From Boltzmann to Euler: Hilbert's 6^{th} Problem Revisited

M. Slemrod

KMRS

KAIST

OCT, 2012

1. BOLTZMANN EQUATION AND EXTENDED THERMODYNAMICS

(see C. M. Dafermos, Hyp. Cons. Laws. 3rd edition)

In contrast to continuum mechanics kinetic theories approach matter as an ensemble of interacting molecules. The state of the system is realized probabilisticly (which was revolutionary view of Maxwell for equilibrium configuration and Boltzmann for the full dynamics)

Of course this probabilistic approach was later continued by Schrodinger in his approach to quantum mechanics

In kinetic theory as noted above the state may be written as a molecular density function

$$f(\xi, x, t)$$

of the velocity $\xi \in \mathbb{R}^3$ of molecules occupying at position $x \in \mathbb{R}^3$ and at time t. Hence f is the probability of finding a molecule with velocity ξ at point x and time t.

In classical kinetic theory of monatomic gases, $f(\xi, x, t)$ satisfies the Boltzmann equation

$$\partial_t f + \xi \cdot \operatorname{grad}_x f = Q(f) \qquad (1.1)$$

where Q stands for a complicated collision integral operator.

Notice the Boltzmann equation when $Q \equiv 0$ just gives the free transport of particles moving with velocity ξ, whereas if if f is independent of x the equation is just a space homogeneous equation

$$\partial_t f = Q(f) \qquad (1.2)$$

only accounting for collisions but not transport.

Some useful results for operator Q are as follows.

<u>Collision invariants</u>

$$\int_{\mathbb{R}^3} Q(f)\,\varphi(\xi)\,d\xi = 0 \qquad (1.3)$$

if and only if a linear combination of

$$\begin{cases} \psi_0 = 1, \\ (\psi_1, \psi_2, \psi_3) = \xi, \\ \psi_4 = |\xi|^2. \end{cases}$$

i.e.

$$\int Q(f)\,(a + \underline{b}\cdot\xi + c|\xi|^2)\,d\xi = 0 \qquad (1.4)$$

for all $a \in \mathbb{R}$, $\underline{b} \in \mathbb{R}^3$, $c \in \mathbb{R}$ constants.

Solution of the equation $Q(f) = 0$

$Q(f) = 0 \iff$

$$f(\xi) = \exp(a + \underline{b}\cdot\xi + c|\xi|^2).$$

Set $c = -\alpha$, $\alpha > 0$ (to force f to be in $L^1(\mathbb{R}^3)$ as a function of ξ),

$\underline{b} = 2\alpha\underline{v}$. Then

$$a + \underline{b}\cdot\underline{\xi} + c|\xi|^2 = a + 2\alpha\underline{v}\cdot\underline{\xi} - \alpha|\xi|^2$$

$$= -\alpha(\underline{\xi} - \underline{v})\cdot(\underline{\xi} - \underline{v}) + \alpha\underline{v}\cdot\underline{v} + a$$

and

$$f(\underline{\xi}) = \exp(a + \underline{b}\cdot\underline{\xi} + c|\underline{\xi}|^2)$$
$$= A\exp(-\alpha(\underline{\xi}-\underline{v})\cdot(\underline{\xi}-\underline{v})), \text{ i.e}$$

$$f(\underline{\xi}) = A\exp(-\alpha|\underline{\xi}-\underline{v}|^2) \qquad (1.5)$$

and A, α, $\underline{v}$ denote a new set of five arbitrary constants.

Hence the solution (1.5) of $Q(f)=0$ is the classical <u>Maxwellian distribution</u>.

Moments of Boltzmann equation

The formal connection between the continuum and kinetic theories of gas dynamics can be seen by considering the family of moments of f:

$$F_{i_1 i_2 \cdots i_N} = \int_{\mathbb{R}^3} \xi_{i_1} \cdots \xi_{i_N} \, f \, d\xi , \qquad (1.6)$$

$$i_1 = 1,2,3, \quad i_2 = 1,2,3, \quad \ldots \quad i_N = 1,2,3.$$

To derive the dynamics of moments we multiply the Boltzmann equation by $1, \underbrace{\xi_1, \xi_2, \xi_3}_{\xi_i}, \underbrace{\xi_1\xi_1, \xi_1\xi_2, \ldots}_{\xi_i \xi_j},$

etc.

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi_i \\ \xi_i \xi_j \\ \xi_i \xi_j \xi_k \\ \vdots \\ \xi_{i_1} \xi_{i_2} \cdots \xi_{i_N} \end{pmatrix} \left(\partial_t f + \xi \cdot \mathrm{grad}_x f - Q(f) \right) d\xi = 0$$

Let us compute the first few moments explicitly.

<u>Zeroth moment</u>:

$$\int_{\mathbb{R}^3} \partial_t f \, d\xi + \int_{\mathbb{R}^3} \xi \cdot \mathrm{grad}_x f \, d\xi = 0$$

since (1.3) implies $\int_{\mathbb{R}^3} Q(f) \, d\xi = 0$.

But

$$\xi \cdot \mathrm{grad}_x f = \xi_i \partial_{x_i} f = \partial_{x_i} (\xi_i f)$$

where $\xi_i \partial_{x_i} = \sum_{i=1}^{3} \xi_i \partial_{x_i}$,

i.e. repeated indices mean summation.

Hence

$$\partial_t \int f \, d\xi + \partial_{x_i} \int_{\mathbb{R}^3} (\xi_i f) \, d\xi = 0$$

or in terms of (1.6)

$$\boxed{\partial_t F + \partial_i F_i = 0}$$

<u>First moments</u>:

$$\int_{\mathbb{R}^3} \xi_i \partial_t f \, d\xi + \int_{\mathbb{R}^3} \xi_i \, \xi \cdot \mathrm{grad}_x f \, d\xi = 0$$

where again we used (1.6). Write

$$\xi_i \, \xi \cdot \mathrm{grad}_x f = \xi_i \xi_j \partial_{x_j} f$$
$$= \partial_{x_j} (\xi_i \xi_j f)$$

and we see

$$\boxed{\partial_t F_i + \partial_j F_{ij} = 0}$$

Second moments

Following the same procedure we see the second moments satisfy

$$\boxed{\partial_t F_{ij} + \partial_{x_k} F_{ijk} = P_{ij}}$$

where

$$P_{ij} = \int_{\mathbb{R}^3} \xi_i \xi_j \, Q(f) \, d\xi .$$

Notice unlike the zeroth and first moments P_{ij} is not identically zero but via (1.6) the trace $P = P_{ii}$ is zero. Thus we do have the relation

$$\boxed{\partial_t F_{ii} + \partial_{x_k} F_{iik} = 0}$$

If we continue taking moments we obtain the infinite system of evolution equations

$$\partial_t F + \partial_i F_i = 0 \qquad i = 1,2,3$$

$$\partial_t F_i + \partial_j F_{ij} = 0 \qquad i,j = 1,2,3$$

$$\partial_t F_{ij} + \partial_{x_k} \cdot F_{ijk} = I_{ij} \qquad i,j,k = 1,2,3$$

$$\left(\partial_t F_{ii} + \partial_{x_k} F_{iik} = 0 \right)$$

$$\partial_t F_{ijk} + \partial_\ell F_{ijk\ell} = I_{ijk}$$

$$\vdots$$

$$\partial_t F_{i_1 \cdots i_N} + \partial_m F_{i_1 \cdots i_N, m} = I_{i_1 \cdots i_N}$$

$$i_1, \ldots, i_N = 1,2,3$$

(1.7)

Since f represents the (probabilistically) the density of molecules moving with velocity ξ at point x at time t, the integral

$$F(x,t) = \int_{\mathbb{R}^3} f(\xi, x, t)\, d\xi = \rho(x,t)$$

would be the density of molecules at point x and time t.

Similarly we write

$$F_i = \rho v_i \quad , \quad i = 1,2,3 \quad , \tag{1.8}$$

$$F_{ij} = \rho v_i v_j - T_{ij} \quad i,j = 1,2,3 \quad , \tag{1.9}$$

$$\frac{1}{2} F_{ii} = \rho \varepsilon + \frac{1}{2} \rho |v|^2 \quad , \tag{1.10}$$

$$\frac{1}{2} F_{iik} = (\rho \varepsilon + \frac{1}{2} \rho |v|^2) v_k - T_{ki} v_i + q_k , \quad k = 1,2,3 \quad . \tag{1.11}$$

Notice we have the identifications

$$\rho = F$$

$$v_i = \frac{F_i}{\rho},$$

$$T_{ij} = -F_{ij} + \rho v_i v_j \qquad (1.12)$$

$$\varepsilon = \frac{1}{2}\frac{F_{ii}}{\rho} - \frac{1}{2}|v|^2,$$

$$q_k = \frac{1}{2}F_{iik} - (\rho\varepsilon + \frac{1}{2}\rho|v|^2)v_k + T_{ki}v_i$$

and each macroscopic quantity:

$(\rho, v_i, T_{ij}, \varepsilon, q_k)$ representing

(density, velocity, Cauchy stress, internal energy, heat flux) can be expressed in terms of the moments of F.

Let us rewrite the Cauchy stress as a superposition of two parts:

$$T_{ij} = -p\,\delta_{ij} + \sigma_{ij} \qquad (1.13)$$

where $\text{trace}\,\sigma = \sigma_{ii} = 0$.

Then we see

$$T_{ii} = -p\,\delta_{ii} = -3p.$$

But from (1.12)

$$\begin{cases} T_{ii} = -F_{ii} + \rho|v|^2 \\ \varepsilon = \frac{1}{2}\frac{F_{ii}}{\rho} - \frac{1}{2}|v|^2 \end{cases}$$

So $2\rho\varepsilon = F_{ii} - \rho|v|^2$

and

$$T_{ii} = -2\rho\varepsilon - \rho|v|^2 + \rho|v|^2, \text{ i.e.}$$

$$T_{ii} = -2\rho\varepsilon \qquad (1.14)$$

Substitute (1.14) into (1.13):

$$T_{ii} = -p\,\delta_{ii} + \sigma_{ii} = -3p \implies$$

$$-2\rho\varepsilon = -3p$$

$$\boxed{\frac{3}{2}p = \rho\varepsilon} \qquad (1.15)$$

Of course (1.15) is the form of familiar constitutive equation of a polytropic gas

$$\varepsilon = c\rho^{\gamma-1}e^{s/c}, \quad p = R\rho^{\gamma}e^{s/c}, \quad \theta = \rho^{\gamma-1}e^{\frac{s}{c}}$$

s specific entropy, θ temperature

$$e^{S/c} = \frac{\varepsilon}{c} \rho^{1-\gamma}$$

$$p = R\rho^{\gamma} \frac{\varepsilon}{c} \rho^{1-\gamma}$$

$$p = \frac{R \varepsilon \rho}{c}.$$

$$\frac{R}{c} = \frac{2}{3} \quad (\text{where } \gamma = 5/3)$$

More familiar perhaps is writing the relations in terms of θ

$$\varepsilon = c\theta,$$

$$p = \frac{R\rho^{\gamma}\theta}{\rho^{\gamma-1}} = R\rho\theta,$$

i.e. internal energy is proportional to temperature, and pressure is proportional to density times temperature.

Truncation of the moment hierarchy (1.7) will result in a finite number of equations. But the resulting system will not be closed. This is because of two facts:

(i) there is always a flux of one higher moment appearing in each truncation and

(ii) the production term P_{ij}, $P_{ijk}, \ldots$ etc. (except for P_{ii}) are undetermined.

<u>Extended thermodynamics</u> is a theory that closes the system by postulating a closure of

(i) the higher moment flux

(ii) the undetermined production terms

in terms of the lower moments.

This program was initiated by H. Grad and developed in more recent years by I. Muller and his collaborators (see I. Muller, T. Ruggeri, Rational Extended Thermodynamics).

Let us consider the simplest case: truncation at 5 moments.

$$\partial_t F + \partial_i F_i = 0 \qquad (1.16)$$

$$\partial_t F_i + \partial_j F_{ij} = 0$$

$$\partial_t F_{ii} + \partial_k F_{iik} = 0$$

Here there is no need to postulate production terms since they have all vanished.

In terms of macroscopic dependent variables we write our system (1.16) as

$$
\begin{aligned}
&\partial_t \rho + \partial_i (\rho v_i) = 0 \\
&\partial_t (\rho v_i) + \partial_j (-T_{ij} + \rho v_i v_j) = 0 \qquad (1.17) \\
&\partial_t (2\rho \varepsilon + \rho |v|^2) \\
&\quad + \partial_k (2q_k - 2T_{ki} v_i + 2\rho \varepsilon + \rho |v|^2 v_k) = 0
\end{aligned}
$$

But recall

$$T_{ij} = -p\,\delta_{ij} + \sigma_{ij}, \quad \text{with}$$

$$\sigma_{ii} = 0,$$

and $\frac{3}{2} p = \rho \varepsilon$. Hence we can write (1.17) as

Oct 2, 2012

$$\boxed{\begin{aligned} &\partial_t \rho + \partial_i (\rho v_i) = 0 \\ &\partial_t (\rho v_i) + \partial_j (p\,\delta_{ij} + \rho v_i v_j) = \partial_j \sigma_{ij} \end{aligned}}$$

1 equ

3 equs

$$\begin{cases} \partial_t \left(\frac{1}{2}\rho |v|^2 + \rho\varepsilon\right) \\ + \partial_k \left(q_k - T_{ki} v_i + \rho\varepsilon + \frac{1}{2}\rho |v|^2 v_k \right) = 0 \end{cases}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} &\partial_t \left(\frac{1}{2}\rho |v|^2 + \rho\varepsilon \right) \\ &+ \partial_k \left(-(p\,\delta_{ij}) + \rho\varepsilon + \frac{1}{2}\rho |v|^2 v_k \right) \\ &\qquad = -\partial_k q_k + \partial_k \sigma_{ki} v_i \end{aligned}}$$

1 equ

with $\boxed{\frac{3}{2} p = \rho\varepsilon}$

5 equs in Total

Conservation of
Mass, Momentum, Energy.
1 + 3 + 1 = 5

Unknowns

$\rho, \quad v_i, \quad \varepsilon \ (\text{or } p)$

plus

$q_k \qquad k = 1, 2, 3 \qquad$ (3 unknowns)

$$\sigma_{ij} \qquad \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \qquad \left(\begin{matrix} 5 \\ \text{unknowns} \end{matrix} \right)$$

$$\sigma_{11} + \sigma_{22} + \sigma_{33} = 0$$

The <u>simplest</u> <u>closure</u> is to take these left over unknowns to be <u>identically</u> <u>zero</u>.

This yields the <u>Euler equations</u> of compressible gas dynamics.

The next level of closure is to use the first three equations of (1.7)

$$\partial_t F + \partial_i F_i = 0$$ 1 equ

$$\partial_t F_i + \partial_j F_{ij} = 0$$ 3 equs

$$\partial_t F_{ij} + \partial_k F_{ijk} = I_{ij}$$ 6 equs

and extract 3 more equations from

$$\partial_t F_{ijk} + \partial_l F_{ijkl} = I_{ijk}$$

by contracting on two of the indices

e.g $i = j$. 3 equs

$1 + 3 + 6 + 3 = 13$ moment equs

in

$$\underbrace{F}_{1}, \underbrace{F_i}_{3}, \underbrace{F_{ij}}_{6}, \underbrace{F_{iik}}_{3} = 13 \text{ unknowns.}$$

Again as before our system will <u>not</u> be closed. Hence we must impose a <u>closure rule</u>. The rule suggested by Muller & Ruggeri is any closure must produce an "entropy" inequality

$$\partial_t \varphi + \partial_i \psi_i \leq 0$$

(φ convex in the ~~fluid~~ extended variables).

Of course this motivated by the second law of thermodynamics when $-\varphi$ would be the physical entropy.

The advantage of this approach is the if φ is convex then the underlying system (by a result of Friedrichs and independently Godunov) can be made symmetric hyperbolic

H Grad (1949) suggested a different closure rule: Just assume f is a linear function of number of moments of the truncated system.

Then plug this f into (1.6) to find the coefficients in the linear combination..

Grad's approach yields an "entropy" inequality only near equilibrium and hence hyperbolicity may be lost. In fact hyperbolicity has been shown to fail away from equilibrium.

Of course implimentation of either closure rule is an unpleasant computation.

Grad's ~~closure~~ ~~for~~ 13 moments equations are in his original paper (GRAD, 1949) and the

Muller – Ruggeri 13 moments are in

C. M. Dafermos, 3rd edition, Hyperbolic Cons. Laws,

p. 67–68) as well as the book of

Muller & Ruggeri, Rational Extended Thermodynamics,

2nd edition). We

will examine this 13 moment

system in § 2.

2. The Chapman-Enskog expansion

Recall the Boltzmann equation has the form

$$\partial_t f + \xi \cdot \mathrm{grad}_x f = Q(f).$$

If we want view the dynamics on a *macroscopic* scale it is convenient to rescale space and time, i.e set

$$x = \frac{x'}{\varepsilon}, \quad t = \frac{t'}{\varepsilon}$$

so that the Boltzmann equation becomes

$$\partial_{t'} f + \xi \cdot \mathrm{grad}_{x'} f = \frac{Q(f)}{\varepsilon}.$$

The parameter ε is *Knudsen number* and measures the mean free path between collisions. For convenience we next drop " ' " in our equation:

$$\partial_t f + \xi \cdot \operatorname{grad}_x f = \frac{Q(f)}{\varepsilon}. \qquad (2.1)$$

D. Hilbert proposed solving (2.1) as a formal ~~power~~ asymptotic expansion

$$f(\xi, x, t) = f_0(\xi, x, t) + \varepsilon f_1(\xi, x, t) + \varepsilon^2 f_2(\xi, x, t) + \cdots \qquad (2.2)$$

which we call the <u>Hilbert</u> <u>expansion</u>.

Obviously balancing orders of ε yields

$$Q(f_0) = 0$$

and we know from § 1 that

$$f_0(\xi) = A \exp(-\alpha|\xi - \underline{v}|^2)$$

but since f_0 may depend on x, t this means $A, \alpha, \underline{v}$ can depend on x, t as well!

$$f_0(\xi, x, t) = A(x,t) \exp(-\alpha(x,t)|\xi - \underline{v}(x,t)|^2).$$

Recall that the macroscopic density, velocity, and temperature can be related to moments of f:

$$\rho(x,t) = \int f \, d\xi$$

$$\rho v_i = \int \xi_i f \, d\xi$$

$$\rho \mathcal{E} = \frac{3}{2} \rho R T = \int |\xi - v|^2 f \, d\xi$$

so at order 1 we can solve for A, α, v in terms of ρ, v_i, θ:

$$\alpha = 3(4e)^{-1} = (2R\theta)^{-1}$$

$$A = \rho \left(\frac{4}{3}\pi e\right)^{-3/2} = (2\pi R\theta)^{-3/2}$$

to see

$$f_0(\xi, x, t) = (2\pi R\theta)^{-3/2} \exp\left((2R\theta)^{-1} |\xi - v|^2\right) \qquad (2.2)$$

and the Maxwellian distribution has been written as a function ξ & x, t where dependence on x, t appears only through the fluid variables ρ, v, θ

Next go back to our five moment truncation (1.16):

$$\partial_t F + \partial_i F_i = 0$$

$$\partial_t F_i + \partial_j F_{ij} = 0 \qquad (1.16)$$

$$\partial_t F_{ii} + \partial_k F_{iik} = 0$$

From the Hilbert expansion we know f to leading order in ε, i.e. f_0 given by (2.2). Hence plugging this <u>Maxwellian distribution</u> allows us to compute F, F_i, F_{ij}, F_{iik} to leading order.

This gives 5 equations in 5 unknowns ρ, v_i, T, i.e. the Euler equations of an ideal gas

$$\left.\boxed{\begin{aligned} &\partial_t \rho + \partial_i (\rho v_i) = 0 \\ &\partial_t (\rho v_i) + \partial_j (p \delta_{ij} + \rho v_i v_j) = 0 \\ &\partial_t \left(\tfrac{1}{2} \rho |v|^2 + \rho \varepsilon \right) + \partial_k \left(- (p \delta_{ij}) + \rho \varepsilon + \tfrac{1}{2} \rho |v|^2 v_k \right) = 0 \end{aligned}}\right| \quad (2\text{-}3)$$

$$\frac{3}{2} p = \rho \varepsilon \quad , \quad \varepsilon = \frac{3}{2} R \theta .$$

So <u>formally</u> if we write (as Hilbert did)

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$$

and let $\varepsilon \to 0$, then $f \to f_0$ and we have passed from the Boltzmann equation to its <u>hydrodynamic</u> <u>limit</u> (2-3), i.e the compressible Euler equations.

This was certainly a strong motivation for Hilbert to propose that this formal derivation he made rigorous in his 1900 address to the International Congress of Mathematicians in Paris.

Closely related to the Hilbert expansion is the Chapman-Enskog expansion which is just the Hilbert expansion written so that dependence (for f) on x, t appears only through the macroscopic fluid variables $\mathcal{M}$

$$\mathcal{M} \equiv (\rho, v_i, \theta)$$

and space derivatives of $\mathcal{M}$..

Of course at order 1 there is no difference between the Hilbert and Chapman-Enskog (C-E) expansions. At order ε, C-E for the Boltzmann equation, will deliver the compressible Navier-Stokes-Fourier (N-S-F) equations for gas dynamics. This was a major breakthrough for the kinetic theory since it yielded what is believed to be the $O(\varepsilon)$ correction to the Euler equations.

The coefficients for the viscosity and heat conduction terms are consistent with experiment. But logically if the $O(\varepsilon)$ term improves Euler, then $O(\varepsilon^2)$ should improve (N-S-F). But this turns out to not to be case. The next level $O(\varepsilon^2)$ (Burnett order) yields a system of equations which is unstable to high frequency perturbations of the rest state.

As a way to both understanding the C-E expansion and the issues of instability let us choose a very simple but useful example: Grad's 13 moment equations.

Again the system has used the scaling

$$x = \frac{x'}{\varepsilon}, \quad t = t'/_{\varepsilon}$$

and dropped the primes for convenience.

32

$$\tfrac{1}{2}F_{iik} = \left(\rho\varepsilon + \tfrac{1}{2}\rho|v|^2\right)v_k - T_{ki}v_i + q_k, \qquad k = 1,2,3, \tag{3.3.58}$$

the first equation in (3.3.53) renders conservation of mass, the second equation renders conservation of linear momentum, and one half the trace of the third equation renders conservation of energy, for a heat-conducting viscous gas with density ρ, velocity v, internal energy ε, Cauchy stress T and heat flux q. We regard T as the superposition, $T = -pI + \sigma$, of a pressure $p = -\frac{1}{3}T_{ii}$ and a shearing stress σ that is traceless, $\sigma_{ii} = 0$. By virtue of (3.3.56) and (3.3.57),

$$\rho\varepsilon = \tfrac{3}{2}p, \tag{3.3.59}$$

which is compatible with the constitutive equations (2.5.20) of the polytropic gas, for $\gamma = 5/3$.

Motivated by the above observations, one may construct a full hierarchy of continuum theories by truncating the infinite system (3.3.53), retaining only a finite number of equations. The resulting systems, however, will not be closed, because the highest order moments, appearing as flux(es) in the last equation(s), and also the production terms on the right-hand side remain undetermined. In the spirit of continuum physics, *extended thermodynamics* closes these systems by postulating that the highest order moments and the production terms are related to the lower order moments by constitutive equations that are determined by requiring that all smooth solutions of the system satisfy identically a certain inequality, akin to the Clausius-Duhem inequality. This induces a companion balance law which renders the system symmetrizable and thereby hyperbolic. The principle of material frame indifference should also be observed by the constitutive relations.

To see how the program works in practice, let us construct a truncation of (3.3.53) with state vector $U = (\rho, v, p, \sigma, q)$, which has dimension 13, as σ is symmetric and traceless. For that purpose, we retain the first three of the equations of (3.3.53), for a total of 10 independent scalar equations, and also extract 3 equations from the fourth equation of (3.3.53) by contracting two of the indices. By virtue of (3.3.54), (3.3.55), (3.3.56), (3.3.57), (3.3.58) and since P, P_i and P_{ii} vanish, we end up with the system

$$\begin{cases} \partial_t\rho + \partial_j(\rho v_j) = 0 \\ \partial_t(\rho v_i) + \partial_j(\rho v_i v_j + p\delta_{ij} - \sigma_{ij}) = 0 \\ \partial_t\left(\rho\varepsilon + \frac{1}{2}\rho|v|^2\right) + \partial_k\left\{\left(\rho\varepsilon + \frac{1}{2}\rho|v|^2 + p\right)v_k - \sigma_{kj}v_j + q_k\right\} = 0 \\ \partial_t\left(\rho v_i v_j - \frac{1}{3}\rho|v|^2\delta_{ij} - \sigma_{ij}\right) + \partial_k\left(F_{ijk} - \frac{1}{3}F_{\ell\ell k}\delta_{ij}\right) = P_{ij}/\varepsilon \\ \partial_t\left\{\left(\rho\varepsilon + \frac{1}{2}\rho|v|^2 + p\right)v_k - \sigma_{kj}v_j + q_k\right\} + \frac{1}{2}\partial_i F_{jjik} = \frac{1}{2}P_{iik}/\varepsilon \end{cases} \tag{3.3.60}$$

This system can be closed by postulating that F_{ijk}, F_{jjik}, P_{ij} and P_{iik} are functions of the state vector $U = (\rho, v, p, \sigma, q)$, which are determined by requiring that all smooth solutions satisfy identically an inequality

$$\partial_t\varphi + \partial_i\psi_i \le 0, \tag{3.3.61}$$

where φ and ψ_i are (unspecified) functions of U, and $\varphi(U)$ is convex. After a lengthy calculation (see the references cited in Section 3.4), one obtains complicated albeit quite explicit constitutive relations:

$$F_{ijk} = \rho v_i v_j v_k + \left(p v_k + \tfrac{2}{5} q_k\right)\delta_{ij} + \left(p v_i + \tfrac{2}{5} q_i\right)\delta_{jk} + \left(p v_j + \tfrac{2}{5} q_j\right)\delta_{ik}\,, \tag{3.3.62}$$

$$\begin{aligned} F_{jjik} = {} & \left(\rho|v|^2 + 7p\right) v_i v_k + \left(p\delta_{ik} - \sigma_{ik}\right)|v|^2 - \sigma_{ij} v_j v_k \\ & - \sigma_{kj} v_j v_i + \frac{14}{5}\left(q_i v_k + q_k v_i\right) + \frac{4}{5} q_j v_j \delta_{ik} + \frac{p}{\rho}\left(5p\delta_{ik} - 7\sigma_{ik}\right), \end{aligned} \tag{3.3.63}$$

$$P_{ij} = \tau_0 \sigma_{ij}\,, \qquad P_{iik} = 2\tau_0 \sigma_{ki} v_i - \tau_1 q_k\,. \tag{3.3.64}$$

To complete the picture, p, τ_0 and τ_1 must be specified as functions of (ρ, θ).

The special vector $U^* = B(U)$, in the notation of Section 1.5, that symmetrizes the system has components

$$U^* = \frac{1}{\theta}\begin{pmatrix} \dfrac{5p}{2\rho} - \theta s - \dfrac{1}{2}|v|^2 + \dfrac{1}{2p}\sigma_{ij} v_i v_j - \dfrac{\rho}{5p^2} q_i v_i |v|^2 \\[2ex] v_i - \dfrac{1}{p}\sigma_{ij} v_j + \dfrac{\rho}{5p^2}\left(|v|^2 q_i + 2 q_j v_j v_i\right) \\[2ex] -1 + \dfrac{2\rho}{3p^2} q_k v_k \\[2ex] -\dfrac{1}{2p}\sigma_{ij} - \dfrac{\rho}{5p^2}\left(v_i q_j + v_j q_i - \tfrac{2}{3} v_k q_k \delta_{ij}\right) \\[2ex] \dfrac{\rho}{5p^2} q_i \end{pmatrix} \tag{3.3.65}$$

In particular, as explained in Section 1.5, truncating the system (3.3.60) by dropping the last two equations should be paired with "freezing" the last two components of U^*, i.e., by setting $q = 0$ and $\sigma = 0$. In that case, the system of the first three equations of (3.3.60) reduces to the system (3.3.29), in the particular situation where $b = 0$, $r = 0$ and $\rho\varepsilon$ and p are related by (3.3.59). If one interprets (ρ, v, p) as the basic state variables and (σ, q) as internal state variables, as explained in Section 2.8, then (3.3.29) becomes the relaxed form of the system (3.3.60).

3.3.8 Nonlinear Electrodynamics:

Another rich source of interesting systems of hyperbolic balance laws is electromagnetism. The underlying system consists of Maxwell's equations

$$\begin{cases} \partial_t B = -\operatorname{curl} E \\ \partial_t D = \operatorname{curl} H - J \end{cases} \tag{3.3.66}$$

We now linearize the 13 moment system about $\underline{v}=0$, $\theta=\theta_0$, $\rho=\rho_0$, i.e.

write

$$\underline{v} = 0 + \bar{v}, \quad \theta = \theta_0 + \bar{\theta}, \quad \rho = \rho_0 + \bar{\rho},$$

substitute into the system, and retain only linear terms. This yields

$$\partial_t \rho + \partial_j (\rho_0 v_j) = 0$$

$$\partial_t (\rho_0 v_i) + \partial_j (p \delta_{ij} - \sigma_{ij}) = 0$$

$$\partial_t \left(\tfrac{3}{2} p\right) + \cancel{\ldots}$$

$$+ \partial_k \left(\tfrac{5}{2} p_0 v_k \; \cancel{\ldots} + q_k \right) = 0$$

$$\partial_t (- \sigma_{ij}) + \partial_k \Big((p_0 v_k + \tfrac{2}{5} q_k) \delta_{ij}$$

$$+ (p_0 v_i + \tfrac{2}{5} q_i) \delta_{jk} + (p_0 v_j + \tfrac{2}{5} q_j) \delta_{ik}$$

$$- \tfrac{5}{3} (p_0 v_k + \tfrac{2}{5} q_k) \delta_{ij} \Big) \cancel{\ldots} = \frac{\tau_0 \sigma_{ij}}{\varepsilon}$$

$\cancel{\partial_t \ldots}$

$\cancel{\partial_t \left(\tfrac{5}{2} p_0 v_k + q_k\right)}$.

$$\partial_t\left(\frac{5}{2}p_0 v_k + q_k\right) + \frac{1}{2}\partial_i\left(2\frac{p_0 p}{\rho_0}5\delta_{ik} - \frac{p_0^2}{\rho_0^2}5\rho\,\delta_{ik}\right.$$

$$\left. - 7\sigma_{ik}\right) = -\frac{\tau_1}{2\varepsilon}q_k$$

We can rewrite the ~~last two equations as~~ next to last equation as

$$-\partial_t\sigma_{ij} + \partial_k\Big(-\frac{2}{3}\big(p_0 v_k + \frac{2}{5}q_k\big)\delta_{ij}$$

$$+ \big(p_0 v_i + \frac{2}{5}q_i\big)\delta_{jk} + \big(p_0 v_j + \frac{2}{5}q_j\big)\delta_{ik}\Big) = \frac{\tau_0}{\varepsilon}\sigma_{ij}.$$

Let us choose the linearization of

$p = R\rho\theta$ i.e

$p = R(\rho_0\theta + \rho\theta_0)$, $\quad p_0 = R\rho_0\theta_0$.

~~with $R\rho_0 = 1$, $R\theta_0 = 1$ i.e. $p = \theta + \rho$.~~

(~~This can be done by a rescaling~~.)

Let us set for convenience $R=1$, $\rho_0 = 1$, $\theta_0 = 1$,

so that $\quad p = \rho + \theta$, $p_0 = 1$

In one space dimension our equations are

$$\partial_t \rho + \partial_x v = 0$$

$$\partial_t v + \partial_x p = \partial_x \sigma$$

$$\frac{3}{2} \partial_t p + \partial_x \left(\frac{5}{2} v \right) = -\partial_x q$$

$$-\partial_t \sigma + \frac{4}{3} \partial_x \left(v + \frac{2}{5} q \right) = \frac{\tau_0}{\varepsilon} \sigma$$

$$\partial_t \left(\frac{5}{2} v + q \right) + \frac{1}{2} \partial_x \left(10p - 5\rho - 7\sigma \right) = -\frac{\tau_1}{2\varepsilon} q$$

Use the second equation to see

$$\frac{5}{2} \partial_t v + \frac{5}{2} \partial_x p = \frac{5}{2} \partial_x \sigma$$

and then the last equation becomes

$$\partial_t q + \partial_x \left(\frac{5}{2} p - \frac{5}{2} \rho - \frac{7}{2} \sigma + \frac{5}{2} \sigma \right) = -\frac{\tau_1}{2\varepsilon} q$$

or

$$\partial_t q + \partial_x \left(\frac{5}{2} (p - \rho) - \sigma \right) = -\frac{\tau_1}{2\varepsilon} q$$

In summary the linearized one dimensional system is

$$\partial_t \rho + \partial_x v = 0$$

$$\left[\begin{array}{l} \partial_t v + \partial_x p = \partial_x \sigma \\ \\ \partial_t p + \frac{5}{3}\partial_x v = -\frac{3}{2} q_x \\ \\ \partial_t \sigma - \frac{4}{3}\partial_x\left(v + \frac{2}{5}q\right) = -\frac{\tau_0}{\varepsilon}\sigma \end{array}\right]$$

$$\partial_t q + \partial_x\left(\frac{5}{2}(p-\rho) - \sigma\right) = -\frac{\tau_1}{\tau\varepsilon} q$$

This <u>exactly</u> the same one dimensional system that we would obtain from Grad's closure rule.

To keep the computations simple let us consider an even simpler subset of these equations by formally setting the heat flux $q \equiv 0$

$$\partial_t v + \partial_x p = \partial_x \sigma \qquad (2.4)$$

$$\partial_t p + \frac{5}{3}\partial_x v = 0 \qquad (2.5)$$

$$\partial_t \sigma - \frac{4}{3}\partial_x v = -\frac{\sigma}{\varepsilon} \qquad (2.6)$$

Our goal is now to apply the Chapman-Enskog procedure to this simple set of three partial differential equations.

Write

$$\sigma_{CE} = \epsilon\sigma^{(0)} + \epsilon^2 \sigma^{(1)} + \epsilon^3 \sigma^{(2)} + \cdots \qquad (2.7)$$

where $\sigma^{(n)}$ depends on the current values of p, v and their <u>space</u> derivatives. Substitute (2.7) into (2.4) – (2.6) to see

$$\partial_t v + \partial_x p = \partial_x (\epsilon \sigma^{(0)} + \epsilon^2 \sigma^{(1)} + \epsilon^3 \sigma^{(2)} + \cdots) \qquad (2.4_\epsilon)$$

$$\partial_t p + \frac{5}{3} \partial_x v = 0 \qquad (2.5_\epsilon)$$

$$\partial_t (\epsilon \sigma^{(0)} + \epsilon^2 \sigma^{(1)} + \epsilon^3 \sigma^{(2)} + \cdots) - \frac{4}{3} \partial_x v = - (\sigma^{(0)} + \epsilon \sigma^{(1)} + \epsilon^2 \sigma^{(2)} + \cdots) \qquad (2.6_\epsilon)$$

$$\Rightarrow \boxed{\sigma^{(0)} = \frac{4}{3} \partial_x v} \qquad (2.8)$$

Insert (2.8) into (2.4_ϵ), (2.5_ϵ) to see

$$\partial_t v + \partial_x p = \partial_x\left(\frac{4}{3}\varepsilon\,\partial_x v + \varepsilon^2\sigma^{(1)} + \varepsilon^3\sigma^{(2)} + \cdots\right) \qquad (2.4_\varepsilon)$$

$$\partial_t\left(\frac{4}{3}\varepsilon\,\partial_x v + \varepsilon^2\sigma^{(1)} + \varepsilon^3\sigma^{(2)} + \cdots\right) = -\left(\varepsilon\sigma^{(1)} + \varepsilon^2\sigma^{(2)} + \cdots\right) \qquad (2.6_\varepsilon)$$

But by (2.4_ε)

$$\partial_t\partial_x v = \partial_x\partial_t v$$

$$= \partial_x\left\{-\partial_x p + \partial_x\left(\varepsilon\frac{4}{3}\partial_x v + \varepsilon^2\sigma^{(1)} + \cdots\right)\right\}$$

$$= -\partial_{xx}p + \partial_{xx}\left(\varepsilon\frac{4}{3}\partial_x v\right)$$

$$+ \partial_x^3\varepsilon^2\sigma^{(1)} + \cdots$$

So that (2.6_ε) becomes

$$\frac{4}{3}\varepsilon\left(-\partial_{xx}p + \frac{4}{3}\varepsilon\,\partial_{xxx}v + \varepsilon^2\partial_x^3\sigma^{(1)} + \cdots\right)$$

$$+\ \partial_t\left(\varepsilon^2\sigma^{(1)} + \varepsilon^3\sigma^{(2)} + \cdots\right)$$

$$= -\varepsilon\sigma^{(1)} - \varepsilon^2\sigma^{(2)} - \varepsilon^3\sigma^{(3)} + \cdots$$

Oct 3, 2012

41

Hence balance $O(\varepsilon)$:

$$\boxed{\sigma^{(1)} = \frac{4}{3}\partial_{xx} p} \qquad (30)$$

and balance $O(\varepsilon^2)$:

$$\frac{4}{3}\left(\frac{4}{3}\right)\partial_{xxx} V + \partial_t \sigma^{(1)} = -\sigma^{(2)} \qquad (31)$$

But

$$\partial_t \sigma^{(1)} = \frac{4}{3}\partial_{xx}\partial_t p \qquad \text{by } (30)$$

$$= \frac{4}{3}\partial_{xx}\left(-\frac{5}{3}\partial_x V\right)$$

$$\partial_t \sigma^{(1)} = -\frac{20}{3\cdot 3}\partial_{xxx} V$$

Hence by (31)

$$\frac{16}{3\cdot 3}\partial_{xxx} V - \frac{20}{3\cdot 3}\partial_{xxx} V = -\sigma^{(2)}$$

and

$$\boxed{\sigma^{(2)} = \frac{4}{3}\left(\frac{1}{3}\right)\partial_{xxx} V} \qquad (32)$$

Hence

$$\sigma_{CE} = \varepsilon \left(\frac{4}{3} \partial_x v + \varepsilon \frac{4}{3} \partial_{xx} p + \varepsilon^2 \frac{4}{3} \left(\frac{1}{3}\right) \partial_{xxx} v + \cdots \right)$$

$$\boxed{\sigma_{CE} = \frac{4}{3} \varepsilon \left(\partial_x v + \varepsilon \partial_{xx} p + \frac{\varepsilon^2}{3} \partial_{xxx} v + \cdots \right) \qquad (32)}$$

and our PDE are

$$\boxed{\begin{aligned} &\partial_t v + \partial_x p = \frac{4}{3} \varepsilon \partial_x \left(\partial_x v + \varepsilon \partial_{xx} p + \frac{\varepsilon^2}{3} \partial_{xxx} v + \cdots \right) \\ &\partial_t p + \frac{5}{3} \partial_x v = 0 \end{aligned}} \qquad (33)$$

TRUNCATION AT $\mathcal{O}(1)$:

$$\begin{cases} \partial_t v + \partial_x p = 0 \\ \partial_t p + \frac{5}{3}\partial_x v = 0 \end{cases}$$

"Euler"

TRUNCATION AT $\mathcal{O}(\varepsilon)$:

$$\begin{cases} \partial_t v + \partial_x p = \frac{4}{3}\varepsilon\, \partial_{xx} v \\ \partial_t p + \frac{5}{3}\partial_x v = 0 \end{cases}$$

"NAVIER-STOKES"

TRUNCATION AT $\mathcal{O}(\varepsilon^2)$:

$$\begin{cases} \partial_t v + \partial_x p = \frac{4}{3}\varepsilon\left(\partial_{xx} v + \varepsilon\, \partial_{xxx} p\right) \\ \partial_t p + \frac{5}{3}\partial_x v = 0 \end{cases}$$

"BURNETT"

TRUNCATION AT $\mathcal{O}(\varepsilon^3)$

$$\begin{cases} \partial_t v + \partial_x p = \frac{4}{3}\varepsilon\left(\partial_{xx} v + \varepsilon\, \partial_{xxx} p + \frac{\varepsilon^2}{3}\partial_{xxxx} v\right) \\ \partial_t p + \frac{5}{3}\partial_x v = 0 \end{cases}$$

"SUPER-BURNETT"

OCT 3, 2012

STABILITY OF REST STATE

PLUG IN $\begin{cases} v = e^{\omega t} e^{ikx} \underline{V} \\ p = e^{\omega t} e^{ikx} \underline{P} \end{cases}$

EULER

$$\begin{cases} \omega \underline{V} + ik\underline{P} = 0 \\ \omega \underline{P} + \frac{5}{3} ik \underline{V} = 0 \end{cases} \Rightarrow$$

$$\omega^2 \underline{V} + ik\omega \underline{P} = 0 \Rightarrow$$

$$\omega^2 \underline{V} + ik\left(-\frac{5}{3} ik\underline{V}\right) = 0$$

$$\left(\omega^2 + \frac{5}{3} k^2\right) \underline{V} = 0$$

$$\omega^2 = -\frac{5}{3} k^2$$

$$\boxed{\omega = \pm i \left(\frac{5}{3}\right)^{1/2} k}$$

NEUTRAL STABILITY AT "EULER" LEVEL

NAVIER - STOKES

$$\omega \underline{V} + ik\underline{P} = -\frac{4}{3} \varepsilon k^2 \underline{V}$$

$$\omega^2 \underline{V} + ik\omega \underline{P} = -\frac{4}{3} \varepsilon k^2 \omega \underline{V}$$

$$\omega \underline{P} + \frac{5}{3} ik \underline{V} = 0$$

$$\Rightarrow \omega^2 \underline{V} + ik\left(-\frac{5}{3} ik \underline{V}\right) = -\frac{4}{3} \varepsilon k^2 \omega \underline{V}$$

$$\left(\omega^2 + \frac{5}{3} k^2 + \frac{4}{3} \varepsilon k^2 \omega\right) \underline{V} = 0$$

QUADRATIC FORMULA $\Rightarrow$

$$\omega = \frac{-\frac{4}{3} \varepsilon k^2 \pm \left(\frac{16}{9} \varepsilon^2 k^4 - 4 \cdot \frac{5}{3} k^2\right)^{1/2}}{2}$$

$$\omega = -\frac{2}{3} \varepsilon k^2 \pm \left(\frac{4}{9} \varepsilon^2 k^4 - \frac{5}{3} k^2\right)^{1/2}$$

$$= -\frac{2}{3} \varepsilon k^2 \pm |k| \left(\frac{4}{9} \varepsilon^2 k^2 - \frac{5}{3}\right)^{1/2}$$

$$\begin{cases} \frac{4}{9}\varepsilon^2 k^2 - 5/3 < 0 & \text{2 complex roots} \\ & \text{Real parts} < 0 \\ \frac{4}{9}\varepsilon^2 k^2 - 5/3 \geq 0 & \text{2 real roots both } \underline{\text{negative}} \end{cases}$$

EXPONENTIAL ASYMPTOTIC STABILITY AT "NAVIER-STOKES" LEVEL

BURNETT

$$\omega V + ik P = -\frac{4}{3}\varepsilon k^2 V + \frac{4}{3}\varepsilon^2 i^3 k^3 P$$

$$\omega P \cancel{*} \frac{5}{3} ik V = 0$$

$$\omega^2 V + ik\omega P = -\frac{4}{3}\varepsilon k^2 \omega V - \frac{4}{3}\varepsilon^2 i k^3 \omega P$$

$$\omega^2 V + ik\left(-\frac{5}{3} ik V\right) = -\frac{4}{3}\varepsilon k^2 \omega V + \frac{4}{3}\cdot\frac{5}{3}\varepsilon^2 ik^3 \frac{5}{3} ik V$$

$$\omega^2 V + 5/3\, k^2 V + \frac{4}{3}\varepsilon k^2 \omega V + \frac{4}{3}\cdot\frac{5}{3}\varepsilon^2 k^4 V = 0$$

$$\left(\omega^2 + \frac{4}{3}\varepsilon k^2 \omega + \frac{5}{3}k^2 + \frac{4}{3}\frac{5}{3}\varepsilon^2 k^4\right) V = 0$$

$$\omega = -\frac{2}{3}k^2\varepsilon \pm \frac{1}{2}\left(\frac{16}{9}\varepsilon^2k^4 - 4\left(\frac{5}{3}k^2 + \frac{20}{9}\varepsilon^2k^4\right)\right)^{1/2}$$

$$\omega = -\frac{2}{3}k^2\varepsilon \pm \left(\frac{4}{9}\varepsilon^2k^4 - \left(\frac{5}{3}k^2 + \frac{20}{9}\varepsilon^2k^4\right)\right)^{1/2}$$

$$\omega = -\frac{2}{3}k^2\varepsilon \pm \left(-\frac{16}{9}\varepsilon^2k^4 - \frac{5}{3}k^2\right)^{1/2}$$

$$\omega = -\frac{2}{3}k^2\varepsilon \pm \frac{i|k|}{3}\left(+16\varepsilon^2 \cancel{k} k^2 + 15\right)^{1/2}$$

2 complex roots, Real parts < 0

EXPONENTIAL ASYMPTOTIC STABILITY AT

"BURNETT LEVEL"

SUPER - BURNETT LEVEL

Oct 3, 2012

$$\omega V + ikP = -\frac{4}{3}\varepsilon^2 k^2 V + \frac{4}{3}\varepsilon^2 i^3 k^3 P + \frac{4}{3}\frac{1}{3} i^4 k^4 V$$

$$\omega P + \frac{5}{3} ik V = 0$$

$$\omega^2 V + ikP\omega = -\frac{4}{3}\varepsilon^2 k^2 \omega V + \frac{4}{3}\varepsilon^2 i k^3 \omega P + \frac{4}{3}\frac{1}{3} k^4 \omega V$$

$$\omega = \frac{2}{9} k^2 (k^2 - 3) \pm \frac{i}{9} |k| \sqrt{4k^6 - 24k^4 - 72k^2 - 135}$$

$k^2 > 3 \quad \Rightarrow \quad \frac{2}{9} k^2 (k^2 - 3) > 0$

large k^2 2 complex roots

Real parts > 0

Exponentially unstable at

Super - Burnett order