$$h_{\mu\nu}(\bar{x}^\lambda) = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}(\bar{x}^\lambda)\,\frac{\partial x^\beta}{\partial \bar{x}^\nu}(\bar{x}^\lambda)\,g_{\alpha\beta}(x^\rho(\bar{x}^\lambda))$$

By asymptotically flat, we mean that $\overset{\circ}{g}_{\mu\nu}$ is H^s asymptotic to some given comparison metric. In practice this means $\overset{\circ}{g}_{\mu\nu}$ is asymptotic to a Schwarzschild type metric $\tilde{g}_{\mu\nu}$, i.e. $\overset{\circ}{g}_{\mu\nu} - \tilde{g}_{\mu\nu} \in H^s$. Here our comparison metric looks like

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \frac{2m}{r}\delta_{\mu\nu}$$

with $\eta_{\mu\nu}$ is the usual Minkowski metric and $m \geq 0$. (Of course a special case is just $m=0$.) The existence theory then guarantees that these asymptotic conditions will automatically be maintained in time.

The symmetric tensor $G_{\mu\nu}$

has 10 independent components so the

Einstein equations comprise 10 algebraically

independent equations for the unknown metric

$g_{\mu\nu}$

$$g = \begin{pmatrix} g_{00} & x & x & x \\ & g_{11} & x & x \\ & & g_{22} & x \\ & & & g_{33} \end{pmatrix}$$

with g having $4+3+2+1=10$

independent components as well. So it

seems that the Einstein eqns should

suffice to determined the $g_{\mu\nu}$ uniquely.

However that is not so.

Although algebraically independent, the 10 $G_{\mu\nu}$ equations are related by four differential identities, the Bianchi identities

$$G^{\mu}_{\nu;\mu} = 0$$

[which follow from

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}_{\mu\nu\eta;\nu} = 0 \qquad \text{(Bianchi)}$$

$$R_{;\eta} - R^{\mu}_{\eta;\mu} - R^{\nu}_{\eta;\nu} = 0 \qquad \text{(contract)}$$

$$(R^{\mu}_{\nu} - \tfrac{1}{2}\delta^{\mu} R)_{;\mu} = 0 \qquad \text{(contract)}$$

$$(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R)_{;\mu} \,] \qquad \text{(rewrite)}$$

]

Thus there are <u>not</u> 10 ~~algebraically~~ functionally independent equations, but only

$$10 - 4 = 6,$$

leaving 4 degrees of freedom in the 10 unknowns $g_{\mu\nu}$.

These degrees of freedom correspond to the fact that if $g_{\mu\nu}$ is a solution of Einstein's equations then so is $g'_{\mu\nu}$, where $g'_{\mu\nu}$ is determined by a general coordinate transformation $x \to x'$. (This is the reason for the way the "uniqueness" problem has been stated).

The failure of Einstein's equations to determine $g_{\mu\nu}$ uniquely is analagous with failure of Maxwell eqns to determine the vector potential uniquely. Recall Maxwell's eqn

$$\frac{\partial B}{\partial t} = - \text{curl } E$$

$$\frac{\partial D}{\partial t} = \text{curl } H - J$$

where in the linear case have

$$D = \varepsilon E \qquad \left(E = \frac{1}{\varepsilon} D\right)$$

$$B = \mu H \qquad \left(H = \frac{1}{\mu} B\right)$$

$$J = \sigma E \qquad \left(J = \frac{\sigma}{\varepsilon} D\right)$$

$$\nabla \cdot B = \nabla \cdot D = 0$$

Substitution of $\cancel{\tfrac{1}{\varepsilon}} D = \varepsilon E$ ~~yields~~ $H = \frac{1}{\mu} B$, $J = \sigma E$

$$\frac{\partial B}{\partial t} = -\operatorname{curl} E$$

$$\varepsilon \frac{\partial E}{\partial t} = \frac{1}{\mu} \operatorname{curl} B - \cancel{\sigma E} - \sigma E$$

curl

$\frac{\partial}{\partial t}$

$$\left\{ \begin{aligned} \frac{\partial B}{\partial t} &= -\operatorname{curl} E \\ \frac{\partial E}{\partial t} &= \frac{1}{\mu\varepsilon} \operatorname{curl} B - \frac{\sigma}{\varepsilon} E \end{aligned} \right\}$$

$$\frac{\partial}{\partial t} \operatorname{curl} B = -\operatorname{curl} \operatorname{curl} E$$

$$\mu\varepsilon \frac{\partial^2 E}{\partial t^2} = \frac{\partial}{\partial t} \operatorname{curl} B - \frac{\sigma}{E} \frac{\partial E}{\partial t}$$

$$\mu\varepsilon \frac{\partial^2 E}{\partial t^2} = -\operatorname{curl} \operatorname{curl} E + \frac{\sigma}{E} \frac{\partial E}{\partial t}$$

$$\nabla \times (\nabla \times E) = \nabla \underbrace{(\nabla \cdot E)}_{= 0} - \nabla \cdot \nabla E$$

$$\boxed{\mu\varepsilon \frac{\partial^2 E}{\partial t^2} = \nabla^2 E + \frac{\sigma}{E} \frac{\partial E}{\partial t}}$$

which of course would determine E and hence J and D as well.

But what about B and H?

Since

$$\frac{\partial B}{\partial t} = -\operatorname{curl} E$$

$$\operatorname{curl} H = J + \frac{\partial D}{\partial t}$$

if we know J and D we only know curl H. and hence H is determined up to a function $\nabla \varphi$

$$\operatorname{curl}(H + \nabla\varphi) = J + \frac{\partial D}{\partial t}$$

if

$$\operatorname{curl} H = J + \frac{\partial D}{\partial t}$$

since $\operatorname{curl} \nabla\varphi = 0$.

One way to specify H uniquely has been provided by the condition

$$\nabla \cdot B = 0$$

$$\Rightarrow \nabla \cdot (\mu H) = 0$$

$$\Rightarrow \quad \nabla \cdot H = 0$$

So if we have 2 soln

$$\tilde{H} = H + \nabla\varphi$$

$$\underbrace{\nabla \cdot \tilde{H}}_{0} = \underbrace{\nabla \cdot H}_{0} + \nabla^2 \varphi \quad \Rightarrow \nabla^2\varphi = 0$$

$\nabla^2\varphi=0$ with appropriate boundary condition will yield $\varphi=0$.
So the condition

$$\nabla\cdot B=0$$

was essential to getting a unique solution to Maxwell's eqns. So $\nabla\cdot B=0$ is a Gauge condition in the language of physics.

In a similar way we will have to add additional conditions to Einstein's equations to get unique solutions.

This is done by choosing harmonic coordinates. This a coordinate system x^α satisfying $\Gamma^\lambda=0$

where $\Gamma^\lambda \equiv g^{\mu\nu}\Gamma^\lambda_{\mu\nu}$.

Can we always choose a coordinate system for which $\Gamma^\lambda=0$ holds? The answer is yes as is seen by the following argument.

Recall that the Christoffel symbols are defined by

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} \left[\frac{\partial g_{\lambda\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\lambda\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\lambda}} \right]$$

In exercise (S. Weinberg, Gravitation and Cosmology, p162) shows that in new x' coordinates

$$\Gamma'^{\lambda} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \Gamma^{\rho} - g^{\rho\sigma} \frac{\partial^2 x'}{\partial x^{\rho} \partial x^{\sigma}}$$

Hence if Γ^{ρ} does not vanish we solve the second order PDE

$$g^{\rho\sigma} \frac{\partial^2 x'}{\partial x^{\rho} \partial x^{\sigma}} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \Gamma^{\rho}$$

which forces $\Gamma'^{\lambda} = 0$ in the new primed coordinates.

The harmonic coordinate condition

$$\Gamma^\lambda = 0$$

can be expressed of course as

$$0 = \Gamma^\lambda = \frac{1}{2} g^{\mu\nu} g^{\lambda\kappa} \left\{ \frac{\partial g_{\kappa\mu}}{\partial x^\nu} + \frac{\partial g_{\kappa\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa} \right\}$$

Recall

$$g^{\lambda\kappa} \frac{\partial g_{\kappa\mu}}{\partial x^\nu} = - g_{\kappa\mu} \frac{\partial g^{\lambda\kappa}}{\partial x^\nu}$$

$$\frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\kappa} = g^{-1/2} \frac{\partial}{\partial x^\kappa} g^{1/2}$$

$$(g = \cancel{\ldots} - \det g_{\mu\nu})$$

This then gives

$$\Gamma^\lambda = - g^{-1/2} \frac{\partial}{\partial x^\kappa} (g^{1/2} g^{\lambda\kappa}) \quad \text{and so}$$

$$0 = \Gamma^\lambda = \frac{\partial}{\partial x^\kappa} (\sqrt{g}\, g^{\lambda\kappa}) = 0$$

The invariant d'Alembertian (Laplace-Beltrami) ~~[illegible]~~ is defined by

$$\Box^2 \phi \equiv (g^{\lambda\kappa} \phi_{;\lambda})_{;\kappa}$$

Using the fact the covariant derivative of a scalar S is just the ordinary derivative

$$S_{;\mu} = \frac{\partial S}{\partial x^\mu}$$

and

$$V^\mu_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu}\left(\sqrt{g}\, V^\mu\right)$$

we see

$$\Box^2 \phi = \left(g^{\lambda\kappa} \phi_{;\lambda}\right)_{;\kappa}$$

$$= \cancel{g^{\lambda\kappa} \phi_{;\lambda\kappa}} +$$

$$= g^{\lambda\kappa} \phi_{;\lambda\kappa} + g^{\lambda\kappa}{}_{;\kappa} \phi_{;\lambda}$$

take $V^\mu = g^{\mu\kappa}$

$$= \underbrace{g^{\lambda\kappa} \frac{\partial^2 \phi}{\partial x^\lambda \partial x^\kappa}}_{\text{since } \phi \text{ is a scalar}} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\kappa}\left(\sqrt{g}\, g^{\lambda\kappa}\right) \underbrace{\phi_{;\lambda}}_{= \frac{\partial \phi}{\partial \lambda}}$$

since ϕ is scalar

$$= g^{\lambda\mu} \frac{\partial^2 \phi}{\partial x^\lambda \partial x^\kappa} - \Gamma^\lambda \frac{\partial \phi}{\partial x^\lambda}$$

Hence if x'^λ satisfies the equation that change $x \to x'$ when $\Gamma'^\lambda = 0$

we see $\Box^2 x'^\lambda = 0$

i.e the "harmonic coordinates" satisfy the wave equation

$$\Box^2 q = 0$$

in each component, i.e each component is a "harmonic" function.

Now the "Harmonic Coordinate" condition

$$\Gamma^\lambda \equiv g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0$$

$\lambda = 0, \underbrace{1, 2, 3}$
4 equations

gives 4 additional equations

which when added to 6 functionally independent equations from the Einstein equations gives 10 eqns for 10 unknowns.

Even more is true. If the "harmonic coordinate" conditions are to play to same role of $\nabla \cdot B = 0$ in Maxwell's equations they need to preserved under the evolution given by the Einstein equations. This is true and was discovered by Y. Fourès-Bruhat in (Theorem d' existence pour certain systèmes d' equations aux derivées partielles non lineaaires, Acta Math 88 141-225 (1952))

Specifically we know that if initial data is consistent with the Einstein vacuum equations

$$G^{0}_{\mu}(0, x^i) = 0$$

(we need this since not all the ~~initial~~ Einstein equations are evolutionary) and consistent with

$$\dot{\Gamma}^{\mu} = 0 \quad \text{at } t=0$$

then

$$\Gamma^{\mu} = 0$$

for $t>0$.

[The requirement that $G^{0}_{\mu}(0, x^i) = 0$

is called the constraint conditions

and these time independent conditions

are also propagated for $t>0$]. These

results are summarized in the following

lemma (Lemma 3.2 of Fischer and Marsden)

<u>Lemma</u> Let $(\mathring{g}, \mathring{k}) \in (\mathcal{L}^s, K^{s-1})$

and suppose $s > \frac{3}{2} + 2$. Assume

$\mathring{\Gamma}^\mu(x^i) = 0$ and $\mathring{G}^0_\mu(x^i) = 0$ (the

superscript ^{0} over Γ and G refer

to the fact that Γ^μ and G^0_μ are

computed from the initial data $(\mathring{g}, \mathring{k})$))

Let $g_{\mu\nu}$ be an H^s space time

satisfying

$$g_{\mu\nu}(0, x^i) = \mathring{g}_{\mu\nu}(0, x^i)$$

$$\frac{\partial g_{\mu\nu}}{\partial t}(0, x^i) = \mathring{k}_{\mu\nu}(0, x^i)$$

and

$$R_{\mu\nu} = 0 \quad \text{(in harmonic coordinates)}$$

Then the harmonic coordinate

condition $\Gamma^\mu(t, x^i) = 0$ for $t > 0$.

The Sobolev spaces $\mathcal{L}^s$, $\mathcal{K}^s$ are just the usual Sobolev space with the asymptotically flat conditions (see p23 of the Fischer and Marsden paper)

Now we can state the main theorem of Fischer and Marsden.

<u>Theorem</u> (Existence). Let $s \geq 4$ (or $s > 3.5$ if not an integer). Let $(g^0_{\mu\nu}, k^0_{\mu\nu}) \in \mathcal{L}^s \times \mathcal{K}^{s-1}$.

Then there exists an $\varepsilon > 0$ and a unique H^s asymptotically flat Lorentz metric $g^L(t, x^i)$ $t \in I = (-\varepsilon, \varepsilon)$ such that $t \mapsto g^L(t, \cdot) \in \mathcal{L}^s$ is C^0, is C^1 into $\mathcal{L}^{s-1}$, $g^L(0, x^i) = \overset{\circ}{g}(x^i)$,

$\dfrac{\partial g^L}{\partial t}(0, x^i) = \overset{\circ}{k}(x^i)$ and

$R_{\mu\nu} = 0$ (in harmonic coordinates)

The solution depends continuously on the initial data in $\mathcal{L}^s \times \mathcal{K}^{s-1}$ and moreover, for T fixed, $|T| < \varepsilon$, the solution is a C^∞ function of the initial data. If the initial data in C^∞, so is the solution.

Case A. If the Cauchy data satisfy $\overset{\circ}{G}_{\mu}{}^{0} = 0$, $\overset{\circ}{\Gamma}{}^{\mu} = 0$ Then $g_{\mu\nu}$ so obtained satisfies

$$R_{\mu\nu} = 0$$

Case B. If the Cauchy data satisfy only $\overset{\circ}{G}_{\mu}{}^{0} = 0$ there exists an H^{s+1} coordinate transformation f such that $\overset{\circ}{\bar{g}} = (f^{-1})^* \overset{\circ}{g}$ has Cauchy data satisfying $\overset{\circ}{\bar{G}}_{\mu}{}^{0} = 0$, $\overset{\circ}{\bar{\Gamma}}{}^{\mu} = 0$. Hence we get an H^s solution $g = f^* \bar{g}$ on $R_{\mu\nu} = 0$

(* denote pull back map, p24 of Fischer and Marsden)

Our main interest in these notes are not the technical details but to see how the Einstein equations can be put in symmetric hyperbolic form.

To see how this is done we need a few computational identities.

(i)

$$R_{\mu\nu} = R^{(h)}_{\mu\nu} + \frac{1}{2}\left(g_{\mu\alpha} \frac{\partial \Gamma^{\alpha}}{\partial x^{\nu}} + g_{\nu\alpha} \frac{\partial \Gamma^{\alpha}}{\partial x^{\mu}} \right)$$

where

(ii)

$$R^{(h)}_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\beta}} + H_{\mu\nu}\left(g_{\alpha\beta}, \frac{\partial g_{\alpha\beta}}{\partial x^{\lambda}} \right)$$

and

(iii)

$$H_{\mu\nu}\left(g_{\alpha\beta}, \frac{\partial g_{\alpha\beta}}{\partial x^{\lambda}} \right) = g^{\alpha\beta} g_{\rho\sigma} \Gamma^{\rho}_{\mu\beta} \Gamma^{\sigma}_{\nu\alpha}$$

$$+ \frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} \Gamma^{\alpha} + g_{\nu\lambda} \Gamma^{\lambda}_{\alpha\beta} g^{\alpha\gamma} g^{\beta\delta} \frac{\partial g_{\gamma\delta}}{\partial x^{\mu}} + g_{\mu\lambda} \Gamma^{\lambda}_{\alpha\beta} g^{\alpha\gamma} g^{\beta\delta} \frac{\partial g_{\gamma\delta}}{\partial x^{\nu}} \right).$$

Thus (u) represents the Ricci curvature in harmonic coordinates and (L) shows that if we are in harmonic coordinates when $\Gamma^\alpha = 0$ then

$$R_{\mu\nu} = R^{(h)}_{\mu\nu}.$$

Note also that $H_{\mu\nu}$ is a homogeneous quadratic in $\frac{\partial g_{\mu\nu}}{\partial x^\alpha}$ and is rational in $g_{\mu\nu}$ with nonzero denominator $\det(g_{\mu\nu})$.

The principal part of the Einstein vacuum system in harmonic coordinates is the operator

$$-\frac{1}{2} g^{\alpha\beta} \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta}$$

operates the same way on each component of $g_{\mu\nu}$ so that the highest order terms are completed uncoupled. Such systems are said to be weakly coupled are a particular case of the strictly hyperbolic systems of Leray (J. Leray, Lectures on hyperbolic equations with variable coefficients, Princeton, N.J. Institute for Advanced Study (1952)) From Leray's theorem for strictly hyperbolic second order systems as improved by Dionne (P. Dionne, Sur les problèmes de Cauchy bien posés. J. d'Analyse Math. Jerusalem 10 1-90 (1962/63))

Y. Choquet-Bruhat (Espaces-temps einsteiniens généraux, chocs gravitationels, Ann. Inst. H. Poincaré 8 327-338 (1968) ; Solution C^∞ d'equations hyperboliques non-lineares, C. R. Acad Sci Paris 272, 386-388 (1971)).

We reiterate the goal here is to use first order symmetric hyperbolic theory and not the second order Leray theory.

To set the first order system we first define $k_{\mu\nu}$ as

$$\boxed{\frac{\partial g_{\mu\nu}}{\partial t} = k_{\mu\nu}}$$ (definition of $k_{\mu\nu}$)

Then differentiate to get

$$\frac{\partial g_{\mu\nu}}{\partial t\, \partial x^i} = \frac{\partial k_{\mu\nu}}{\partial x^i}$$ (equality of cross partials)

and then contract

$$\boxed{g^{ij} \frac{\partial g_{\mu\nu,i}}{\partial t} = g^{ij} \frac{\partial k_{\mu\nu}}{\partial x^i}}$$

(since g^{ij} is nonsingular the contracted and previous non contracted version are equivalent)

Now lets write out

$$R_{\mu\nu}^{(4)} = 0$$

separating the time derivatives from the space ~~derivatives~~ derivatives.

$$0 = -\frac{1}{2} g^{00} \frac{\partial^2 g_{\mu\nu}}{\partial t^2} - g^{0j} \frac{\partial^2 g_{\mu\nu}}{\partial t\, \partial x^j}$$

$$-\frac{1}{2} g^{ij} \frac{\partial^2 g_{\mu\nu}}{\partial x^i \partial x^j} + H_{\mu\nu}\left(g_{\alpha\beta}, \frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\right)$$

or

$$0 = -\frac{1}{2} g^{00} \frac{\partial k_{\mu\nu}}{\partial t} - g^{0j} \frac{\partial k_{\mu\nu}}{\partial x^j}$$

$$-\frac{1}{2} g^{ij} \frac{\partial g_{\mu\nu,i}}{\partial x^j} + H_{\mu\nu}\left(g_{\alpha\beta}, g_{\alpha\beta,i}, k_{\alpha\beta}\right)$$

(where the t derivative is the $k_{\alpha\beta}$ term)

or

$$\boxed{-g^{00} \frac{\partial k_{\mu\nu}}{\partial t} = 2 g^{0j} \frac{\partial k_{\mu\nu}}{\partial x^j} + g^{ij} \frac{\partial g_{\mu\nu,i}}{\partial x^j} - 2 H_{\mu\nu}}$$

Let's put the boxed equation in a list

$$\frac{\partial g_{\mu\nu}}{\partial t} = K_{\mu\nu}$$

$$g^{ij} \frac{\partial g_{\mu\nu,i}}{\partial t} = g^{ij} \frac{\partial K_{\mu\nu}}{\partial x^j}$$

$$-g^{00} \frac{\partial K_{\mu\nu}}{\partial t} = 2g^{0j} \frac{\partial K_{\mu\nu}}{\partial x^j} + g^{ij} \frac{\partial g_{\mu\nu,i}}{\partial x^j} - 2H_{\mu\nu}$$

which are 50 equations for the 50 unknowns.

$g_{\mu\nu}$, $g_{\mu\nu,i}$ $K_{\mu\nu}$

We express this system in the form

$$A^0(V) \frac{\partial V}{\partial t} = A^i(V) \frac{\partial V}{\partial x^i} + B(V)$$

with

$$A^0(g_{\mu\nu}, g_{\mu\nu,i}, x_{\mu\nu}) =$$

$$\begin{bmatrix} I^{10} & 0^{10} & 0^{10} & 0^{10} & 0^{10} \\ 0^{10} & g^{11} I^{10} & g^{12} I^{10} & g^{13} I^{10} & 0^{10} \\ 0^{10} & g^{12} I^{10} & g^{22} I^{10} & g^{23} I^{10} & 0^{10} \\ 0^{10} & g^{13} I^{10} & g^{23} I^{10} & g^{33} I^{10} & 0^{10} \\ 0^{10} & 0^{10} & 0^{10} & 0^{10} & -g^{00} I^{10} \end{bmatrix}$$

$$A^i\,(g_{\mu\nu},\ g_{\mu\nu,i}\ k_{\mu\nu}) =$$

$$\begin{bmatrix}
O^{10} & O^{10} & O^{10} & O^{10} & O^{10} \\
O^{10} & O^{10} & O^{10} & O^{10} & g^{j1}\ I^{10} \\
O^{10} & O^{10} & O^{10} & O^{10} & g^{j2}\ I^{10} \\
O^{10} & O^{10} & O^{10} & O^{10} & g^{j3}\ I^{10} \\
O^{10} & g^{1j}\ I^{10} & g^{2j}\ I^{10} & g^{3j}\ I^{10} & 2g^{j0}\ I^{10}
\end{bmatrix}$$

where O^{10} is the 10 by 10 zero matrix

and I^{10} is the 10 by 10 identity matrix

$$B(g_{\mu\nu},\ g_{\mu\nu,i}\ ,\ k_{\mu\nu}) = \begin{bmatrix} k_{\mu\nu} \\ O^{30} \\ -2\,H_{\mu\nu}(g_{\mu\nu},\ g_{\mu\nu,i}\ ,\ k_{\mu\nu}) \end{bmatrix}$$

and O^{30} is the 30 component

zero vector.

Since $g^{00} < 0$ ~~ϕ~~, g^{ij} ~~ϕ~~ is positive definite, we see A^0 is positive definite and A^0, A^j are symmetric. Thus the system is symmetric hyperbolic and the Fischer – Marsden existence theorem applies giving a unique solution $g_{\mu\nu}$ in H^s satisfying $R^{(h)}_{\mu\nu} = 0$. Furthermore from our Lemma if the initial data satisfies $\overset{\circ}{C}e^{\circ}_{\mu} = 0$, $\overset{\circ}{\Gamma}^{\mu} = 0$ then $\Gamma^{\mu} = 0$ and Case A is proven.

We refer the reader to the Fischer and Marsden paper for Case B.