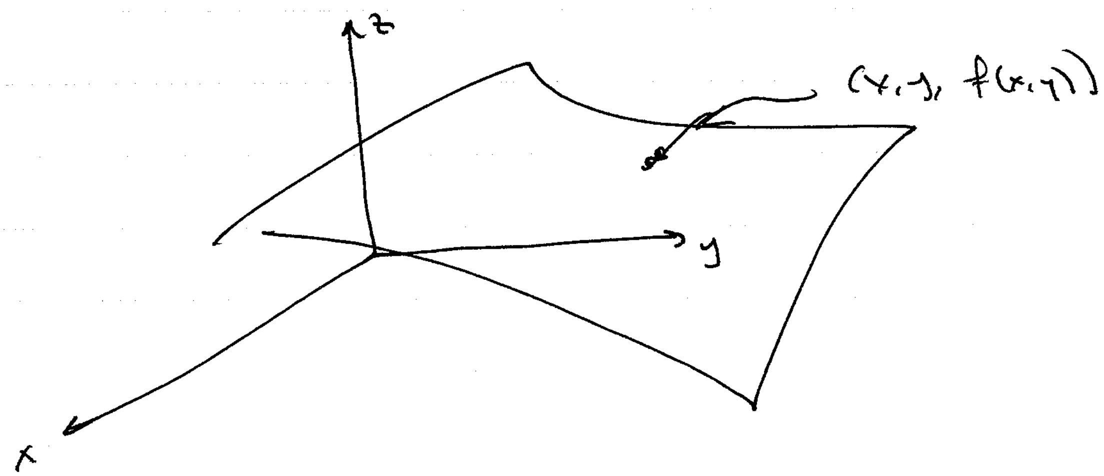
$$\operatorname{curl} \Phi = \frac{\partial \Phi_\alpha}{\partial x_\beta} - \frac{\partial \Phi_\beta}{\partial x_\alpha}, \quad 1 \le \alpha, \beta \le m,$$

is an $m \times m$ matrix.

9. The problem of isometric embedding

In the earlier section on elastic sheets and I have mentioned the problem of isometric embedding. In this section I return to the issue.

First recall some elementary results of introductory calculus. Let a surface in $\mathbb{R}^3$ be represented as the graph $(x, y, f(x,y))$



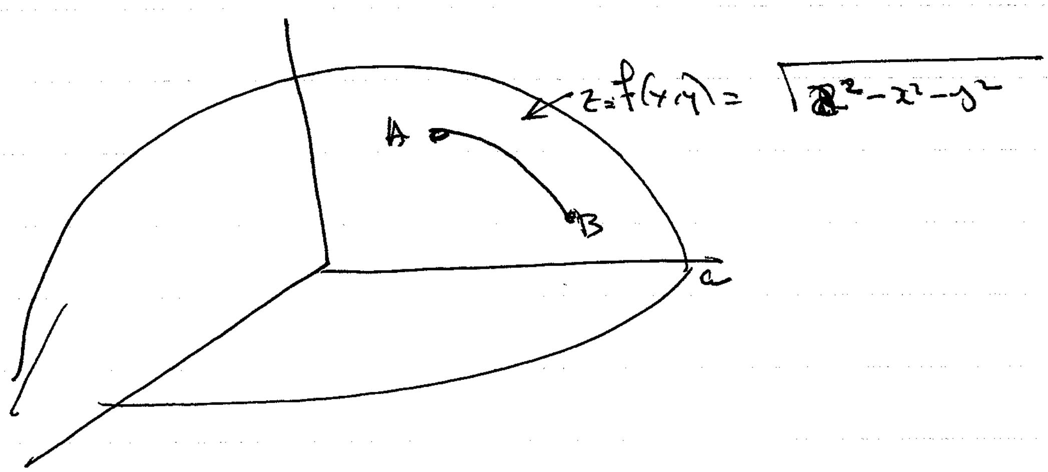
So on the surface $z = f(x,y)$ and distance is given as usual by

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2$$
$$= (dx)^2 + (dy)^2 + \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right)^2$$
$$= \left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right) dx^2 + 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \, dx\, dy + \left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right) dy^2$$

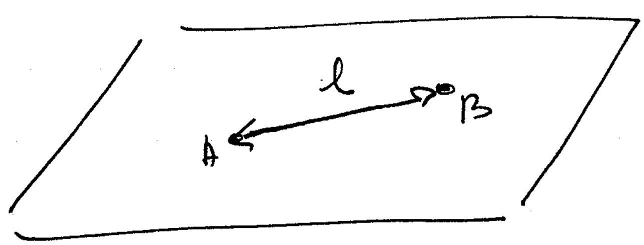
and our metric is just $g_{11} = \left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)$

$g_{12} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$, $g_{22} = 1 + \left(\frac{\partial f}{\partial y}\right)^2$.

Thus for example if we want to represent distance on a globe of the world



when projected on a two dimension map of the world



we cannot use the Euclidean distance between A and B on the flat 2 dimensional map but must instead use weighted metric $ds^2 = g_{11}dx^2 + 2g_{12}\,dx\,dy + g_{22}\,dy^2$

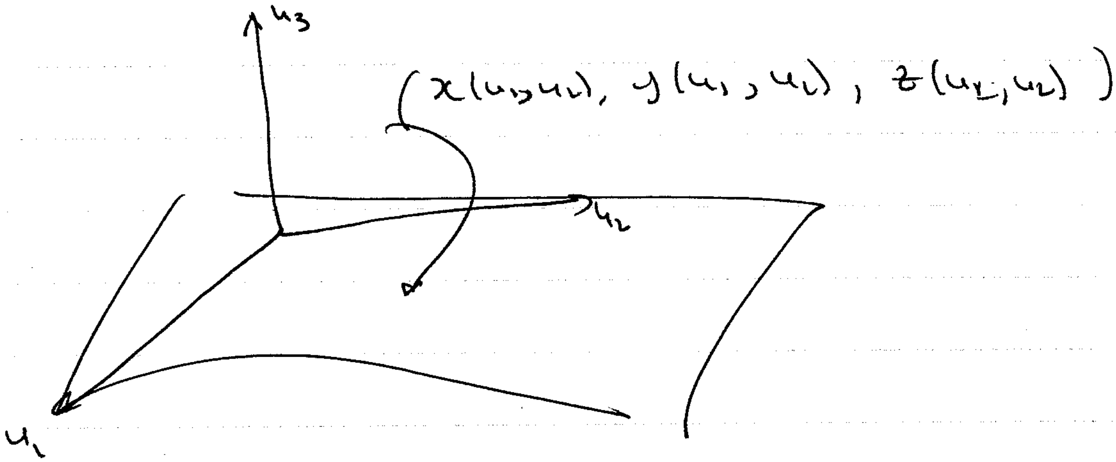
as given above. With this metric distances on the globe and distances on the flat map are identical, i.e an isometry. Furthermore is quite trivial that the metric g_{ij} given on the flat plane corresponds isometrically to the surface of a sphere where distances are given by the usual Euclidean distance

$$ds^2 = dx^2 + dy^2 + dz^2,$$

i.e simply set $z = f(x,y) = \sqrt{a - x^2 - y^2}$.
Hence we have satisfied the equation

$$dx^2 + dy^2 + dz^2 = \left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right) dx^2 + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) dx\,dy + \left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right) dy^2$$

More generally if we write our graph as



So

$$dx = \frac{\partial x}{\partial u_1}\,du_1 + \frac{\partial x}{\partial u_2}\,du_2$$

$$dy = \frac{\partial y}{\partial u_1}\,du_1 + \frac{\partial y}{\partial u_2}\,du_2$$

$$dz = \frac{\partial z}{\partial u_1}\,du_1 + \frac{\partial z}{\partial u_2}\,du_2$$

and

$$(dx)^2 + (dy)^2 + (dz)^2 = \left(\frac{\partial x}{\partial u_1}du_1 + \frac{\partial x}{\partial u_2}du_2\right)^2 + \left(\frac{\partial y}{\partial u_1}du_1 + \frac{\partial y}{\partial u_2}\right)^2$$
$$+ \left(\frac{\partial z}{\partial u_1}du_1 + \frac{\partial z}{\partial u_2}du_2\right)^2$$

$$(dx)^2+(dy)^2+(dz)^2 = \left(\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2\right)(du_1)^2$$

$$+ 2\left(\frac{\partial x}{\partial u_1}\right)\left(\frac{\partial x}{\partial u_2}\right) du_1\, du_2$$

$$+ 2\left(\frac{\partial y}{\partial u_1}\right)\left(\frac{\partial y}{\partial u_2}\right) du_1\, du_2$$

$$+ 2\left(\frac{\partial z}{\partial u_1}\right)\left(\frac{\partial z}{\partial u_2}\right) du_1\, du_2$$

$$+ \left(\left(\frac{\partial x}{\partial u_2}\right)^2 + \left(\frac{\partial y}{\partial u_2}\right)^2 + \left(\frac{\partial z}{\partial u_2}\right)^2\right)(du_2)^2$$

So given a metric we can attempt to find the surface via the equations

$$g_{11} = \left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2$$

$$g_{12} = \frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2} + \frac{\partial y}{\partial u_1}\frac{\partial y}{\partial u_2} + \frac{\partial z}{\partial u_1}\frac{\partial z}{\partial u_2}$$

$$g_{22} = \left(\frac{\partial x}{\partial u_2}\right)^2 + \left(\frac{\partial y}{\partial u_2}\right)^2 + \left(\frac{\partial z}{\partial u_2}\right)^2$$

In short, the isometric embedding or realization problem is:

<u>Given the metric g_{ij}, find the surface $x(u_1,u_2)$, $y(u_1,u_2)$, $z(u_1,u_2)$</u>

The issue remains the same for higher dimensional embeddings of (M^n, g) (an n-dimensional Riemannian manifold) into $\mathbb{R}^m$ (with m-dimensional Euclidean norm).

The issue here is then quite the opposite of the Einstein equations of §7. There we are given Einstein's relation on Riemann curvature tensor (in fact the Ricci tensor) which then makes the problem

<u>Given the Einstein relation</u>, $\underline{g^{ij} R_{jk} - \frac{1}{2} R \delta^i_k = 0}$ <u>find the metric g_{ij}.</u>

The Einstein vacuum equations are clearly balance equations, whereas the isometric embedding problem becomes a system of balance equations is phrased in terms of solvability of the Gauss-Codazzi equations. Hence it is the Gauss-Codazzi equations we turn to in the next section.