A sketch of the proof of Blum's theorem.

1° From the Codazzi equs

$$\nabla_j H^\mu_{ia} - \nabla_i H^\mu_{ja} + A^\nu_{\mu i} H^\nu_{aj} - A^\nu_{\mu j} H^\nu_{ai} = 0$$

we have in particular

$$\nabla_1 H^\mu_{23} - \nabla_3 H^\mu_{21} + A^\nu_{\mu 3} H^\nu_{21} - A^\nu_{\mu 1} H^\nu_{23} = 0 ,$$

$$\nabla_1 H^\mu_{32} - \nabla_2 H^\mu_{31} + A^\nu_{\mu 2} H^\nu_{31} - A^\nu_{\mu 1} H^\nu_{32} = 0 . \tag{1.7}$$

Subtract to see

$$\nabla_2 H^\mu_{31} - \nabla_3 H^\mu_{21} - A^\nu_{\mu 2} H^\nu_{31} + A^\nu_{\mu 3} H^\nu_{21} = 0 . \tag{1.8}$$

Thus the Codazzi relation (1.8) is implied by the first two (1.7). For $n=3$, $m=6$ this reduces the number of independent Codazzi relations by 3.

2° We can rewrite the Codazzi equations as

$$e_{\ell jk} \nabla_j H^{\mu}_{ka} \;\overset{\text{minus}}{\cancel{+}}\; \varepsilon_{\ell jk} A^{\nu}_{\mu i} H^{\nu}_{ag} = 0$$

Recall the relation

$$\operatorname{cof} H^{\mu}_{i\ell} = \varepsilon_{ijk}\, \varepsilon_{\ell mn} H^{\mu}_{kn} H^{\mu}_{jm}$$

Hence

$$\nabla_\ell \operatorname{cof} H^{\mu}_{i\ell} = \varepsilon_{ijk}\, \varepsilon_{\ell mn} (\nabla_\ell H^{\mu}_{kn}) H^{\mu}_{jm}$$

$$= \frac{\partial^2 W}{\partial H^{\mu}_{i\ell}\, \partial H^{\mu}_{kn}} \nabla_\ell H^{\mu}_{kn} \quad \text{(by lemma 2)} \quad + \varepsilon_{ijk}\, \varepsilon_{\ell mn} H^{\mu}_{kn} (\nabla_\ell H^{\mu}_{jm}). \quad \text{(no sum on } \mu) \uparrow (1.9)$$

$$= \underbrace{\varepsilon_{ijk}\, \varepsilon_{\ell mn}}\, H^{\mu}_{jm} \nabla_\ell H^{\mu}_{kn} \qquad \begin{matrix} j \leftrightarrow k \\ m \leftrightarrow n \end{matrix}$$

From the Codazzi equations

$$\varepsilon_{\ell mn} \nabla_\ell H^{\mu}_{kn} \;\overset{(-)}{\cancel{+}}\; \varepsilon_{\ell mn} A^{\nu}_{\mu n} H^{\nu}_{k\ell} = 0, \quad (1.10)$$

$$\varepsilon_{\ell mn} \nabla_\ell H^{\mu}_{jm} \;\overset{(-)}{\cancel{+}}\; \varepsilon_{\ell mn} A_{\mu m} H^{\nu}_{j\ell} = 0 \quad (1.10)$$

and substitution of these relations into (1.9) yields

$$\nabla_\ell (\operatorname{cof} H^{\mu}_{i\ell}) \;\overset{(-)}{\cancel{+}}\; \varepsilon_{ijk}\, \varepsilon_{\ell mn} A^{\nu}_{\mu n} H^{\nu}_{k\ell} H^{\mu}_{jm}$$

$$\overset{(-)}{+}\; \varepsilon_{ijk}\, \varepsilon_{\ell mn} A^{\nu}_{\mu m} H^{\nu}_{j\ell} H^{\mu}_{kn} = 0$$

(no sum on μ)

Interchange m, n and j, k in the above expression to see

$$\nabla_\ell \left(\operatorname{cof} H_{i\ell}^{\mu}\right) \cancel{+4}\,(-2)\, \varepsilon_{ijk}\, \varepsilon_{\ell mn}\, A^{\nu}_{\mu m} H^{\nu}_{j\ell} H^{\mu}_{kn} = 0,$$

(no sum on μ).

Now sum on μ

$$\sum_{\mu=4}^{6} \nabla_\ell \left(\operatorname{cof} H_{i\ell}^{\mu}\right) \cancel{+4}\,(-)\,2\, \varepsilon_{ijk}\, \varepsilon_{\ell mn} \sum_{\mu=4}^{6} A^{\nu}_{\mu m} H^{\nu}_{j\ell} H^{\mu}_{kn} = 0$$

The Gauss equations may be written as

$$\sum_{\mu=4}^{6} \operatorname{cof} H_{i\ell}^{\mu} = R_{i\ell}$$

where

$$R = \begin{bmatrix} R_{2323} & R_{2331} & R_{2312} \\ R_{2331} & R_{3131} & R_{3112} \\ R_{2312} & R_{3112} & R_{1212} \end{bmatrix}$$

Thus our equation for divergence of the cofactors becomes for $n=3$, $m=6$:

$$\nabla_\ell R_{i\ell} \cancel{+} (-2)\, \epsilon_{ijk}\, \epsilon_{mn\ell} \sum_{\mu=4}^{6} A^{\nu}_{\mu mn} H^{\nu}_{j\ell} H^{\mu}_{kn} = 0 \qquad i=1,2,3$$

The first term on the left hand side is zero by the second Bianchi identity, i.e

$$\nabla_\ell(R_{1\ell})=0\,,\quad \nabla_\ell(R_{2\ell})=0\,,\quad \nabla_\ell(R_{3\ell})=0.$$

The ~~right~~ second term on the left hand side is zero from the skew symmetry of $A^{\nu}_{\mu n}$ in μ,ν. Hence we have shown a ~~linear~~ combination of of the Codazzi equations combined with the Gauss relation gives three trivial relations $0=0$, and thus we have reduced the number of independent Codazzi equations by an additional 3

5e

3° Now write the Codazzi and Ricci equations as

$$C^{\mu}_{k\alpha} \overset{\text{def.}}{=} \varepsilon_{ijk} \nabla_j H^{\mu}_{i\alpha} + \varepsilon_{ijk} A^{\nu}_{\mu i} H^{\nu}_{\alpha j} = 0 ,$$

$$K^{\mu\nu}_{k} \overset{\text{def.}}{=} \varepsilon_{ijk} \nabla_i A^{\nu}_{\mu j} + \varepsilon_{ijk} A^{\nu}_{\eta i} A^{\eta}_{\mu j} \quad (= \varepsilon_{ijk} A^{\eta}_{\nu i} A^{\eta}_{\mu j})$$

$$- g^{pq} \varepsilon_{ijk} H^{\mu}_{ip} H^{\nu}_{jq} = 0$$

Apply ∇_k to the Codazzi system

$$\varepsilon_{ijk} \nabla_k \nabla_j H^{\mu}_{i\alpha} \overset{-}{+} \varepsilon_{ijk} (\nabla_k A^{\nu}_{\mu i}) H^{\nu}_{\alpha j} \overset{-}{+} \varepsilon_{ijk} A^{\nu}_{\mu i} (\nabla_k H^{\nu}_{\alpha j}) = 0$$

The last term can be rewritten using Codazzi as

$$\varepsilon_{ijk} \nabla_k H^{\nu}_{\alpha j} = \overset{+}{-} \varepsilon_{ijk} A^{\eta}_{\nu j} H^{\eta}_{\alpha k} .$$

Thus

$$\varepsilon_{ijk} \nabla_k \nabla_j H^{\mu}_{i\alpha} \; \cancel{+ \varepsilon_{ijk} (\nabla_k A_{\mu i})}$$

$$\overset{-}{+} \varepsilon_{ijk} \nabla_k A^{\nu}_{\mu i} H^{\nu}_{\alpha j} - \varepsilon_{ijk} A^{\eta}_{\nu j} H^{\eta}_{\alpha k} A^{\nu}_{\mu i}$$

$$= 0$$

or interchanging j, k in the last term

$$\epsilon_{jk} \nabla_k \nabla_j H^\mu_{ia} - \epsilon_{jk} H^\nu_{aj} \left(\nabla_k A_{\mu i}^{\nu} - A_{\mu k}^{\eta} A^{\eta}_{\mu i} \right) = 0$$

Finally use the formula for the commutator $\epsilon_{jk} \nabla_k \nabla_j H^\mu_{ia}$ and the Gauss equations and we recover H^ν_{aj} multiplying the Ricci equations, i.e. ~~If H^ν_{aj} is of full rank then the Ricci equations~~ hold.

$$H^\nu_{aj} \quad K^\nu_{i\mu} = 0$$

where $K^\mu_{i\mu} = 0$. Since $1 \le \alpha, j \le 3$

this gives 9 equations in the nine unknowns $K^\nu_{i\mu}$. Blum's rank condition asserts

that this system has the unique solution $K^\nu_{i\mu} = 0$ and the Ricci equations are satisfied. In other words

~~the covariant divergence~~

$$\cancel{\nabla_k C^\mu_{ka}} = 0$$

~~if and~~

if the Gauss and Codazzi equations (of which there are only 2∇ independent equations)

then the covariant divergence of $C^{\mu}_{\kappa\alpha}$ is

zero,

$$\nabla_\kappa C^{\mu}_{\kappa\alpha} = 0$$

and since

$$\nabla_\kappa C^{\mu}_{\kappa\alpha} = H^{\nu}_{\alpha j} K^{\mu}_{i\mu}$$

we have $K^{\nu}_{i\mu} = 0$ under Blum's rank

condition. Hence we may say $K^{\nu}_{i\mu} = 0$

is an <u>involution system</u> carried along by

the Codazzi and Gauss equations.