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From Boltzmann to Euler: Hilbert's 6th problem revisited

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ABSTRACT

This article addresses the hydrodynamic limit of the Boltzmann equation, namely the compressible Euler equations of gas dynamics. An exact summation of the Chapman–Enskog expansion originally given by Gorban and Karlin is the key to the analysis. An appraisal of the role of viscosity and capillarity in the limiting process is then given where the analogy is drawn to the limit of the Korteweg–de Vries–Burgers equations as a small parameter tends to zero.

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Introduction

In his 1900 address to the International Congress of Mathematicians in Paris, Hilbert proposed the problem of developing "mathematically the limiting process …which leads from the atomistic view to the laws of motion of continua" [1]. Historically a canonical interpretation of this "6th problem of Hilbert" has been taken to mean passage from the kinetic Boltzmann equation for a rarefied gas to the continuum Euler equations of compressible gas dynamics as the Knudsen number ε approaches zero. As of this moment, success in the problem has been limited to cases where the limiting continuum equations possess smooth solutions, i.e. before shock formation. (An excellent up to date survey may be found in the book of Saint-Raymond [1].) The goal of this short note is to address one question: why? More specifically is there an inherent mathematical reason that makes Hilbert's problem unattainable and can we pinpoint this reason in a simple mathematical form? Needless to say, I believe that the answer is that the difficulty with Hilbert's problem can indeed be readily identified and an elementary mathematical explanation is available: the limiting hydrodynamics is analogous to passage to the scalar conservation law $u_t + uu_x = 0$ from the Korteweg–de Vries–Burgers equation

$$u_t + uu_x = \varepsilon u_{xx} - K\varepsilon^2 u_{xxx} \quad \text{as } \varepsilon \to 0.$$

In this elementary example there is a competition between the viscosity ε and capillarity $K\varepsilon^2$, and the delicate balance between the two determines whether a subsequence of solutions of the KdV–Burgers equation will converge to a solution of $u_t + uu_x = 0$. The rest of the paper is devoted to deriving this analogy. The key tool is reformulating the hydrodynamic limit problem for the Boltzmann equation as a limit process for a local system of balance laws. This is done via the elegant exact summation of the Chapman–Enskog expansion for Grad's moment equations originally given by Gorban and Karlin [2–4].

The paper is divided into five sections after this Introduction:

1. Hilbert's problem and the Chapman–Enskog expansion.
2. Results of Gorban and Karlin.
3. The energy identity.
4. Implications of Gorban and Karlin's summation for Hilbert's 6th problem
5. References.

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1. Hilbert's problem and the Chapman–Enskog expansion

Since the canonical starting point for resolution of Hilbert's challenge is the Boltzmann equation, we begin there as well. The Boltzmann equation is

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f = \frac{Q(f, f)}{\varepsilon} \tag{1}$$

where $f = f(t, x, \xi)$ is the probability of finding a molecule of gas at point $x \in \mathbb{R}^3$, at time t, moving with velocity $\xi \in \mathbb{R}^3$. We note each ξ_i varies from $-\infty$ to ∞ and hence particles are allowed to have infinite velocities. The function f determines macroscopic fluid variables of density, momentum and temperature via its moments. Denote the density, momentum and temperature by ρ, $\rho\mathbf{u}$ and Θ. The Chapman–Enskog expansion is a formal power series in the Knudsen number $\varepsilon > 0$ for the function f in terms of these macroscopic variables (which we denote by M). Hence we write

$$f_{CE}(M, \xi) = f^{(0)}(M, \xi) + \varepsilon f^{(1)}(M, \xi) + \varepsilon^2 f^{(2)}(M, \xi) + \cdots \tag{2}$$

where $f^{(0)}(M, \xi)$ is the usual local Maxwellian. Furthermore, truncation at the zeroth order yields the balance laws of mass, momentum and energy for an elastic fluid, i.e. compressible gas dynamics of an ideal gas. Truncation at order ε yields the Navier–Stokes–Fourier equations, truncation at order ε^2 yields the Burnett equations, truncation at order ε^3 yields the super-Burnett equations and so forth.

The success of the Chapman–Enskog expansion in delivering the well known Navier–Stokes–Fourier theory at order ε has motivated many to view the Navier–Stokes–Fourier theory as fundamental in computing shock structure (see, for example, Courant and Friedrichs [5]). But as seen from a simple one-dimensional analogy, this justification is questionable. If one wishes to determine the shock structure for the scalar conservation law

$$u_t + uu_x = 0,$$

by imposition of a viscous term one gets

$$u_t + uu_x = \varepsilon u_{xx},$$

i.e. Burgers' equation. Next we investigate traveling wave solutions of the Burgers' equation

$$u = u\left(\frac{x - ct}{\varepsilon}\right)$$

and thereby we recover the ordinary differential equation

$$-cu' + uu' = u''.$$

Thus in any study of non-smooth solutions to our original scalar conservation laws via imposition of higher gradient terms we see that there is no concept of "small" or "negligible" higher derivative terms, i.e. all terms are *a priori* of the same magnitude. The ε has been scaled out. Hence any study of non-smooth hydrodynamics via a truncation of the Chapman–Enskog expansion, while convenient, is an illegitimate use of the Boltzmann equation. Nevertheless the Chapman–Enskog expansion is an appealing tool since it allows us to cast Hilbert's question in the language and tools of partial differential equations. But if truncations of C–E (Chapman–Enskog) expansion are illegitimate for non-smooth solutions of the fluid equations, that leaves us with only one course, i.e. summation of the entire Chapman–Enskog expansion. This is exactly what Gorban and Karlin have done in a remarkable series of papers [2–4]. In Section 2 we review their work.

2. Results of Gorban and Karlin

Gorban and Karlin [2–4] follow Grad and take the first 13 moments of the Boltzmann equations and then close the system by Grad's closure rule for f. This yields Grad's 13-moment system, which when linearized about the rest state is

$$\begin{aligned}
&\partial_t \rho = -\nabla \cdot \mathbf{u}, \\
&\partial_t \mathbf{u} = -\nabla \rho - \nabla \Theta - \nabla \cdot \boldsymbol{\sigma}, \\
&\partial_t \Theta = -\frac{2}{3}\left(\nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{q}\right), \\
&\partial_t \boldsymbol{\sigma} = -\left((\nabla \mathbf{u}) + (\nabla \mathbf{u})^T - \frac{2}{3}\nabla \cdot \mathbf{u}\,\mathbf{I}\right) - \frac{2}{3}\left((\nabla \mathbf{q}) + (\nabla \mathbf{q})^T - \frac{2}{3}\nabla \cdot \mathbf{q}\,\mathbf{I}\right) - \boldsymbol{\sigma}, \\
&\partial_t \mathbf{q} = -\frac{5}{3}\nabla \Theta - \nabla \cdot \boldsymbol{\sigma} - \frac{2}{3}\mathbf{q}, \\
&p = \rho + \Theta.
\end{aligned}$$

A simplification is obtained in the 10-moment theory in one space dimension (which is sufficient for our purposes):

$$\partial_t p = -\frac{5}{3}\partial_x u, \tag{3}$$

$$\partial_t u = -\partial_x p - \partial_x \sigma, \tag{4}$$

$$\partial_t \sigma = -\frac{4}{3}\partial_x u - \frac{\sigma}{\varepsilon}. \tag{5}$$

Here we have in fact rescaled $x = x'/\varepsilon$, $t = t'/\varepsilon$ and then dropped the prime. In this way the role of the Knudsen number ε becomes apparent, i.e. it is an ordering tool for the Chapman–Enskog expansion. In fact we write the C–E expansion for (3)–(5) as

$$\sigma_{CE} = \varepsilon\sigma^{(0)} + \varepsilon^2\sigma^{(1)} + \varepsilon^3\sigma^{(2)} + \cdots$$

where the $\sigma^{(n)}$ depend on p, u and their space derivatives. As shown by Gorban and Karlin,

$$\sigma_{CE} = -\frac{4}{3}\left(\varepsilon\partial_x u + \varepsilon^2\partial_x p + \frac{\varepsilon^3}{3}\partial_x^3 u + \cdots\right)$$

and truncation at orders ε, ε^2, ε^3 yields the dispersion relations

$$\omega_\pm = -\frac{2}{3}k^2 \pm \frac{i|k|}{3}\sqrt{4k^2 - 15}$$

(Navier–Stokes order),

$$\omega_\pm = -\frac{2}{3}k^2 \pm \frac{i|k|}{3}\sqrt{8k^2 + 15}$$

(Burnett order),

$$\omega_\pm = -\frac{2}{9}k^2(k^2 - 3) \pm \frac{i|k|}{9}\sqrt{4k^6 - 24k^4 - 72k^2 - 135}$$

(super-Burnett order).

Note that at super-Burnett order we have a Bobylev instability [12] for $k^2 > 3$, k being the frequency in Fourier space.

Of course the goal of Gorban and Karlin was summation not truncation. They accomplish the summation of the C–E expansion for (3)–(5) by taking the Fourier transform of σ_{CE} and summing the series. This yields

$$\hat{\sigma}_{CE} = \sum_{n=0}^{\infty} -ika_n(-k^2)^n\hat{u} + \sum_{n=0}^{\infty} -k^2 b_n(-k^2)^n\hat{p} \tag{6}$$

when the Fourier transform is defined by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx.$$

It is convenient to write (6) as

$$\hat{\sigma}_{CE} = -ikA(k^2)\hat{u} - k^2B(k^2)\hat{p}$$

where

$$A(k^2) = \sum_{n=0}^{\infty} -ika_n(-k^2)^n,$$

$$B(k^2) = \sum_{n=0}^{\infty} b_n(-k^2)^n.$$

The good fortune in this example is that the sums A, B are related via the formula

$$A = \frac{B}{1 - k^2B} \tag{7}$$

and that if B is written as $C = k^2B$, then C satisfies the fundamental cubic equation

$$-\frac{5}{3}(1 - C)^2\left(C + \frac{4}{5}\right) - \frac{C}{k^2} = 0. \tag{8}$$

Eq. (8) has one real and two complex roots. The real root is the one of interest to us and is negative for $k^2 > 0$, $C(0) = 0$, and monotone decreasing in k^2, with asymptotic limit $C \to -4/5$ as $k^2 \to \infty$. Hence A, B are known functions $A, B < 0$ for $k^2 > 0$.

3. The energy identity

One implication of the Gorban and Karlin summation is the energy identity

$$\frac{1}{2}\partial_t \int_{-\infty}^{\infty} \frac{3}{\rho}|p|^2 + |u|^2 dx + \frac{1}{2}\partial_t \int_{-\infty}^{\infty} -\frac{3}{\rho}k^2 B(k^2)|\hat{p}|^2 dk = \int_{-\infty}^{\infty} k^2 A(k^2)|\hat{u}|^2 dk. \quad (9)$$

(This follows from taking the Fourier transform of (3), (4) and Parseval's identity.) Since A, B are negative for $k^2 > 0$ we can interpret (9) as

$$\partial_t(\text{MECHANICAL ENERGY}) + \partial_t(\text{CAPILLARITY ENERGY}) = \text{VISCOUS DISSIPATION}$$

just as given in Korteweg's theory of capillarity [6,7]. Thus exact summation of the Chapman–Enskog expansion has yielded a non-local version of Korteweg's theory and not Navier–Stokes theory. Why is the theory non-local? The answer is that we have non-localness not because of any physical mechanism but because of the truncation of our moment expansion. In fact Boillat and Ruggeri [8] have shown that maximum wave speeds for moment truncation satisfy the inequality

$$V_{\max}^2 \geq \frac{6}{5}\sqrt{\frac{5}{3}\frac{k}{m}\Theta}(N - 1/2)$$

where N = number of moments.

Hence, we expect, as the number of moments goes to infinity, our non-local Korteweg theory to approach a linearization of Korteweg's local theory:

$$\partial_t \rho + \partial_i(\rho u_i) = 0,$$
$$\partial_t(\rho u_i) + \partial_j(\rho u_i u_j) = \partial_j T_{ij},$$

where the Cauchy stress for Korteweg's theory is

$$\begin{aligned}
&T = T^E + T^V + T^K,\\
&T_{ij}^E = -\rho\psi'(\rho)\delta_{ij}, \qquad \rho^2\psi'(\rho) = p(\rho),\\
&T_{ij}^V = \lambda(\mathrm{tr}D)\delta_{ij} + 2\mu D_{ij}\\
&D_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right),\\
&\lambda = -\frac{2}{3}\mu, \quad \mu > 0,\\
&T_{ij}^K = \alpha\rho\partial_i(\rho\partial_j\rho) - \alpha\rho\partial_i\rho\partial_j\rho, \quad \alpha > 0.
\end{aligned} \quad (10)$$

The full nonlinear energy balance equation is

$$\begin{aligned}
&\partial_t\left(\frac{1}{2}\rho|u|^2 + \rho\psi(\rho) + \frac{\alpha}{2}\rho\partial_i\rho\partial_i\rho\right) + \partial_j\left[u_j\left(\frac{1}{2}\rho|u|^2 + \rho\psi(\rho) - \frac{\alpha}{2}\rho\partial_i\rho\partial_i\rho\right)\right]\\
&\quad +\partial_j[\alpha\rho(\partial_t\rho\partial_j\rho + u_i\partial_i\rho\partial_j\rho)] + \partial_j(u_i T_{ij}) + \mu(\partial_j(u_i\partial_i u_j) - \partial_i(u_i\partial_j u_j))\\
&\ = -(\lambda+\mu)(\partial_i u_i)^2 - \mu(\partial_j u_i)(\partial_j u_i) \leq 0.
\end{aligned} \quad (11)$$

(Here repeated indices imply summation.)

In summary, Gorban and Karlin's summation has shown us that we may reasonably conjecture that the sum of the Chapman–Enskog expansion will yield a local version of Korteweg's theory of capillarity.

4. Implications of Gorban and Karlin's summation for Hilbert's 6th problem

The implication of the exact summation of C–E by Gorban and Karlin now becomes clear. The whole issue may be seen in Eq. (11), the energy balance. If we put the Knudsen number scaling into (11), the coefficient α is actually a term $\alpha_0\varepsilon^2$ and to recover the classical balance of energy of the Euler equation would require the sequence

$$\varepsilon^2\rho^\varepsilon\partial_i\rho^\varepsilon\partial_i\rho^\varepsilon \to 0$$

in the sense of distributions as $\varepsilon \to 0$. This would require a strong interaction with viscous dissipation. The natural analogy is given by the use of the KdV–Burgers equation:

$$u_t + uu_x = \varepsilon u_{xx} - K\varepsilon^2 u_{xxx} \quad (12)$$

where at a more elementary level we see the competition between viscosity and capillarity. The result in (12) is known but far from trivial. Specifically in the absence of viscosity we have the KdV equation

$$u_t + uu_x = -K\varepsilon^2 u_{xxx} \quad (13)$$

and we know from the results of Lax and Levermore [9] that as $\varepsilon \to 0$ the solution of (13) will not approach the solution of the conservation law

$$u_t + uu_x = 0 \tag{14}$$

after the breakdown time of smooth solutions of (14). On the other hand, addition of viscosity which is sufficiently strong, i.e. K sufficiently small in (12), will allow passage as $\varepsilon \to 0$ to a solution of (14). This has been proven in the paper of Schonbek [10]. So, the next question is whether we are in the Lax–Levermore case (13) or the Schonbek case (12) with K sufficiently small. In my paper [11] I noted the C–E summation of Gorban and Karlin for the Grad 10-moment system leads to a rather weak viscous dissipation, i.e. Eqs. (5.10), (5.11) of [11]. At the moment, this is all we have to go on and I can only conclude that things are not looking too promising for a possible resolution of Hilbert's 6th problem. It appears that in the competition between viscosity and capillarity (mathematically, dissipation of oscillation versus generation of oscillation), capillarity has become a very dogged opponent, and the capillarity energy will not vanish in the limit as $\varepsilon \to 0$. Hilbert's hope may have been justified in 1900, but as a result of the work of Gorban, Karlin, Lax, Levermore, and Schonbek, I think that serious doubts are now apparent.

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