Lecture note on

Riemann Geometry and nonlinear conservation laws

(Figures are not given yet.)

1 Preliminaries on Differential Geometry

1.1 Curves in $\boldsymbol{R}^n$

Let $\mathbf{x}(t)$ be a parameterized curve in $\boldsymbol{R}^n$. Then,

$$\mathbf{v}(t) := \dot{\mathbf{x}}(t), \quad \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

are the velocity and the unit tangent vectors, respectively. The curvature $\boldsymbol{\kappa}$ is the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length, i.e.,

$$\boldsymbol{\kappa} = \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt} \Big/ \frac{ds}{dt} = \frac{1}{|\mathbf{v}|}\dot{\mathbf{T}}.$$

Since

$$\mathbf{T}\cdot\mathbf{T} = 1 \quad \Rightarrow \quad \dot{\mathbf{T}}\cdot\mathbf{T} = 0 \quad \Rightarrow \quad \boldsymbol{\kappa}\cdot\mathbf{T} = 0,$$

the curvature vector is orthogonal to $\mathbf{T}$. The scalar curvature is simply,

$$\kappa = |\boldsymbol{\kappa}|.$$

Of course since

$$\frac{ds}{dt} = |\mathbf{v}|, \quad \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{T},$$

we have

$$\boldsymbol{\kappa} = \frac{d^2\mathbf{x}}{ds^2}.$$

As usual in the plane, if we take

$$\mathbf{x} = (t, f(t)),$$

then,

$$\frac{d\mathbf{x}}{ds} = (\frac{dt}{ds}, f'(t)\frac{dt}{ds}) = (1, f'(t))/\frac{ds}{dt} = \frac{(1, f'(t))}{\sqrt{1+(f'(t))^2}}.$$

The computation for the curvature $\boldsymbol{\kappa}$ is more complicate and gives

$$\boldsymbol{\kappa} = \frac{d^2\mathbf{x}}{ds^2} = \frac{d}{ds}\frac{(1, f'(t))}{\sqrt{1+(f'(t))^2}} = \cdots = \frac{f''(t)(-f'(t), 1)}{(1+(f'(t))^2)^2}.$$

The scalar curvature is

$$\kappa = \left|\frac{d^2\mathbf{x}}{ds^2}\right| = \frac{|f''(t)|}{(1+(f'(t))^2)^{3/2}}.$$

The usual calculus formula is quite complicate even for a curve in the plan given as a graph.

2 Surfaces in $\boldsymbol{R}^3$

Let $\mathcal{S}$ be a C^2 surface in $\boldsymbol{R}^3$ (See Figure in page 5). For convenience, we assume that the surface $\mathcal{S}$ is given by a graph, i.e.,

$$\mathcal{S} \equiv (x, y, f(x, y)).$$

Let $T_p\mathcal{S}$ be the tangent space of vectors which are tangent to $\mathcal{S}$ at a point $p \in \mathcal{S}$. To study curvature of $\mathcal{S}$, we slice $\mathcal{S}$ by planes containing the normal vector $\mathbf{n}$ and the point $p \in \mathcal{S}$ and consider the curvature $\boldsymbol{\kappa}$ of the resulting curves (see Figure page 6).

Since, as noted in Section 1.1, curvature vector is always orthogonal to $\mathbf{v}$ which lies in $T_p\mathcal{S}$, $\boldsymbol{\kappa}$ lies in direction of $\mathbf{n}$. Hence,

$$\boldsymbol{\kappa} = \kappa \mathbf{n}.$$

(If the direction is given, then the interest is only on the scalar curvature κ.) The largest and the smallest curvatures lie in two orthogonal directions(why?). These are $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2$, the principal curvatures.

Choose orthonormal coordinates on $\boldsymbol{R}^3$ with origin at p, $\mathcal{S}$ tangent to the x, y plane, $\mathbf{n}$ pointing in positive z direction as shown above (see Figure in page 6). The vectors $\mathbf{v}, \mathbf{n}$ define a plane as shown which intersects $\mathcal{S}$ in a curve C. The curvature κ of this curve at p which is called the curvature in the direction $\mathbf{v}$ is the second derivative

$$\kappa = (D^2 f)_p(\mathbf{v}, \mathbf{v}) \equiv \mathbf{v} \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(p) & \frac{\partial^2 f}{\partial x \partial y}(p) \\ \frac{\partial^2 f}{\partial x \partial y}(p) & \frac{\partial^2 f}{\partial y^2}(p) \end{pmatrix} \mathbf{v}^t.$$

For example, if $\mathbf{v} = (1, 0)$, then $\kappa = \frac{\partial^2 f}{\partial x^2}(p)$.

The bilinear form $(D^2 f)_p$ on $T_p\mathcal{S}$ is called the *second fundamental form* Π of $\mathcal{S}$ at p. In our local coordinates,

$$\Pi = (D^2 f) \equiv \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Of course the formula is good only at a point where the surface is taking to the x, y plane. For the second fundamental form we will always use orthogonal coordinates.

Since Π is symmetric, we may choose coordinates x, y such that Π is diagonal:

$$\Pi = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

Then the curvature κ in direction $\mathbf{v} = (\cos\theta, \sin\theta)$ is given by Euler's formula (1760), i.e.,

$$\kappa = \Pi(\mathbf{v}, \mathbf{v}) = \mathbf{v}\Pi\mathbf{v}^t = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

Definition 2.1. *At a point p in a surface $\mathcal{S} \subset \boldsymbol{R}^3$ the eigenvalues κ_1, κ_2 of the second fundamental form Π are called the principal curvatures and the corresponding eigenvectors, which are determined uniquely unless $\kappa_1 = \kappa_2$, are called the principal directions or directions of the curvatures. The trace of $\Pi = \kappa_1 + \kappa_2$ is called the mean curvature H. The determinant of $\Pi = \kappa_1\kappa_2$ is called the Gauss curvature K.*

Note that the signs of Π and H but not K depend on a choice of unit normal $\mathbf{n}$. Also sometimes (but not here) $H = (\kappa_1 + \kappa_2)/2$.

2.1 Coordinates, length, metric

Local coordinates or parameters u_1, u_2 on a C^2 surface $S \subset \boldsymbol{R}^3$ are provided by a C^2 diffeomorphism (or parametrization) between a domain in the u_1, u_2 plane and a portion of $\mathcal{S}$.

Example 2.2 (Sphere of radius a)**.** *The standard spherical coordinates φ, θ provide local coordinates on all of the sphere of radius a except for the poles where longitude θ is undefined and the latitude φ is not differentiable. The position vector is determined by the coordinates*

$$(x, y, z) = (a \sin\varphi \cos\theta, a \sin\varphi \sin\theta, a\cos\varphi).$$

In general the position is a function of the coordinates, say u_i. Along a curve these coordinates are function of a single parameter t. We use the the following notations. The subscript i after comma indicates the partial derivative, e.g.,

$$\mathbf{x}_{,i} = \mathbf{x}_{u_i} = \frac{\partial \mathbf{x}}{\partial u_i} = (\frac{\partial x}{\partial u_i}, \frac{\partial y}{\partial u_i}, \frac{\partial z}{\partial u_i}).$$

The dot $\dot{}$ denotes the differentiation with respect to the (time) parameter t. Then, the chain rule gives

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \sum_{i=1}^{2} \mathbf{x}_{,i}\dot{u}_i.$$

The arc length of a curve in the surface with coordinate system $\mathbf{u}(t)$ is given by

$$L = \int_{t_0}^{t_1} |\dot{\mathbf{x}}(t)| dt = \int_{t_0}^{t_1} |\mathbf{x}_{,1}\dot{u}_1 + \mathbf{x}_{,2}\dot{u}_2| dt = \int_{t_0}^{t_1} (\sum g_{ij}\dot{u}_i\dot{u}_j)^{1/2} dt,$$

where

$$(g_{ij}) = (\mathbf{x}_{,i} \cdot \mathbf{x}_{,j}) \equiv \left(\frac{\partial \mathbf{x}}{\partial u_i} \cdot \frac{\partial \mathbf{x}}{\partial u_j}\right) = \begin{pmatrix} |\mathbf{x}_{,1}|^2 & \mathbf{x}_{,1} \cdot \mathbf{x}_{,2} \\ \mathbf{x}_{,1} \cdot \mathbf{x}_{,2} & |\mathbf{x}_{,2}|^2 \end{pmatrix}.$$

In other words the arc length L is given by $L = \int ds$ where $ds^2 = \sum g_{ij} du_i du_j$.

Example 2.3 (Sphere of radius a)**.** *For the standard spherical coordinates system $(x, y, z) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$, the arc length is given by*

$$ds^2 = a^2 d\varphi^2 + a^2 \sin^2\varphi d\theta^2 = a^2(\dot{\varphi}^2 + (\dot{\theta}\sin\varphi)^2)dt^2,$$

as

$$g_{11} = a^2, \quad g_{22} = a^2\sin^2\varphi, \quad g_{12} = g_{21} = 0.$$

The matrix $g = (g_{ij})$ is called the *first fundamental form* or metric. It is an *intrinsic* from, i.e., only depends on information on the surface since the metric relates to measurements inside the surface. For many surfaces in $\boldsymbol{R}^3$, it is convenient to use x, y as local coordinates. Then

$$\mathbf{x}_{,1} = (1, 0, z_x), \quad \mathbf{x}_{,2} = (0, 1, z_y).$$

Proposition 2.4. *For any local coordinates u_1, u_2 about a point p in a C^2 surface in $\boldsymbol{R}^3$, the second fundamental form II at p is similar to*

$$g^{-1}(D^2\mathbf{x}) \cdot \mathbf{n} \equiv g^{-1} \begin{pmatrix} \mathbf{x}_{,11} \cdot \mathbf{n} & \mathbf{x}_{,12} \cdot \mathbf{n} \\ \mathbf{x}_{,12} \cdot \mathbf{n} & \mathbf{x}_{,22} \cdot \mathbf{n} \end{pmatrix}, \tag{2.1}$$

where

$$\mathbf{x}_{,ij} = \frac{\partial^2 \mathbf{x}}{\partial u_i \partial u_j}, \quad \mathbf{n} = \frac{\mathbf{x}_{,1} \times \mathbf{x}_{,2}}{|\mathbf{x}_{,1} \times \mathbf{x}_{,2}|}.$$

Proof. After an appropriate translation and a rotation one may assume that the point p is the origin and the xy-plane is the tangent plane for the surface S and $p = 0$. First take the local coordinate system is just $u_1 = x,\ u_2 = y$ and let the surface be given as the graph $z = f(x, y)$. Then, $f_x(0) = f_y(0) = 0$ and $\mathbf{n} = (0, 0, 1)$. For these particular local coordinates $\mathbf{x} = (x, y, f(x, y))$, we have

$$\mathbf{x}_{,11} \cdot \mathbf{n} = f_{xx},\ \mathbf{x}_{,22} \cdot \mathbf{n} = f_{yy},\ \mathbf{x}_{,12} = \mathbf{x}_{,21} = f_{xy}.$$

The metric $g_{ij} = \mathbf{x}_{,i} \cdot \mathbf{x}_{,j}$ is given by

$$\begin{aligned} g_{11} &= (1, 0, f_x(0, 0)) \cdot (1, 0, f_x(0, 0)) = 1, \\ g_{22} &= (0, 1, f_y(0, 0)) \cdot (0, 1, f_y(0, 0)) = 1, \\ g_{12} &= (1, 0, f_x(0, 0)) \cdot (0, 1, f_y(0, 0)) = 0. \end{aligned}$$

Hence, $g(0)$ is the identity matrix. Hence the matrix in (2.1) is actually the second fundamental form Π.

Now let u_1, u_2 be any local coordinates and J denote the Jacobian at $p = 0$, i.e.,

$$J := \begin{pmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} \end{pmatrix}$$

Now write

$$x = x(u_1, u_2),\ y = y(u_1, u_2),\ \mathbf{x} = (x(u_1, u_2), y(u_1, u_2), f(x(u_1, u_2), y(u_1, u_2))\,).$$

Then, since

$$\frac{\partial f}{\partial u_i}(0, 0) = f_x(0, 0)\frac{\partial x}{\partial u_i} + f_y(0, 0)\frac{\partial y}{\partial u_i} = 0,$$

the metric under these local coordinates are given by

$$g_{ij} = \mathbf{x}_{,i} \cdot \mathbf{x}_{,j} = (\frac{\partial x}{\partial u_i}, \frac{\partial y}{\partial u_i}, 0) \cdot (\frac{\partial x}{\partial u_j}, \frac{\partial y}{\partial u_j}, 0) \quad \text{or} \quad g = J^T J.$$

The second order terms are

$$\mathbf{x}_{,ij} = \Big(\frac{\partial^2 x}{\partial u_i \partial u_j}, \frac{\partial^2 y}{\partial u_i \partial u_j}, f_{xx}\big(\frac{\partial x}{\partial u_i}\big)^2 + f_{yy}\big(\frac{\partial y}{\partial u_i}\big)^2 + 2f_{xy}\frac{\partial x}{\partial u_i}\frac{\partial y}{\partial u_i}\Big).$$

Therefore, since $\mathbf{n} = (0, 0, 1)$ and $J^T J = g$, one obtains

$$\begin{pmatrix} \mathbf{x}_{,11} \cdot \mathbf{n} & \mathbf{x}_{,12} \cdot \mathbf{n} \\ \mathbf{x}_{,12} \cdot \mathbf{n} & \mathbf{x}_{,22} \cdot \mathbf{n} \end{pmatrix} = J^T \Pi J = g J^{-1} \Pi J.$$

Therefore, $g^{-1}(D^2\mathbf{x}) \cdot \mathbf{n}$ is similar to the fundamental form Π, which completes the proof of the proposition. □

Consequently,

$$H = trace(g^{-1}(D^2\mathbf{x}) \cdot \mathbf{n}) = \frac{|\mathbf{x}_{,2}|^2\mathbf{x}_{,11} - 2(\mathbf{x}_{,1} \cdot \mathbf{x}_{,2})\mathbf{x}_{,12} + |\mathbf{x}_{,1}|^2\mathbf{x}_{,22}}{|\mathbf{x}_{,1}|^2|\mathbf{x}_{,2}|^2 - (\mathbf{x}_{,1} \cdot \mathbf{x}_{,2})^2} \cdot \mathbf{n}, \tag{2.2}$$

$$K = det(g^{-1}(D^2\mathbf{x}) \cdot \mathbf{n}) = \frac{(\mathbf{x}_{,11} \cdot \mathbf{n})(\mathbf{x}_{,22} \cdot \mathbf{n}) - (\mathbf{x}_{,12} \cdot \mathbf{n})^2}{|\mathbf{x}_{,1}|^2|\mathbf{x}_{,2}|^2 - (\mathbf{x}_{,1} \cdot \mathbf{x}_{,2})^2}. \tag{2.3}$$

Remark 2.5. *(A) If we solve for the eigenvalue of* Π *and eliminate in in form* H *and* K*, we have principal curvatures:*

$$\boldsymbol{\kappa} = \frac{H \pm \sqrt{H^2 - 4K}}{2}. \tag{2.4}$$

(B) If the surface is a graph, $\mathbf{x} = (x, y, f(x, y))$*, then*

$$H = \frac{|\mathbf{x}_{,2}|^2 \mathbf{x}_{,11} - 2(\mathbf{x}_{,1} \cdot \mathbf{x}_{,2})\mathbf{x}_{,12} + |\mathbf{x}_{,1}|^2 \mathbf{x}_{,22}}{|\mathbf{x}_{,1}|^2 |\mathbf{x}_{,2}|^2 - (\mathbf{x}_{,1} \cdot \mathbf{x}_{,2})^2} \cdot \mathbf{n}, \tag{2.5}$$

$$K = \frac{(\mathbf{x}_{,11} \cdot \mathbf{n})(\mathbf{x}_{,22} \cdot \mathbf{n}) - (\mathbf{x}_{,12} \cdot \mathbf{n})^2}{|\mathbf{x}_{,1}|^2 |\mathbf{x}_{,2}|^2 - (\mathbf{x}_{,1} \cdot \mathbf{x}_{,2})^2}. \tag{2.6}$$

Example 2.6 (Catenoid)**.** *The catenoid is given by*

$$\sqrt{x^2 + y^2} = \cosh z.$$

Instead of using (x, y)*, we take the* $u_1 = z$ *and the polar angle* $u_2 = \theta$ *as our local coordinates. Then* $x = r\cos\theta,\ y = r\sin\theta$ *imply that the surface is given by* $r = \cosh z$ *and hence,*

$$\mathbf{x} = (\cosh z \cos\theta, \cosh z \sin\theta, z).$$

Show that $H = 0$ *and* $K = -\cosh^{-4} z$ *and hence the principle curvatures are* $\kappa_1 = \cosh^{-2} z$ *and* $\kappa_2 = -\cosh^{-2} z$*.*

2.2 Gauss's Theorem Egregium

To say that $g = I$ at $p \in S$ to first order means that $g(p) = I$ and

$$g_{ij,k}(p) \equiv \frac{\partial g_{ij}}{\partial u_k}(p) = 0.$$

Theorem 2.7 (Theorema Egregium)**.** *The Gauss curvature* K *is intrinsic. Specifically, there are local coordinates* u_1, u_2 *about any point* p *in a* C^2 *surface* S *in* $\boldsymbol{R}^3$ *such that the first fundamental form* g *at* p *is* I *to first order. For such a coordinate system, the Gauss curvature is*

$$K = \frac{\partial^2 g_{12}}{\partial u_1 \partial u_2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2^2}. \tag{2.7}$$

Proof. Locally S is the graph of a function f over its tangent plane. Choose orthogonal coordinates x, y on the tangent plane. Then, since $f_x(0) = f_y(0) = 0$,

$$g = \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix} = I + \text{quadratic terms}.$$

Hence, g equals I to the first order in neighborhodd of $(0, 0)$. Now a direct computation shows

$$\frac{\partial^2}{\partial x \partial y}(g_{12}) = \frac{\partial^2}{\partial x \partial y}(f_x f_y) = f_{xx} f_{yy} + f_{xy}^2,$$

$$\frac{\partial^2}{\partial x^2}(g_{22}) = \frac{\partial^2}{\partial x^2}(f_y^2) = 2f_{xy}^2, \quad \frac{\partial^2}{\partial y^2}(g_{11}) = \frac{\partial^2}{\partial y^2}(f_x^2) = 2f_{xy}^2.$$

Therefore,

$$\frac{\partial^2 g_{12}}{\partial u_1 \partial u_2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2^2} = f_{xx} f_{yy} - f_{xy}^2 = \det \Pi = K$$

at $x = y = 0$. But any coordinates u_1, u_2 for which the metric at p is I agree with the special x, y coordinates on the tangent plane to first order will not change the result. □

Example 2.8. *The principle curvatures of a plat piece of paper are $\kappa_1 = 0 = \kappa_2$. Hence the Gauss curvature is $K = 0$. Assume the paper is bent but not stretched. It is clear that the distance on the surface between two points on the paper is not changed after the bending. Such a distance preserving deformation is called isometry. The Gauss's Theorem Egregium implies that the Gaous curvature K is not changed after such a deformation.*

Consider a cylinder of radius $r > 0$. The principle curvatures are $\kappa_1 = 1/r$ and $\kappa_2 = 0$. The mean curvature $H = 1/r$ is not intrinsic. The Gauss curvature $K = 0$ is intrinsic. The curvatures κ_1, κ_2 are extrinsic and are not determined by metric alone.

Example 2.9. *The surface of a sphere of radius $a > 0$ may given by local coordinates ϕ, θ, i.e.,*

$$x = a\sin\phi\cos\theta,\ \ y = a\sin\phi\sin\theta,\ \ z = a\cos\phi.$$

The metric g and its determinant are given by

$$g = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\phi \end{pmatrix}, \quad |g| = a^4\sin^2\phi.$$

The surface area A of the polar cap with angle ϕ_1 is given by

$$A = \int_0^{2\pi}\int_0^{\phi_1} a^2\sin\phi\, d\phi d\theta = \int_0^{2\pi}\int_0^{\phi_1}\sqrt{|g|}\, d\phi d\theta.$$

Confirm Theorem Egregium from this example.

2.3 The Gauss-Weingarten Equation

Recall for a space curve C. We can set up a local orthnormal system of coordinates T, unit tangent, N, unit normal, and B, unit binormal vectors. If s denote arc length along the curve C, the following Frenet-Serret formulas are satisfied:

$$\frac{dT}{ds} = \kappa N, \tag{2.8}$$

$$\frac{dN}{ds} = -\kappa T + \tau B, \tag{2.9}$$

$$\frac{dB}{ds} = -\tau N, \tag{2.10}$$

where κ is the curvature and τ is the torsion.

There are similar relations for surface in $\boldsymbol{R}^3$ which are called Gauss-Weingarten equations. Let u_1, u_2 be local local coordinates and the surface is given by $\mathbf{x} = (x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$. The the triple

$$\mathbf{x}_{,1},\ \mathbf{x}_{,2},\ \mathbf{n} = \frac{\mathbf{x}_{,1}\times\mathbf{x}_{,2}}{|\mathbf{x}_{,1}\times\mathbf{x}_{,2}|}$$

is called the Gauss frame. The frame satisfies the following Gauss-Weingarten equations.

$$\begin{aligned}
\mathbf{x}_{,11} &= \Gamma^1_{11}\mathbf{x}_{,1} + \Gamma^2_{11}\mathbf{x}_{,2} + L\mathbf{n},\\
\mathbf{x}_{,12} &= \Gamma^1_{12}\mathbf{x}_{,1} + \Gamma^2_{12}\mathbf{x}_{,2} + M\mathbf{n},\\
\mathbf{x}_{,22} &= \Gamma^1_{22}\mathbf{x}_{,1} + \Gamma^2_{22}\mathbf{x}_{,2} + N\mathbf{n},\\
\mathbf{n}_{,1} &= \alpha_1\mathbf{x}_{,1} + \beta_1\mathbf{x}_{,2},\\
\mathbf{n}_{,2} &= \alpha_2\mathbf{x}_{,1} + \beta_2\mathbf{x}_{,2},
\end{aligned}$$

where Γ^k_{ij} are called the Christoffel symbols of the second kind, which are given by

$$\begin{aligned}
2|g|\Gamma^1_{11} &= \quad GE_{,1} + FE_{,2} - 2FF_{,1}, & 2|g|\Gamma^2_{11} &= -FE_{,1} + EE_{,2} - 2EF_{,1},\\
2|g|\Gamma^1_{12} &= \quad GE_{,2} - FG_{,1}, & 2|g|\Gamma^2_{12} &= \quad EG_{,1} - FE_{,2},\\
2|g|\Gamma^1_{22} &= -FG_{,2} - GG_{,1} + 2GF_{,2}, & 2|g|\Gamma^2_{22} &= \quad EG_{,2} + FG_{,1} - 2FF_{,2},
\end{aligned}$$

and $\alpha_i,\ \beta_i$ are

$$\begin{aligned}
|g|\alpha_1 &= \quad MF - LG, & |g|\alpha_2 &= LF - ME\\
|g|\beta_1 &= \quad NF - MG, & |g|\beta_2 &= MF - NE.
\end{aligned}$$

Here we have reverted to the classical notation for the first and second fundamental forms for surfaces in $\boldsymbol{R}^3$:

$$g_{11} = E,\ g_{12} = F,\ g_{22} = G,\ \Pi_{11} = L,\ \Pi_{12} = M,\ \Pi_{22} = N.$$

2.4 Gauss-Codazzi Equations

In the case of space curves the Frenet-Serret equations say that given curvature and torsion, $\kappa(s)$ and $\tau(s)$, a space curve is determined uniquely up to position and orientation by the usual ODE existence, uniqueness theorem. We can ask the same question for surfaces. Is there a set of functions that define a surface to within a position in space? Specifically, do the first and second fundamental forms as functions of u_1, u_2 define a surface up to a position in space? Do the Gauss-Weingarten equations themselves provide the appropriate generalization of the Frenet-Serret formulas?

The answer is no and the reason is obvious from one glance at the Gauss-Weingarten system, i.e., we must satisfy equivalence of mixed partial derivatives

$$\mathbf{x}_{,112} = \mathbf{x}_{,121}, \quad \mathbf{x}_{,212} = \mathbf{x}_{,221}.$$

The mixed partials equalities given above when applied to the Gauss-Weingarten system satisfied if and only if the first and second fundamental forms satisfy the Codazzi equations:

$$\begin{aligned}
L_{,2} - M_{,1} &= L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11},\\
M_{,2} - N_{,1} &= L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12},
\end{aligned}$$

and the Gauss equation:

$$LN - M^2 = F((\Gamma^2_{22})_{,1} - (\Gamma^1_{22})_{,2} + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{12}).$$

But recall the Christoffel symbols are defined in terms of first order derivatives of the metric, i.e., the first fundamental form. The Gauss's Theorem Egregium relates the second derivatives of the first fundamental form to the Gauss curvature K. So it is no surprise that right hand side of Gauss equation above which is second order derivatives of the first fundamental form is nothing more than $K(EF - G^2)$. Hence the Gauss equation is

$$LN - M^2 = K(EF - G^2).$$

Theorem 2.10 (Fundamental Theorem of Surfaces)**.** *If the first and the second fundamental forms are sufficiently differentiable in u_1, u_2, satisfy Gauss-Codazzi equations and $EG - F^2 > 0,\ E > 0,\ G > 0$, then there exists a surface uniquely determined up to its position in space which has respectively the given first and second fundamental forms.*

3 Surfaces in $\boldsymbol{R}^n$

We extend the results of the previous section from surfaces in $\boldsymbol{R}^3$ to surfaces in $\boldsymbol{R}^n$. As before, choose orthonormal coordinates on $\boldsymbol{R}^n$ with the origin at p and S tangent to x_1x_2 plane at p. Locally S is the graph of a function $(f_3, \cdots, f_n) : T_pS(= \boldsymbol{R}^2) \to T_pS^\perp(= \boldsymbol{R}^{n-2})$. Let

$$\mathbf{x}(x_1, x_2) = (x_1, x_2, f_3(x_1, x_2), \cdots, f_n(x_1, x_2)\,).$$

Any unit vector $\mathbf{v}$ tangent to S at p, together with the vectors normal to S at p, spans a hyperplane, which intersects S in a curve. The curvature vector $\boldsymbol{\kappa}$ of this curve, which we call the curvature in the direction $\mathbf{v}$, is just the second derivative

$$\boldsymbol{\kappa} = (D^2\mathbf{f}(p))(\mathbf{v}, \mathbf{v}) = \mathbf{v}^T \Pi \mathbf{v}, \quad \Pi = \begin{pmatrix} \mathbf{x}_{,11} & \mathbf{x}_{,12} \\ \mathbf{x}_{,12} & \mathbf{x}_{,22} \end{pmatrix}. \tag{3.1}$$

Notice now the entries of the 2×2 matrix Π are not scalar but have values in $T_pS^\perp \subset \boldsymbol{R}^n$, where Π is called the second fundamental tensor of S at p in local coordinates x_1, x_2. More generally, the second fundamental tensor is written as

$$\Pi = (D^2\mathbf{f})_p.$$

Again the formula (3.1) is good only at the point where the surface S is tangent to the x_1, x_2 plane.

Example 3.1. *The generalization of this section should be identical to the ones in the previous section for surfaces in $\boldsymbol{R}^3$ or naturally connected to them. For $n = 3$, the second fundamental tensor is given as*

$$\Pi = \begin{pmatrix} (0, 0, \frac{\partial^2 f}{\partial x_1^2}) & (0, 0, \frac{\partial^2 f}{\partial x_1 \partial x_2}) \\ (0, 0, \frac{\partial^2 f}{\partial x_1 \partial x_2}) & (0, 0, \frac{\partial^2 f}{\partial x_2^2}) \end{pmatrix}.$$

Hence the trace of Π is

$$(0, 0, \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}) = (\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2})\mathbf{n} = H\mathbf{n} =: \mathbf{H},$$

where H is the mean curvature. We call $\mathbf{H}$ the mean curvature vector. The scalar quantity

$$(0, 0, \frac{\partial^2 f}{\partial x_1^2}) \cdot (0, 0, \frac{\partial^2 f}{\partial x_2^2}) - (0, 0, \frac{\partial^2 f}{\partial x_1 \partial x_2}) \cdot (0, 0, \frac{\partial^2 f}{\partial x_1 \partial x_2}) = \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \Big(\frac{\partial^2 f}{\partial x_1 \partial x_2}\Big)^2$$

is just the Gauss curvature.

In general case of $\boldsymbol{R}^n$ we still call $\mathbf{H} := \mathrm{trace}(\Pi)$ the mean curvature vector and $G := \det \Pi$ the Gauss curvature. Neither $\mathbf{H}$ nor G depend on the choice of orthonormal coordinates.

Remark 3.2 (Gram-Schmidt)**.** *The tangent space T_pS, its orthogonal compliment $T_pS^\perp$ and the projection function $\mathbf{P}$ change from point to point. In general one may always take an orthonormal coordinate system $\mathbf{x}_i$, $i = 1, \cdots n$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ are from the tangent space T_pS using the Gram-Schmidt process. For example let $\mathbf{y}_1, \mathbf{y}_2 \in T_pS$ be two linearly independent vectors. Then*

$$\mathbf{x}_1 := \mathbf{y}_1/\|\mathbf{y}_1\|, \quad \mathbf{x}_2 := (\mathbf{y}_2 - (\mathbf{y}_2 \cdot \mathbf{x}_1)\mathbf{x}_1)/\|\mathbf{y}_2 - (\mathbf{y}_2 \cdot \mathbf{x}_1)\mathbf{x}_1\|,$$

are unit vectors which are orthogonal to each other. Then, the projection mapping is given by

$$\mathbf{P}(\mathbf{x}) = \mathbf{x} - (\mathbf{x} \cdot \mathbf{x}_1)\mathbf{x}_1 - (\mathbf{x} \cdot \mathbf{x}_2)\mathbf{x}_2.$$

Proposition 3.3. *For any local coordinates u_1, u_2 and a point p in a C^2 surface $S \subset \boldsymbol{R}^n$, the second fundamental tensor* Π *at $p \in S$ is similar to*

$$g^{-1}\mathbf{P}(D^2\mathbf{x}) \cdot \mathbf{n} \equiv g^{-1}\begin{pmatrix} \mathbf{P}(\mathbf{x}_{,11}) & \mathbf{P}(\mathbf{x}_{,12}) \\ \mathbf{P}(\mathbf{x}_{,12}) & \mathbf{P}(\mathbf{x}_{,22}) \end{pmatrix}, \tag{3.2}$$

where $\mathbf{P}$ *denotes the projection onto* $T_pS^\perp$. *(Note that the second fundamental tensor* Π *is given by the parametrization of the surface given in Remark 3.2.) Consequently,*

$$\mathbf{H} = trace(g^{-1}\mathbf{P}(D^2\mathbf{x})\,) = \frac{\mathbf{P}(|\mathbf{x}_2|^2\mathbf{x}_{,11} - 2(\mathbf{x}_{,1}\cdot\mathbf{x}_{,2})\mathbf{x}_{,12} + |\mathbf{x}_{,1}|^2\mathbf{x}_{,22})}{|\mathbf{x}_{,1}|^2|\mathbf{x}_{,2}|^2 - (\mathbf{x}_{,1}\cdot\mathbf{x}_{,2})^2},$$

$$G = \det(g^{-1}\mathbf{P}(D^2\mathbf{x})\,) = \frac{\mathbf{P}(\mathbf{x}_{,11})\cdot\mathbf{P}(\mathbf{x}_{,22}) - |\mathbf{P}(\mathbf{x}_{,12})|^2}{|\mathbf{x}_{,1}|^2|\mathbf{x}_{,2}|^2 - (\mathbf{x}_{,1}\cdot\mathbf{x}_{,2})^2}.$$

The proof of the proposition is same as the case of $\boldsymbol{R}^3$, Proposition 2.4. Now we consider an example.

Example 3.4. Consider the surface $\{(z,w) \in \boldsymbol{C}^2 : w = e^z\}$. Write $x = Re(z)$ and $y = Im(z)$. Then,

$$w = e^{x+iy} = e^x(\cos y + i\sin y).$$

We may view the surface as a one in $\boldsymbol{R}^4$ taking $\mathbf{x} = (x, y, e^x\cos y, e^x\sin y)$. Then,

$$\mathbf{x}_{,1} = (1, 0, e^x\cos y, e^x\sin y),\ \ \mathbf{x}_{,2} = (0, 1, -e^x\sin y, e^x\cos y),$$
$$\mathbf{x}_{,11} = (0,0,e^x\cos y, e^x\sin y), \mathbf{x}_{,12} = (0,0,-e^x\sin y, e^x\cos y),\ \ \mathbf{x}_{,22} = (0,0,-e^x\cos y, -e^x\sin y).$$

Since $|\mathbf{x}_{,1}|^2 = |\mathbf{x}_{,2}|^2 = 1 + e^{2x}$, $\mathbf{x}_{,1}\cdot\mathbf{x}_{,2} = 0$, $\mathbf{x}_{,11} + \mathbf{x}_{,22} = 0$, we have $\mathbf{H} = 0$.

To compute the Gauss curvature G, we need to find a orthogonal basis for T_pS. Since $\mathbf{x}_{,1}\cdot\mathbf{x}_{,2} = 0$, we take these vectors. Then,

$$\begin{aligned}\mathbf{P}(\mathbf{x}_{,11}) \quad &= \mathbf{x}_{,11} - \frac{\mathbf{x}_{,11}\cdot\mathbf{x}_{,1}}{1+e^{2x}}\mathbf{x}_{,1} - \frac{\mathbf{x}_{,11}\cdot\mathbf{x}_{,2}}{1+e^{2x}}\mathbf{x}_{,2} = \mathbf{x}_{,11} - \frac{e^{2x}}{1+e^{2x}}\mathbf{x}_{,1} - \frac{0}{1+e^{2x}}\mathbf{x}_{,2} \\ &= \frac{1}{1+e^{2x}}(-e^{2x}, 0, e^x\cos y, e^x\sin y).\end{aligned}$$

Similarly,

$$\mathbf{P}(\mathbf{x}_{,12}) = \frac{1}{1+e^{2x}}(0, -e^{2x}, -e^x\sin y, e^x\cos y),\ \ \mathbf{P}(\mathbf{x}_{,22}) = \frac{1}{1+e^{2x}}(e^{2x}, 0, -e^x\cos y, -e^x\sin y).$$

Hence

$$G = -2e^{2x}/(1+e^{2x})^3$$

Gauss's Theorem Egregium still holds. In other words, if the matric in the local coordinates satisfies $g = I$ to first order, then

$$G = \frac{\partial^2 g_{12}}{\partial u_1\partial u_2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2^2}.$$

The metric of the local coordinate in the example satisfies $g = \frac{1}{1+e^{2x}}I$ to the first order (identically in this example) and

$$\frac{\partial^2 g_{12}}{\partial u_1\partial u_2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2^2} = -2e^{2x} = G(1+e^{2x})^3.$$

Lemma 3.5. *If $g = aI$ to the first order, then*

$$G = a^3\Big(\frac{\partial^2 g_{12}}{\partial u_1\partial u_2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u_1^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u_2^2}\Big).$$

4 m-dimensional surfaces in $\boldsymbol{R}^n$

We now extend the theory to C^2 m-dimensional surfaces $S \subset \boldsymbol{R}^n$. As before, origin at $p \in S$ and S tangent to the $x_1, \cdots, x_m$ plane at p. Locally S is the graph of the function

$$(f_{m+1}, \cdots, f_n) : T_pS(\equiv \boldsymbol{R}^m) \to (T_pS)^\perp(\equiv \boldsymbol{R}^{n-m}).$$

Again a unit vector $\mathbf{v}$ tangent to S at p together with vectors normal to S at p spans a plane, which intersects S in a curve. The curvature vector $\boldsymbol{\kappa}$ of the curve which we call the curvature in direction $\mathbf{v}$ is just the second derivative

$$\boldsymbol{\kappa} = (D^2\mathbf{f})_p(\mathbf{v}, \mathbf{v}) = \mathbf{v}^T(D^2\mathbf{f}(p))\mathbf{v},$$

where

$$\mathbf{f} = (x_1, \cdots, x_m, f_{m+1}, \cdots, f_n).$$

The bilinear form $(D^2\mathbf{f})_p$ on T_pS with values in $(T_pS)^\perp$ is called the second fundamental tensor Π of S at p given in coordinates as a symmetric $m \times m$ matrix with entries in $(T_pS)^\perp$:

$$\Pi = \Big(\frac{\partial^2 \mathbf{f}}{\partial x_i \partial x_j}\Big).$$

The trace of Π is called the mean curvature vector $\mathbf{H}$. Sometimes people use a scalar factor and write $\mathbf{H} = \mathrm{trace}\Pi$.

In the special case of hypersurfaces when $n = m+1$, Π is just the unit normal $\mathbf{n}$ times a scalar matrix, called the second fundamental form and also denoted by Π. $\mathbf{H} = H\mathbf{n}$ where H is the scalar mean curvature. If we choose coordinates to make the second fundamental form diagonal, then

$$\Pi = (\kappa_1, \cdots \kappa_m)I, \quad H = \kappa_1 + \cdots + \kappa_m.$$

4.1 moment

Lemma 4.1. *Let $S \subset \boldsymbol{R}^3$ be a 2-dimensional C^2 surface given by $x_3 - f(x_1, x_2) = 0$ and $\mathbf{n} = (n_1, n_2, n_3)$ be a unit vector field which is orthogonal to the surface. Then,*

$$n_{1,1} = -\kappa_1, \quad n_{2,2} = -\kappa_2, \quad n_{3,3} = 0, \quad i.e., \quad H = -\mathrm{div}(\mathbf{n}).$$

Proof. The unit normal vector $\mathbf{n}$ is given by

$$\mathbf{n} = (-f_{x_1}, -f_{x_2}, 1)\Big/\sqrt{1 + f_{x_1}^2 + f_{x_2}^2}.$$

Clearly, $n_{3,3} = 0$. Consider a curve $\mathbf{x}(p_1 + t, p_2, f(p_1 + t, p_2))$ on the surface. Then the curve is on a plane normal to x_2 axis. Hence the formula (2.9) is written as

$$\frac{d\mathbf{n}}{ds} = \frac{d\mathbf{n}}{dx_1}\frac{dt}{ds} = -\kappa_1 T - = \kappa_1(1, 0, f_{x_1})\Big/\frac{ds}{dt}.$$

Therefore, from the first component, we obtain $n_{1,1} = -\kappa_1$. Similarly one obtains $n_{2,2} = -\kappa_2$ and hence the scalar mean curvature is $H = \kappa_1 + \kappa_2 = -\mathrm{div}(\mathbf{n})$ □

Theorem 4.2. *Let S be a C^2 m-dimensional surface in $\boldsymbol{R}^n$. The first variation of area S with respect to a compactly supported C^2 vector field $\mathbf{v}$ on S is given by integrating $\mathbf{v}$ against the mean curvature vector:*

$$\delta^1(S) = -\int_S \mathbf{v} \cdot \mathbf{H} d\sigma.$$

4.2 Sectional and Riemann curvature

The sectional curvature K_{sect} of S at p assign to every 2-plane $P \subset T_pS$ the Gauss curvature of the 2-dimensional surface

$$S \cap (P \oplus T_pS^{\perp}).$$

This is easier to state than to sketch since we can only draw a sketch where the tangent space T_pS is already a two dimensional surface. If $\mathbf{v}, \mathbf{w}$ are 2 vectors that give an orthonormal basis for the 2-plane P, we can find the components of Π by just remembering that components of a matrix are defined by how the liniear operator acts on basis vectors, i.e.,

$$\Pi = \begin{pmatrix} \Pi(\mathbf{v}, \mathbf{v}) & \Pi(\mathbf{v}, \mathbf{w}) \\ \Pi(\mathbf{v}, \mathbf{w}) & \Pi(\mathbf{w}, \mathbf{w}) \end{pmatrix} \tag{4.1}$$

and hence taking determinant

$$K_{sect}(P) = \Pi(\mathbf{v}, \mathbf{v})\Pi(\mathbf{w}, \mathbf{w}) - \Pi(\mathbf{v}, \mathbf{w})\Pi(\mathbf{v}, \mathbf{w}). \tag{4.2}$$

(Here, we are abusing the notations. The Π on the left side of (4.1) is the second fundamental tensor for the 2-dimensional surface $S \cap (P \oplus T_pS^{\perp})$ and the ones on the right side is the second fundamental tensor for the whole surface S.)

Example 4.3. *Consider a 2-dimensional surface $S \subset \boldsymbol{R}^n$. Let*

$$\Pi = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Since T_pS is also a 2-dimensional plane, the only case we may take is $\mathbf{v} = \mathbf{e}_1 = (1, 0)$ and $\mathbf{w} = \mathbf{e}_2 = (0, 2)$. Therefore,

$$K_{sect}(P) = a_{11}a_{22} - a_{12}a_{21}.$$

Q: *Show that the sectional curvature is well defined by showing that the value is independent of choice of the orthonormal basis $\mathbf{v}$ and $\mathbf{w}$.*

Now we compute the sectional curvature in (4.2) explicitly. Let $\mathbf{v} = (v_1, \cdots, v_n$, $\mathbf{w} = (w_1, \cdots, w_n)$ and $\Pi = (a_{ij})$. Then,

$$\mathbf{v}^T\Pi\mathbf{v} = \sum\sum a_{ik}v_iv_k, \quad \mathbf{w}^T\Pi\mathbf{w} = \sum\sum a_{jl}w_jw_l, \quad \mathbf{v}^T\Pi\mathbf{w} = \sum\sum a_{il}v_iw_l.$$

Therefore, the formula (4.2) becomes

$$K_{sect}(P) = (\sum\sum a_{ik}v_iv_k)(\sum\sum a_{jl}w_jw_l) - (\sum\sum a_{il}v_iw_l)(\sum\sum a_{jk}v_jw_k).$$

Using summation cancelation for repeated indices gives

$$K_{sect}(P) = R_{ijkl}v_iw_jv_kw_l, \quad R_{ijkl} = a_{ik}a_{jl} - a_{il}a_{jk}.$$

The tensor $\mathbf{R}$ is called the Riemann curvature tensor. If we write

$$\Pi = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

we see

$$R_{1234} = a_{13}a_{24} - a_{14}a_{23}.$$

One may simply think that R_{ijkl} is the determinant of the 2×2 matrix whose elements are from i'th and j'th rows and k'th and l'th columns. If $i > j$ or $k > l$, then one should multiply -1. We also see that the Riemann curvature tensor is just the 2×2 matirx of the second fundamental tensor Π.

Immediately, we see exchanging 2 columns or rows changes the sign, i.e.,

$$R_{jikl} = R_{ijlk} = -R_{ijkl}. \tag{4.3}$$

Since Π is symmetric, we have

$$R_{ijkl} = R_{klij}. \tag{4.4}$$

In addition we can check that the permutation of the last 3 indices makes zero, which is called Brancki's identity:

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0. \tag{4.5}$$

(This identity does not look trivial. Is there a simple proof?) The Ricci curvature is

$$R_{jl} = R_{ijil} \tag{4.6}$$

The scalar curvature is

$$R = R_{ii}.$$

(Again using summation curvature.)

5 Covariant derivatives

Let's start with a simple example. Consider the unit tangent vector $T(t) = (-\sin t, \cos t)$ to the unit circle $\mathbf{x}(t) = (\cos t, \sin t)$. As we move around the circle the unit tangent vector changes with derivative

$$\frac{dT}{dt}(t) = (-\cos t, -\sin t),$$

which of course is no longer in the tangent line to the circle. The covariant derivative is just the projection of the derivative back into the tangent line. In this example, since $\frac{dT}{dt}(t)$ is normal to the tangent line, the covariant derivative is zero for all t. The covariant derivative of any unit tangent vector field is obviously zero.

A second example is $a(t) = (0, 1 + \sin t)$ at $\mathbf{x}(t) = (\cos t, \sin t)$. Then, the regular derivative is $b(t) = a'(t) = (0, \cos t)$. The covariant derivative is $c(t) = (b(t) \cdot T(t))T(t) = \cos^2(t)T(t)$.

Of course the "intrinsic" notion is the covariant derivative since we then are differentiating only along the surface, in this case the unit circle. Another way to think about this is in the following simple picture

6 Some casual remarks on general relativity

First let's recall some of the well-known observations that can motivate the search for a new metric g_{ij} to resolve the appearance of gravitational force. If we consider any metric of the form

$$ds^2 = a_1 dx^2 + a_2 dy^2 + a_3 dy^2 + a_4 dt^2,$$

then the length of the path $\mathbf{x}(t) = (x(t), y(t), z(t), t)$ in space-time is

$$J = \int_{t_0}^{t_1} |\frac{ds}{dt}| dt = \int_{t_0}^{t_1} \sqrt{a_1(\frac{dx}{dt})^2 + a_2(\frac{dy}{dt})^2 + a_3(\frac{dz}{dt})^2 + a_4}\ dt.$$

The path with the shortest distance is obtained by the usual variational method. Set

$$x(t) = \bar{x}(t) + \epsilon x_1(t), y(t) = \bar{y}(t) + \delta y_1(t), z(t) = \bar{z}(t) + \gamma z_1(t)$$

and

$$J(\epsilon, \delta, \gamma) := \int_{t_0}^{t_1} \sqrt{a_1(\frac{d\bar{x}}{dt} + \epsilon\frac{dx_1}{dt})^2 + a_2(\frac{d\bar{y}}{dt} + \delta\frac{dy_1}{dt})^2 + a_3(\frac{d\bar{z}}{dt} + \gamma\frac{dz_1}{dt})^2 + a_4}\ dt.$$

Here we assume that $(\bar{x}(t), \bar{y}(t), \bar{z}(t), t)$ is the length minimizing path that connecting two points $\mathbf{x}(t_0)$ and $\mathbf{x}(t_1)$. Then $(x(t), y(t), z(t))$ is considered as its perturbation in the direction $\mathbf{x}_1 = (x_1, y_1, z_1)$ where $\mathbf{x}_1(t_0) = \mathbf{x}_1(t_1) = 0$. Then, $J(0,0,0)$ is the minimum value and so

$$\frac{\partial J}{\partial \epsilon} = \frac{\partial J}{\partial \delta} = \frac{\partial J}{\partial \gamma} = 0 \qquad \text{at} \quad (0,0,0).$$

Therefore,

$$0 = \frac{\partial J}{\partial \epsilon}(0,0,0) = \int_{t_0}^{t_1} \Big(a_1(\frac{d\bar{x}}{dt})^2 + a_2(\frac{d\bar{y}}{dt})^2 + a_3(\frac{d\bar{z}}{dt})^2 + a_4\Big)^{-1/2} \frac{d\bar{x}}{dt}\frac{dx_1}{dt} dt$$

Since it holds for all x_1, the integration by parts gives

$$\frac{d}{dt}\Big[\Big(a_1(\frac{d\bar{x}}{dt})^2 + a_2(\frac{d\bar{y}}{dt})^2 + a_3(\frac{d\bar{z}}{dt})^2 + a_4\Big)^{-1/2}\frac{d\bar{x}}{dt}\Big] = 0. \tag{6.1}$$

Consider the motion of a particle in space with constant velocity $\mathbf{v} = (a, b, c)$

$$x(t) = at + x_0,\ y(t) = bt + y_0,\ z(t) = ct + z_0.$$

Then one can easily see that (6.1) is satisfied and hence gives the shortest path length. However, it contradicts for an accelerated system. ...

Finally I quote from Einstein's address in Kyoto (Dec. 1922) as in given on P. 2 of "Subtle is the Lord. The science and life of Albert Einstein" by Abraham Pais, Oxford Univ. Press (1982).

"If all [accelerated] systems are equivalent, then Euclidean geometry cannot hold in all of them. To throw out geometry and keep [physical] laws is equivalent to describing thoughts without words. We must search for words before we can express thoughts. What must be search for at this point? This problem remained unsolvable to me until 1912, when I suddenly realized that Gauss's theory of surfaces holds the key for unlocking this mystery. I realized the Gauss's surface coordinates had a profound significance. However, I did not know at that time the Riemann had studied the foundation of geometry in an even more profound way. I suddenly remembered Gauss's theory was contained in the geometry course given by Geiser when I was a student.... I realized that the foundation of geometry have physical significance. My dear friend Grossman was there when I returned from Pragul to Zürich. From him I learned for the first time about Ricci and later about Riemann. So I asked my friend whether any problem could be solved by Riemann's theory, namely, whether the invariants of the line element could completely determine the quantities I have been looking for." (Of course what Einstein means by line element is just the metric g_{ij}.)