Lectures $\left\{\begin{array}{l}\text{Oct 23, 2012}\\ \text{Oct 25, 2012}\end{array}\right\}$

Euler & Geometry

1. Dynamics

Recall in previous lectures on Boltzmann we derived the classical Euler equations of gas dynamics:

$$
\begin{cases}
\partial_t \rho + \partial_i(\rho v_i) = 0 & (1)\\
\partial_t(\rho v_i) + \partial_j(p\,\delta_{ij} + \rho v_i v_j) = \partial_j \sigma_{ij} & (2)\\
\partial_t\left(\frac{1}{2}\rho|v|^2 + \rho\varepsilon\right) + \partial_k\Big(-(p\delta_{ij}) + \rho\varepsilon & (3)\\
\qquad + \frac{1}{2}\rho|v|^2 v_k\Big) = - \partial_k q_k + \partial_k \sigma_{ki} v_i &
\end{cases}
$$

where

$$\frac{3}{2}p = \rho\varepsilon$$

$$\varepsilon = c\rho^{\gamma-1} e^{s/c}$$

$$p = R\rho^{\gamma} e^{s/\varepsilon}$$

$$\theta = \rho^{\gamma-1} e^{s/\varepsilon}$$

$$\partial_t \left(\tfrac{1}{2} \rho |v|^2 + \rho \Psi(\rho) \right) \qquad (7)$$

$$+ \partial_j \left(v_j \cdot \left(\tfrac{1}{2} \rho |v|^2 + \rho \Psi(\rho) \right) \right) \leq 0$$

(in the sense of distributions).

where

$\rho^2 \Psi'(\rho) = p(\rho)$.

Of course the construction of De Lellis & Szekelyhidi is based on finding very "wild" initial data (not smooth) that leads to this non-uniqueness.

So motivates

Q4 We know that (7) is insufficient for proving uniqueness of weak solutions. How should we replace (7)?

A.4. Open!

So (4),(5) is a system of four equations
in four unknowns $(\rho, \rho v_i)$, $i=1,2,3$,
with p given by $p = \frac{\rho^\gamma}{\gamma}$.

Q1 Does the initial value problem possess smooth solutions local in time?

A1 Yes. Use contraction maps or artificial viscosity. See for example the book of C. M. Dafermos for many references.

Q2 Does the initial value problem possess smooth solutions global in time for say small smooth data?

A2 ~~Fa~~ No. There is blow-up.
If data can have unbounded support we can just think about 1D.
If data can have compact support then harder and T. Sideris has ~~the~~ proven blow-up.

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Q3 Does the initial value problem have global weak solutions?

A3 Peter Lax has called the issue here the great "scandal of mathematics". After all the years of research on this problem we still don't know. In fact recent results of de Lellis & Szekelyhidi has there are possibly an infinite number of weak solutions to the initial value problem all satisfying an entropy inequality (see C. deLellis, L. Szekelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. Archive for Rational Mechanics and Analysis 195 (2010), 225-260.):

One simplifying assumption is to assume $S = \text{const}$ i.e isentropic gas dynamics. That can be done if q_k is chosen to exact balance the energy equation. We also choose $\sigma_{ij} \equiv 0$ which would mean no viscous terms. Then (1), (2) become

$$\begin{cases} \partial_t \rho + \partial_i (\rho v_i) = 0 & (4) \\ \partial_t (\rho v_i) + \partial_j (p\, \delta_{ij} + \rho v_i v_j) = 0 & (5) \end{cases}$$

with $p = R\rho^{\gamma}\,\text{const}$.

To keep things simple we can assume $\text{const} = \frac{1}{R\gamma}$ so that

$$p = \frac{\rho^{\gamma}}{\gamma}, \quad \gamma > 1 \qquad (6)$$

2. Steady flow

Steady flow is a simpler problem

but not easy.

The steady Euler equations are

$$\partial_i(\rho v_i) = 0 \tag{7}$$

$$\partial_j(p\,\delta_{ij} + \rho v_i v_j) = 0 \tag{8}$$

$$p = \frac{\rho^\gamma}{\gamma}$$

i.e. isentropic, steady flow of an ideal gas

Let us consider the 2-D case

where is addition the vorticity = curl $\underline{v}$

is identically zero:

$$\text{curl } \underline{v} = 0 \tag{9}$$

Write

$$\underline{V} = (u, v)$$

i.e $v_1 = u$, $v_2 = v$.

Then we have

$$\begin{cases} \partial_x (\rho u) + \partial_y (\rho v) = 0 & \text{(cons of mass)} \quad (10) \\ (i=1) \quad \partial_x (p\,\delta_{11} + \rho u^2) + \partial_y (\rho u v) = 0 & (11) \\ (i=2) \quad \partial_x (\rho u v) + \partial_y (p\,\delta_{22} + \rho v^2) = 0 & (12) \\ \text{(cons of linear momentum)} & \\ u_y - v_x = 0 & \text{(irrotational motion)} \quad (13) \end{cases}$$

where $p = \frac{\rho^\gamma}{\gamma}$.

Notice (10), (11), (12), (13) gives 4 equations in 3 unknowns ρ, u, v, hence they cannot be independent.

3. Bernoulli equation

Let us consider (11), (12), (13).

Write (11), (12) as

$$\begin{cases} p'(\rho)\rho_x + \underline{\underline{(\rho u)_x}}\, u + \rho u u_x + \underline{\underline{(\rho v)_y}}\, u + \rho v u_y = 0 \\ \underline{\underline{(\rho u)_x}}\, v + \rho u v_x + p'(\rho)\rho_y + \underline{\underline{(\rho v)_y}}\, v + \rho v v_y = 0 \end{cases}$$

Notice the $=$ terms cancel by (10).

Now divide by ρ to see

$$\begin{cases} \frac{p'(\rho)}{\rho}\rho_x + u u_x + v u_y = 0 \\ \frac{p'(\rho)}{\rho}\rho_y + v v_y + u v_x = 0 \end{cases}$$

and now use the irrotationality

$u_y - v_x = 0$.

This gives us with

$$p(\rho) = \frac{\rho^\gamma}{\gamma}, \quad p'(\rho) = \rho^{\gamma-1}, \quad \frac{p'(\rho)}{\rho} = \rho^{\gamma-2}$$

$$\frac{p'(\rho)}{\rho}\rho_x = \partial_x\left(\frac{\rho^{\gamma-1}}{\gamma-1}\right), \quad \frac{p'(\rho)}{\rho}\rho_y = \partial_y\left(\frac{\rho^{\gamma-1}}{\gamma-1}\right)$$

$$\partial_x\left(\frac{\rho^{\gamma-1}}{\gamma-1}+\frac{1}{2}q^2\right)=0,$$
$$\partial_y\left(\frac{\rho^{\gamma-1}}{\gamma-1}+\frac{1}{2}q^2\right)=0, \quad (14)$$

where $q^2 \equiv u^2+v^2$, q is the speed of the flow.

Hence from (14) we have the Bernoulli relation

$$\frac{\rho^{\gamma-1}}{\gamma-1}+\frac{1}{2}q^2 = \text{const}.$$

For convenience choose the const $= \frac{1}{\gamma-1}$ so that Bernoulli relation is just

$$\frac{\rho^{\gamma-1}}{\gamma-1}+\frac{1}{2}q^2 = \frac{1}{\gamma-1}. \quad (15)$$

That is we have ρ as a function of (u,v).

So we can choose two equations from (10), (11), (12), (13) to solve. Traditionally people chose (10), (13)

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0 & \text{(\underline{Cons of mass})} \quad (10) \\ u_y - v_x = 0 & \text{(\underline{irrotationality})} \quad (13) \end{cases}$$

subject to <u>Bernoulli</u>

$$\frac{\rho^{\gamma-1}}{\gamma-1} + \frac{1}{2} q^2 = \frac{1}{\gamma-1}. \quad (15)$$

why? The simplest reason is that if ρ were constant then (10), (13) become

$$u_x + v_y = 0 \qquad \text{(Cauchy-Riemann)}$$
$$u_y - v_x = 0$$

which are Cauchy-Riemann equations and are satisfied by any complex analytic function's real and imaginary parts.

Hence there was a great effort by analysts to extend the use of complex variable techniques to case ρ not constant. A survey is found in the book of L. Bers,

Subsonic & Transonic Flow, published by the American Math. Society.

What is the nature of the equations? This is the topic of next section.

4. Nature of the equations for 2-D, steady isentropic flow.

First let us solve the Bernoulli relation (15) explicitly:

$$\rho^{\gamma-1} + \frac{1}{2} q^2 (\gamma-1) = 1,$$

$$\rho^{\gamma-1} = 1 - \frac{1}{2} q^2 (\gamma-1),$$

$$\rho = \left(1 - \frac{q^2(\gamma-1)}{2}\right)^{\frac{1}{\gamma-1}}. \qquad (16)$$

Notice when

$$\frac{q^2(\gamma-1)}{2} = 1$$

we have $\rho = 0$, i.e cavitation and a vacuum would form. We set

$$q_{cav} = \sqrt{\frac{2}{\gamma-1}}.$$

Furthermore let us set the (sound speed)^2

$$c^2 \equiv p'(\rho).$$

We define the sound speed

$$c = \sqrt{p'(\rho)}$$

$$c^2 = p'(\rho)$$

and with $p = \frac{\rho^\gamma}{\gamma}$:

$$\boxed{c^2 = \rho^{\gamma-1}}$$

The ratio of speed q to sound speed c is defined as

$$\boxed{M := \frac{q}{c}}$$

and called the <u>Mach</u> number.

When $c^2 (= p'(\rho) = \rho^{\gamma-1})$

$$q^2 = c^2$$
$$= 1 - \left(\frac{\gamma-1}{2}\right) q^2$$

we have

$$q^2 \left(1 + \left(\frac{\gamma-1}{2}\right)\right) = 1$$

$$q^2 \left(\frac{1}{2} + \frac{\gamma}{2}\right) = 1$$ and we call this q_{cr}.

$$\boxed{q_{cr} = \sqrt{\frac{2}{1+\gamma}}}$$

When
$$\begin{cases} q < q_{cr} & \text{subsonic} \\ q = q_{cr} & \text{sonic} \\ q > q_{cr} & \text{supersonic} \end{cases}$$

Lf

~~It is convenience to introduce~~

~~polar coordinates for the dependent variable~~

~~$u = q\cos\theta, \quad v = q\sin\theta$~~

$$\Rightarrow (\rho u)_x + (\rho v)_y = 0 \quad \Rightarrow$$

$$\rho'(q) q_x u + \rho u_x + \rho'(q) q_y v + \rho v_y = 0$$

$$q^2 = (u^2+v^2)$$

$$q_x = \frac{1}{q}(u u_x + v v_x)$$

$$q_y = \frac{1}{q}(u u_y + v v_y)$$

$$\frac{\rho'(q)}{q}(u u_x + v v_x)u + \rho u_x$$

$$+ \frac{\rho'(q)}{q}(u u_y + v v_y)v + \rho v_y = 0$$

$$\frac{\rho^{\gamma-1}}{\gamma-1} - \frac{1}{2} q^2 = \frac{1}{\gamma-1}$$

$$\rho^{\gamma-1} = 1 - \frac{1}{2}(\gamma-1) q^2$$

$$\frac{d}{dq} \Rightarrow \quad (\gamma-1)\rho^{\gamma-2}\frac{d\rho}{dq} = -(\gamma-1) q$$

$$\rho^{\gamma-2}\frac{d\rho}{dq} = -q$$

$$\frac{d\rho}{dq} = -\frac{q}{\rho^{\gamma-2}}$$

$$-\frac{q}{\rho^{\gamma-2}} \frac{(u u_x + v v_x)u}{q} + \rho u_x$$

$$-\frac{q}{\rho^{\gamma-2}} \frac{(u u_y + v v_y)}{q} + \rho v_y = 0$$

$$-\cancel{\rho}(u u_x + v v_x)u + \rho^{\gamma-1} u_x$$

$$-\cancel{\rho}(u u_y + v v_y)v + \rho^{\gamma-1} v_y = 0$$

Since $c^2 = \rho^{\gamma-1}$ this means

$$-\cancel{\rho}(u u_x + v v_x)\, u + c^2 u_x$$

$$-\cancel{\rho}(u u_y + v v_y)\, v + c^2 v_y = 0$$

$$\boxed{(c^2 - u^2) u_x - 2uv(v_x + u_y) + (c^2 - v^2) v_y = 0}$$

(17)

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If we introduce stream function

φ

so that $u = \varphi_x, \; v = \varphi_y$

then $u_y = v_x$ automatically and

we have

$$\underbrace{(c^2-u^2)}_{A}\varphi_{xx} \; \underbrace{-2uv}_{B}(\varphi_{xy}) + \underbrace{(c^2-v^2)}_{C}\varphi_{yy} = 0$$

Discriminant

$$B^2 - 4AC \quad \begin{array}{l} > \text{ hyperbolic} \\ = \\ < \text{ elliptic} \end{array}$$

$$4u^2v^2 - 4(c^2-v^2)(c^2-u^2) =$$

$$4u^2v^2 - 4(c^4 - c^2(u^2+v^2) + u^2v^2) =$$

$$-4c^2(c^2 - (u^2+v^2)) =$$

$$-4c^2(c^2-q^2)$$

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$$\begin{cases} -4c^2(c^2-q^2) > 0 & \text{hyperbolic} \\ -4c^2(c^2-q^2) < 0 & \text{elliptic} \end{cases}$$

$$\rightarrow \begin{cases} c^2-q^2 < 0 & \text{hyperbolic} \\ c^2-q^2 > 0 & \text{elliptic} \end{cases}$$

$$\cancel{\frac{q^2}{c^2}} \quad \begin{cases} c^2 < q^2 & \text{hyperbolic} \\ c^2 > q^2 & \text{elliptic} \end{cases}$$

$$\boxed{\begin{array}{ll} 1 < \left(\frac{q}{c}\right)^2 = M^2 & \underline{\text{hyperbolic}} \\ 1 > \left(\frac{q}{c}\right)^2 = M^2 & \underline{\text{elliptic}} \end{array}} \quad (18)$$

$$\boxed{\begin{cases} \tilde{q} > q_{cr} & \text{supersonic} \quad \text{hyperbolic} \\ q < q_{cr} & \text{subsonic} \quad \text{elliptic} \end{cases}}$$

(19)

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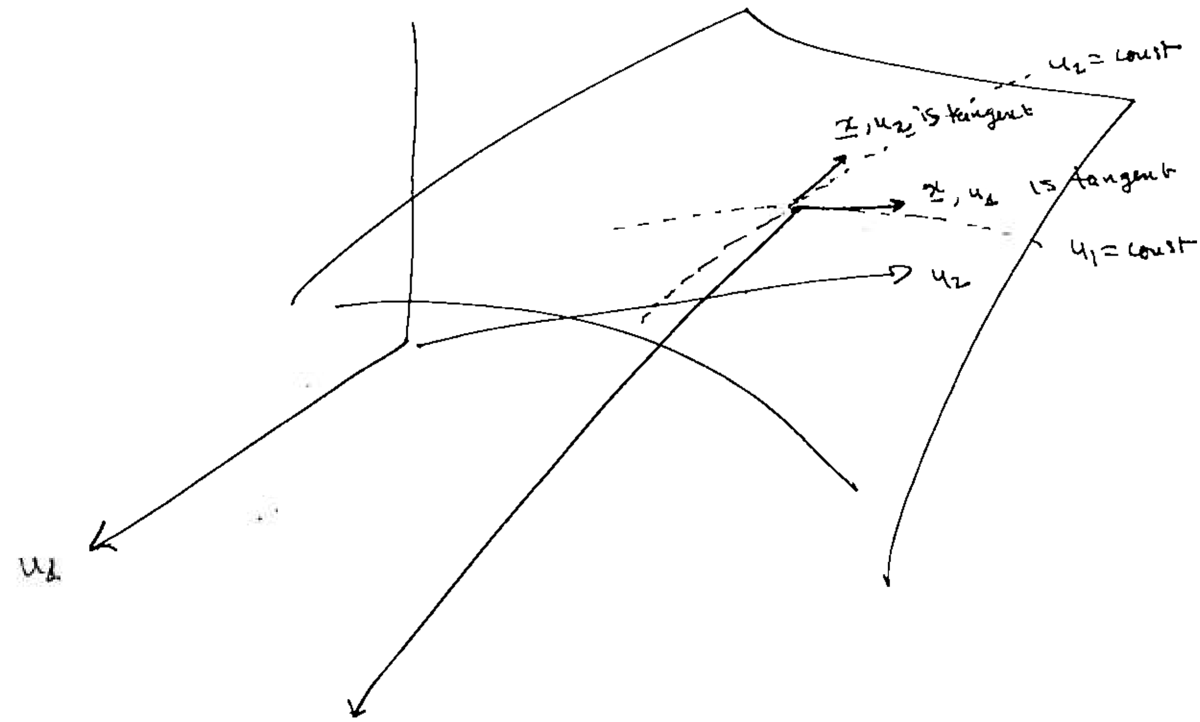
5. Equations of isometric embedding

Review



point on surface

$$\left(x_1(u_1,u_2),\ x_2(u_1,u_2),\ x_3(u_1,u_2) \right)$$

$\frac{\partial \underline{x}}{\partial u_1}$, $\frac{\partial \underline{x}}{\partial u_2}$ are tangent to surface

$$\underline{n} = \frac{\frac{\partial \underline{x}}{\partial u_1} \times \frac{\partial \underline{x}}{\partial u_2}}{\left| \frac{\partial \underline{x}}{\partial u_1} \times \frac{\partial \underline{x}}{\partial u_2} \right|}$$

is unit normal

The triple

$$\left(\frac{\partial \underline{x}}{\partial u_1}, \frac{\partial \underline{x}}{\partial u_2}, \underline{n} \right)$$

is called the <u>Gauss frame</u>. Hence any vector in $\mathbb{R}^3$ can be decomposed into a linear combination of these three vectors. This decomposition gives us the <u>Gauss-Weingarten equations</u>:

$$\text{(a)} \quad \frac{\partial^2 \underline{x}}{\partial u_1^2} = \Gamma_{11}^1 \frac{\partial \underline{x}}{\partial u_1} + \Gamma_{11}^2 \frac{\partial \underline{x}}{\partial u_2} + L \underline{n}$$

$$\text{(b)} \quad \frac{\partial^2 \underline{x}}{\partial u_1 \partial u_2} = \Gamma_{12}^1 \frac{\partial \underline{x}}{\partial u_1} + \Gamma_{12}^2 \frac{\partial \underline{x}}{\partial u_2} + M \underline{n} \qquad (201)$$

$$\text{(c)} \quad \frac{\partial^2 x}{\partial u_2^2} = \Gamma_{22}^1 \frac{\partial \underline{x}}{\partial u_1} + \Gamma_{22}^1 \frac{\partial \underline{x}}{\partial u_2} + N \underline{n}$$

$$\frac{\partial \underline{n}}{\partial u_1} = \alpha_1 \frac{\partial \underline{x}}{\partial u_1} + \alpha_2 \frac{\partial \underline{x}}{\partial u_2}$$

$$\frac{\partial \underline{n}}{\partial v_2} = \beta_1 \frac{\partial \underline{x}}{\partial u_1} + \beta_2 \frac{\partial \underline{x}}{\partial u_2}$$

Of course there is no $\underline{n}$ component in the last two equations in (20) since $\underline{n}$ is a unit vector:

$$\begin{cases} \dfrac{\partial}{\partial u_1}\, \underline{n}\cdot\underline{n} = 0 & \Rightarrow \quad \underline{n}\cdot\dfrac{\partial \underline{n}}{\partial u_1} = 0 \\ \dfrac{\partial}{\partial u_2}\, \underline{n}\cdot\underline{n} = 0 & \Rightarrow \quad \underline{n}\cdot\dfrac{\partial \underline{n}}{\partial u_2} = 0 \end{cases}$$

Also from (20) we see

$$\begin{aligned} L &= \frac{\partial^2 \underline{x}}{\partial u_1^2}\cdot \underline{n} \\ M &= \frac{\partial^2 \underline{x}}{\partial u_1 \partial u_2}\cdot \underline{n} \\ N &= \frac{\partial^2 \underline{x}}{\partial u_2^2}\cdot \underline{n} \end{aligned} \qquad (21)$$

which is a definition of the component L, M, N of the <u>second</u> <u>fundamental</u> <u>form</u>:

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix}.$$

Equality of cross partials

$$\frac{\partial}{\partial u_2}(a) = \frac{\partial}{\partial u_1}(b)$$

$$\frac{\partial}{\partial u_2}(b) = \frac{\partial}{\partial u_1}(c) \qquad (21)$$

i.e

$$\frac{\partial}{\partial u_2}\left(\Gamma_{11}^{1}\frac{\partial x}{\partial u_1} + \Gamma_{11}^{2}\frac{\partial x}{\partial u_2} + L\,\underline{n}\right) =$$
$$\frac{\partial}{\partial u_1}\left(\Gamma_{12}^{1}\frac{\partial x}{\partial u_1} + \Gamma_{12}^{2}\frac{\partial x}{\partial u_2} + M\,\underline{n}\right) \qquad (22)$$

$$\frac{\partial}{\partial u_2}\left(\Gamma_{12}^{1}\frac{\partial x}{\partial u_1} + \Gamma_{12}^{2}\frac{\partial x}{\partial u_2} + M\,\underline{n}\right) = \qquad (23)$$
$$\frac{\partial}{\partial u_1}\left(\Gamma_{22}^{1}\frac{\partial x}{\partial u_1} + \Gamma_{22}^{2}\frac{\partial x}{\partial u_2} + N\,\underline{n}\right)$$

(22) $\Rightarrow$

$$\Gamma_{11}^{1}\frac{\partial^2 x}{\partial u_2 \partial u_1} + \Gamma_{11}^{2}\frac{\partial^2 x}{\partial u_2^2} + \frac{\partial L}{\partial u_2}\cdot \underline{n}$$
$$+ \frac{\partial \Gamma_{11}^{1}}{\partial u_2}\frac{\partial x}{\partial u_1} + \frac{\partial \Gamma_{11}^{2}}{\partial u_2}\frac{\partial x}{\partial u_2} + L\frac{\partial n}{\partial u_2} = \qquad (24)$$
$$\Gamma_{12}^{1}\frac{\partial^2 x}{\partial u_1^2} + \Gamma_{12}^{2}\frac{\partial^2 x}{\partial u_1 \partial u_2} + \frac{\partial M}{\partial u_1}\cdot \underline{n}$$
$$+ \frac{\partial \Gamma_{12}^{1}}{\partial u_1}\frac{\partial x}{\partial u_1} + \frac{\partial \Gamma_{12}^{2}}{\partial u_1}\frac{\partial x}{\partial u_2} + M\frac{\partial n}{\partial u_1}$$

and a similar equation from (23).

Let take the dot product of (24) with $\underline{n}$. This gives

$$\Gamma_{11}^{1}\,\frac{\partial^2 \underline{x}}{\partial u_2\,\partial u_1}\cdot\underline{n} + \Gamma_{11}^{2}\,\frac{\partial^2 \underline{x}}{\partial u_2^2}\cdot\underline{n} + \frac{\partial L}{\partial u_2} =$$

$$\Gamma_{12}^{1}\,\frac{\partial^2 \underline{x}}{\partial u_1^2}\cdot\underline{n} + \Gamma_{12}^{2}\,\frac{\partial^2 \underline{x}}{\partial u_1\,\partial u_2}\cdot\underline{n} + \frac{\partial M}{\partial u_1} \qquad (25)$$

where the other terms are zero by orthogonality.

From (21), (25) can be rewritten as

$$\Gamma_{11}^{1}\,M + \Gamma_{12}^{2}\,N + \frac{\partial L}{\partial u_2} =$$

$$\Gamma_{12}^{1}\,L + \Gamma_{12}^{2}\,M + \frac{\partial M}{\partial u_1} \qquad (26)$$

If we do the same computation for (23) we obtain

$$\Gamma_{12}^{1}\,M + \Gamma_{12}^{2}\,N + \frac{\partial M}{\partial u_2} = \qquad (27)$$

$$\Gamma_{22}^{1}\,L + \Gamma_{22}^{2}\,M + \frac{\partial N}{\partial u_1}$$

These are the Codazzi equations or alternatively

$$\boxed{\begin{aligned} \frac{\partial L}{\partial u_2} - \frac{\partial M}{\partial u_1} &= L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{12}) - N\Gamma^2_{11} \\ \frac{\partial M}{\partial u_2} - \frac{\partial N}{\partial u_1} &= L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12} \end{aligned}} \qquad (28)$$

CODAZZI

The Codazzi are equations are two balance laws in three unknowns L, M, N.

But we close the system by

1) using the definition of the Gauss curvature,
2) Gauss's theorem egregium.

1) Gauss curvature K:

$$K = \frac{\det \begin{bmatrix} L & M \\ M & N \end{bmatrix}}{\det g}$$

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

$$\boxed{K = \frac{(LN - M^2)}{(EG - F^2)}} . \qquad (29)$$

Gauss relation

Here g is the known metric of manifold (M^2, g)

2) Gauss's theorem egregium tells us the ~~K~~ is known function of g and its first and second derivatives.

Hence if we wish to recover a surface in $\mathbb{R}^3$ from its metric i.e. solve

$$\frac{\partial \underline{x}}{\partial u_i} \cdot \frac{\partial \underline{x}}{\partial u_j} = g_{ij} \qquad i, j = 1, 2$$

for $x_1(u_1, u_2)$, $x_2(u_1, u_2)$, $x_3(u_1, u_2)$ given g_{ij} ($g_{11} = E$, $g_{12} = F$, $g_{22} = G$)

we can equivalently solve the Gauss (eqn. (29)) and Codazzi equations (eqns (28)).

This equivalence is given as the fundamental theorem of surface theory.

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Fundamental Theorem of Surface Theory

If the first and second fundamental forms are sufficiently differentiable in u_1, u_2 and satisfy the Gauss-Codazzi equations and $EG-F^2>0$, $E>0$, $G>0$, then there exists a surface uniquely determined ~~by~~ up to its position in space which has respectively the given first and second fundamental forms.

SUMMARY

EULER (STEADY) IRROTATIONAL

$$\partial_1 (\rho v_1^2 + p(\rho)) + \partial_2 (\rho v_1 v_2) = 0$$

$$\partial_1 (\rho v_1 v_2) + \partial_2 (\rho v_2^2 + p(\rho)) = 0$$

$$p = \frac{\rho^\gamma}{\gamma}$$

BERNOULLI : $\frac{\rho^{\gamma-1}}{\gamma-1} + \frac{1}{2} q^2 = \frac{1}{\gamma-1}$

CODAZZI

$$\frac{\partial N}{\partial u_1} - \frac{\partial M}{\partial u_2} = -\left(L \Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N \Gamma_{12}^2\right)$$

$$-\frac{\partial M}{\partial u_1} + \frac{\partial L}{\partial u_2} = +\left(L \Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{12}^1) - N \Gamma_{11}^2\right)$$

GAUSS : $LN - M^2 = \mathcal{R}$

$$\mathcal{R} = (E G - F^2) K$$

NATURAL IDENTIFICATION

$$\begin{cases} N = \rho v_1^2 + p \\ M = -\rho v_1 v_2 \\ L = \rho v_2^2 + p \end{cases}$$

Let us see what Gauss becomes.

$$LN - M^2 = K \Longrightarrow$$

$$(\rho v_2^2 + p)(\rho v_1^2 + p) - \rho^2 v_1^2 v_2^2 = K$$

$$\underline{\underline{\rho^2 v_1^2 v_2^2}} + p\rho(v_1^2 + v_2^2) + p^2 - \underline{\underline{\rho^2 v_1^2 v_2^2}} = K$$

set $v_1^2 + v_2^2 = q^2$

$$p\rho q^2 + p^2 = K$$

We have not made a choice for p. Let's choose

$$p = \frac{\rho^{\gamma}}{\gamma}$$

as before but now with $\gamma = -1$.

This is called a <u>Chaplygin gas</u> : $p = -\frac{1}{\rho}$.

Hence

$$p\rho q^2 + p^2 = k$$

$$-\frac{1}{\rho}\rho q^2 + \frac{1}{\rho^2} = k$$

$$\boxed{-q^2 + \frac{1}{\rho^2} = k}$$

Geometric Bernoulli equation (30)

$$\frac{1}{\rho^2} = k + q^2$$

$$\left\{\begin{aligned} \rho &= \frac{1}{(q^2+k)^{1/2}} \\ p &= -(q^2+k^2)^{1/2} \end{aligned}\right.$$

Sound speed

$$c^2 = p'(\rho) \qquad p = -\frac{1}{\rho}$$
$$= \frac{1}{\rho^2}$$

$M^2 = \frac{q^2}{c^2}$ is geometric Mach number

$$= \frac{\frac{1}{\rho^2} - k}{\left(\frac{1}{\rho^2}\right)} = \rho^2\left(\frac{1}{\rho^2} - k\right) = 1 - \rho^2 k$$

$$\Rightarrow \quad M^2 = 1 - \rho^2 k$$

$$M^2 - 1 = -\rho^2 k$$

$\therefore \begin{cases} M^2 - 1 < 0 & k > 0 & \text{subsonic} & \text{elliptic} \\ M^2 - 1 = 0 & k = 0 & \text{sonic} & \\ M^2 - 1 > 0 & k < 0 & \text{supersonic} & \text{hyperbolic} \end{cases}$

Since $k = K(EG - F^2)$

$\begin{cases} K > 0 & \text{subsonic} & \text{elliptic} \\ K = 0 & \text{sonic} & \\ K < 0 & \text{supersonic} & \text{hyperbolic} \end{cases}$

But if CODAZZI - GAUSS is now written as

$$\partial_1(\rho v_1^2 + p(\rho)) + \partial_2(\rho v_1 v_2) = \cdots \quad (30)$$

$$\partial_1(\rho v_1 v_2) + \partial_2(\rho v_2^2 + p(\rho)) = \cdots \quad (31)$$

with
$$p(\rho) = -\frac{1}{\rho}$$
and "Bernoulli" $-q^2 + \frac{1}{\rho^2} = K$

what happens to
$$\frac{\partial v_1}{\partial u_2} - \frac{\partial v_2}{\partial u_1} \ ?$$
the vorticity ? (geometric)

what happen to
$$\frac{\partial}{\partial u_1}(\rho v_1) + \frac{\partial}{\partial u_2}(\rho v_2) = \ ?$$
the continuity equ ? (geometric)

Now consider (30) (31) and do the following computation :

$$v_1 \left(\partial_1 (\rho v_1^2 + p) + \partial_2 (\rho v_1 v_2) = B_1 \right)$$

$$v_2 \left(\partial_1 (\rho v_1 v_2) + \partial_2 (\rho v_2^2 + p) = B_2 \right)$$

(30) $\cdot v_1$
(31) $\cdot v_2$

and set

$$-\sigma = \partial_1 v_2 - \partial_2 v_1$$

$$v_1 \left(\partial_1 (\rho v_1) v_1 + \rho v_1 \partial_1 v_1 + \partial_1 p + \partial_2 (\rho v_2) v_1 + \partial_2 v_1 \rho v_2 \right)$$

$$= B_1 v_1$$

$$v_2 \left(\partial_1 (\rho v_1) v_2 + \rho v_1 \partial v_2 + \partial_2 (\rho v_2) v_2 + \rho v_2 \partial_2 v_2 + \partial_2 p \right)$$

$$= B_2 v_2$$

$$v_1^2 \left[\partial_1(\rho v_1) + \partial_2(\rho v_2) \right] + \rho v_1^2 \partial_1 v_1 + \rho v_1 v_2 \partial_2 v_1 + v_1 \partial_1 p = B_1 v_1$$

$$v_2^2 \left[\partial_1(\rho v_1) + \partial_2(\rho v_2) \right] + \rho v_1 v_2 \partial_1 v_2 + \rho v_2^2 \partial_2 v_2 + v_2 \partial_2 p = B_2 v_2$$

Set $\begin{cases} \underline{\underline{\partial_2 v_1}} = \partial_1 v_2 + \sigma & \text{in first eqn} \\ \underline{\underline{\partial_1 v_2}} = \partial_2 v_1 - \sigma & \text{in second eqn} \end{cases}$

and $\begin{cases} \partial_1 p = \partial_1(\rho^{-1}) = \frac{1}{\rho^2} \partial_1 \rho \\ \partial_2 p = \frac{1}{\rho^2} \partial_2 \rho \end{cases}$

$$-q^2 + \frac{1}{\rho^2} = k$$

$$\frac{1}{\rho^2} = k + q^2$$

$$\begin{cases} \partial_1 p = \frac{1}{\rho^2} \partial_1 \rho = (k + q^2)\partial_1 \rho \\ \partial_2 p = \frac{1}{\rho^2} \partial_2 \rho = (k + q^2)\partial_2 \rho \end{cases}$$

$$\rho = (k + q^2)^{-1/2}$$

$$\partial_1 \rho = -\frac{1}{2}(k+q^2)^{-3/2} (\partial_1 k + 2q \partial_1 q)$$

$$\partial_2 \rho = -\frac{1}{2}(k+q^2)^{-3/2} (\partial_2 k + 2q \partial_2 q)$$

$$\begin{cases} \partial_1 p = -\frac{1}{2}(k+q^2)^{-1/2} (\partial_1 k + 2q \partial_1 q), \\ \partial_2 p = -\frac{1}{2}(k+q^2)^{-1/2} (\partial_2 k + 2q \partial_2 q). \end{cases}$$

$$\begin{cases} \partial_1 p = -\frac{1}{2} \rho (\partial_1 k + 2q \partial_1 q) \\ \partial_2 p = -\frac{1}{2} \rho (\partial_2 k + 2q \partial_2 q) \end{cases}$$

$$\begin{cases} v_1^2 \left[\partial_1(\rho v_1) + \partial_2(\rho v_2) \right] + \rho v_1^2 \partial_1 v_1 + \rho v_1 v_2 \underline{\underline{\partial_2 v_1}} \\ \quad + v_1 \left(-\frac{1}{2} \rho \partial_1 k \right) - v_1 \rho q \partial_1 q = B_1 v_1 \\ \\ v_2^2 \left[\partial_1(\rho v_1) + \partial_2(\rho v_2) \right] + \rho v_1 v_2 \underline{\underline{\partial_1 v_2}} \\ \quad + \rho v_2^2 \partial_2 v_2 + v_2 \left(-\frac{1}{2} \rho \partial_2 k \right) \\ \quad - v_2 \rho q \partial_2 q = B_2 v_2 \end{cases}$$

$$v_1^2 \left(\partial_1(\rho v_1) + \partial_2(\rho v_2) \right) + \widetilde{\rho v_1^2} \partial_1 v_1$$

$$+ \rho v_1 v_2 \left(\widetilde{\partial_1 v_2} + \sigma \right) - \frac{v_1}{2} \rho \partial_1 k - \widetilde{v_1 \rho q} \partial_1 q = B_1 v_1$$

$$v_2^2 \left(\partial_1(\rho v_1) + \partial_2(\rho v_2) \right) + \rho v_1 v_2 \left(\widetilde{\partial_2} v_1 - \sigma \right)$$

$$+ \rho v_2^2 \widetilde{\partial_2} v_2 - \frac{v_2}{2} \rho \partial_2 k - \widetilde{v_2 \rho q \partial q} = B_2 v_2$$

But

$$\rho v_1^2 \partial_1 v_1 + \rho v_1 v_2 \partial v_2 - \rho v_1 q \partial q =$$

$$\frac{1}{2} \rho v_1 \left[\partial_1 (v_1^2 + v_2^2) - \partial q^2 \right] =$$

$$\frac{1}{2} \rho v_1 \partial_1 (v_1^2 + v_2^2 - q^2) = 0$$

and $\sim$ terms cancel.

Hence

$$\begin{cases} v_1^2 \left(\partial_1(\rho v_1) + \partial_2(\rho v_2) \right) + \rho v_1 v_2 \sigma - \frac{v_1 \rho}{2} \partial_1 k = B_1 v_1 \\ v_2^2 \left(\partial_1(\rho v_1) + \partial_2(\rho v_2) \right) - \rho v_1 v_2 \sigma - \frac{v_2}{2} \rho \partial_2 k = B_2 v_2 \end{cases}$$

ADD

$$(v_1^2 + v_2^2) \left[\partial_1(\rho v_1) + \partial_2(\rho v_2) \right] = \frac{v_1 \rho}{2} \partial_1 k + \frac{v_2 \rho}{2} \partial_2 k + B_1 v_1 + B_2 v_2$$

GEOMETRIC CONTINUITY EQN

$$\boxed{\partial_1(\rho v_1) + \partial_2(\rho v_2) = \frac{v_1 \rho}{2q^2} \partial_1 k + \frac{v_2 \rho}{2q^2} \partial_2 k + \frac{B_1 v_1}{q^2} + \frac{B_2 v_2}{q^2}} \tag{37}$$

DIVIDE BY $\left\{ \begin{matrix} v_1^2 \\ v_2^2 \end{matrix} \right\}$ and SUBTRACT

$$\partial_1 (\rho v_1) + \partial_2(\rho v_2) + \sigma \frac{v_2}{v_1} - \frac{\rho}{v_1} \partial_1 k = \frac{B_1}{v_1}$$

$$\partial_1(\rho v_2) + \partial_2(\rho v_2) - \sigma \frac{v_1}{v_2} - \frac{\rho}{v_2} \partial_2 k = \frac{B_2}{v_2}$$

$\Longrightarrow$

$$\sigma \left(\frac{v_2}{v_1} + \frac{v_1}{v_2} \right) = \frac{B_1}{v_1} + \frac{\rho}{v_1} \partial_1 k - \frac{B_2}{v_2} - \frac{\rho}{v_2} \partial_2 k$$

$$\sigma = \frac{\frac{B_1}{v_1} + \frac{\rho}{v_1} \partial_1 k - \frac{B_2}{v_2} - \frac{\rho}{v_2} \partial_2 k}{\left(\frac{v_2}{v_1} + \frac{v_1}{v_2} \right)} \cdot \left(\frac{v_1 v_2}{v_1 v_2} \right)$$

$$\sigma = \frac{B_1 v_2 + \rho v_2 \partial_1 k - B_2 v_1 - \rho v_1 \partial_2 k}{v_2^2 + v_1^2}$$

GEOMETRIC ROTATIONALITY EQN

$$\boxed{\partial_1 v_2 - \partial_2 v_1 = \frac{1}{q^2} \left\{ B_1 v_2 + \rho v_2 \partial_1 k - B_2 v_1 - \rho v_1 \partial_2 k \right\}} \quad (38)$$

Thus starting with

$$\begin{cases} \text{CODAZZI} & (31), (32) \quad \text{GEOM (LIN MOM)} \quad \text{GEOM (BERNOULLI} \\ \text{GAUSS} & (30) \end{cases}$$

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$$\begin{cases} \text{GEOM CONTINUITY} & (33) \\ \text{GEOM ROTATIONALITY} & (34) \end{cases}$$

In language of theory of conservation laws

(33), (34) are

"entropy" equations in the sense of Lax. This means that (33), (34) were <u>additional</u> <u>balance</u> <u>laws</u> implied by the balance laws (31), (32) and constitutive equation (30) (for smooth solns to PDEs).