Lectures on the Isometric Embedding Problem

$(M^n, g) \to \mathbb{R}^m, \quad m = \frac{n}{2}(n+1)$

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Abstract

This work derives the basic balance laws of Codazzi, Ricci, and Gauss for the isometric embedding of an n-dimensional Riemannian manifold into $m = \frac{n}{2}(n+1)$ dimensional Euclidean space. It is shown how the balance laws can be placed in quasi-linear symmetric form and weak solutions for the linearized problem can be obtained from the Lax-Milgram theorem.

0 Introduction

While the classical isometric embedding for 2-dimensional Riemannian manifold into 3-dimensional Euclidean space is quite well studied and discussed eloquently in the recent book of Han and Hong [1], the more general case of embedding n-dimensional Riemannian manifolds into $\frac{n}{2}(n+1)$ Euclidean space has a comparatively small literature. The main results in the case of $n = 3$ have been given in the papers of Bryant, Griffiths and Yang [2], Nakamura and Maeda [3, 4], Goodman and Yang [5], and most recently Poole [6] and the related and more general case $m \geq 3$ by Han and Klum [8]. All of these papers rely on a linearization of the full non-linear system

$$\partial_i y \cdot \partial_j y = g_{ij}$$

for the embedding problem where g_{ij} is the given metric of Riemannian manifold and y is the desired embedding. Applied analysts more familiar with continuum mechanics and quasi-linear balance laws might find a presentation of the embedding problem in a symmetric quasi-linear form more appealing since in that context there is an extensive literature originating with Friedrichs [7] and others which is nicely presented by Han and Hong [1]. However since as far as I know no-one has shown that the isometric embedding problem $(M^n, g) \to \mathbb{R}^\mu$, $m = \frac{n}{2}(n+1)$, the case of critical Janet dimension m, can be written in symmetric quasi-linear form, I thought it worth developing in a self-contained set of lecture notes. These notes are presented here.

The table of contents is as follows.

1 The basic isometric embedding equations

Let $(\mathbf{X}, g)$ be an n-dimensional Riemannian manifold. Roughly Riemann was thinking about extending the idea of a surface in Euclidean space without the necessity of having an underlying Euclidean space. If indeed the manifold $(\mathbf{X}, g)$ can be embedded globally into $\mathbb{R}^m$ (or locally in which case the word "immersion" is used) then we can write a coordinate patch $(y^1, \ldots, y^m)$ on the manifold.

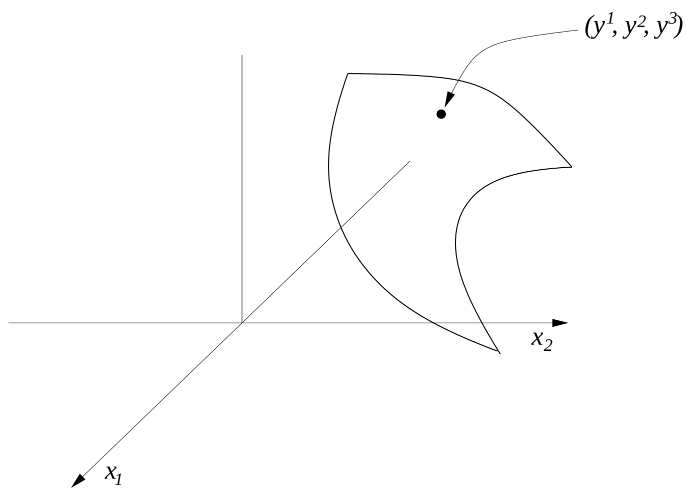


Figure 1: (M^2, g) embedded in $\mathbb{R}^3$

Example (M^2, g) embedded into $\mathbb{R}^3$

$$n = 2, \qquad m = 3$$

Distances on the manifold are computed according to the metric g

$$\partial_i y \cdot \partial_j y = g_{ij}, \qquad 1 \leq i, j \leq n, \tag{1.1}$$

where $\partial_i = \dfrac{\partial}{\partial x_i}$, x_i are the local coordinates.

The two dimensional Riemannian manifold when viewed as a surface in $\mathbb{R}^3$ is very instructive.

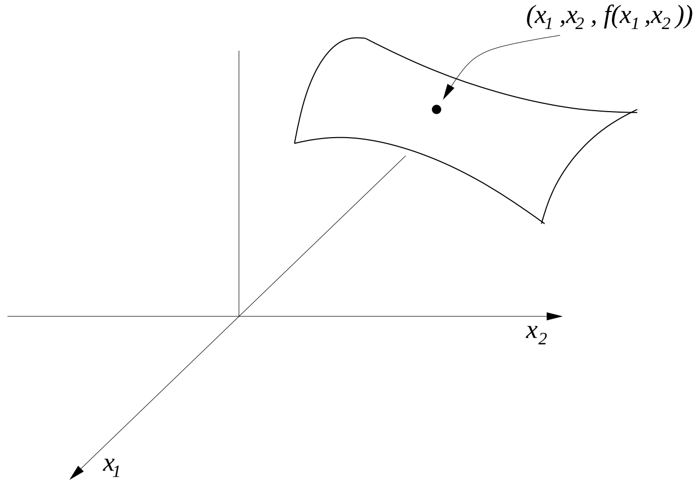


Figure 2: $y^1 = x_1,\ y^2 = x_2, y^3 = f(x_1, x_2)$

In introductory courses we write the distance along the surface (by the Pythagorean Theorem) as

$$\begin{aligned} ds^2 &= (dx_1)^2 + (dx_2)^2 + (df)^2 \\ &= (dx_1)^2 + (dx_2)^2 + \left(\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2\right)^2 \\ (ds)^2 &= \left(\mathbf{1} + \left(\frac{\partial f}{\partial x_1}\right)^2\right)(dx_1)^2 + 2\,\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}(dx)(dx_2) + \left(\mathbf{1} + \left(\frac{\partial f}{\partial x_2}\right)^2\right)(dx_2)^2 \end{aligned}$$

and hence our metric along the surface is

$$\begin{aligned} 1 + \left(\frac{\partial f}{\partial x_1}\right)^2 &= g_{11}, \\ 2\,\frac{\partial f}{\partial x_1}\,\frac{\partial f}{\partial x_2} &= 2g_{12}, \qquad (g_{12} = g_{21}) \\ 1 + \left(\frac{\partial f}{\partial x_2}\right)^2 &= g_{22}\,. \end{aligned} \tag{1.2}$$

If we ask the inverse problem: given the metric can we find the surface we see (1.2) is a system of nonlinear partial differential equations. More generally this is reflected when we write (1.1):

$$\partial_i y \cdot \partial_j y = g_{ij}.$$

Since $1 \leq i, j \leq 2$ in the case of (M^2, g) embedded into $\mathbb{R}^3$

$$g = \begin{bmatrix} g_{11} & & g_{12} \\ & \ddots & \\ g_{21} & & g_{22} \end{bmatrix}$$

we have an equation for each component of g. By symmetry in this case we have **3** equations for the **three** unknowns y^1, y^2, y^3. Thus we have a **determined** system. On the other hand embedding (M^2, g) into $\mathbb{R}^2$ we would have just (y^1, y^2) and we would have 3 equations in 2 unknowns (**overdetermined case**) and embedding (M^2, g) into $\mathbb{R}^4$ we would have $(y^1, \ldots, y^4)$ and we would have 3 equations in 4 unknowns (**undetermined case**).

Since for an n-dimensional Riemannian manifold we have

$$g = \overbrace{\begin{bmatrix} g_{11} & \cdots & g_{1n} \\ & \ddots & \\ g_{n1} & \cdots & g_{nn} \end{bmatrix}}^{n} \Big\} \, n$$

and an $n \times n$ symmetric matrix has $\frac{1}{2}\, n(n+1)$ entries on and above the diagonal in general the isometric embedding problem (recovering the "surface" from the metric) is

undetermined	$m > \frac{n}{2}\,(n+1),$
determined	$m = \frac{n}{2}\,(n+1),$
overdetermined	$m < \frac{n}{2}\,(n+1),$

where m will be the number of unknowns $(y^1, \ldots, y^m)$ and $\frac{n}{2}\,(n+1)$ will be the number of equations. The number

$$\frac{n}{2}\,(n+1)$$

is of course crucial and is called the **Janet dimension**.

Now for the over determined case we would not expect too many solutions and mathematicians have pursued the problem of uniqueness. On the other hand for the over determined case we have the flexibility of more unknowns than equations and it is here that we see in some sense that Riemann's concept of an abstraction of surfaces becomes superfluous. Specifically for m sufficiently large (M^n, g) embeds globally and smoothly into $\mathbb{R}^m$ and (M^n, g) looks exactly like a surface:

Theorem 1. *(John F Nash, Jr [9])*

A C^k, $3 \le k \le \infty$, Riemannian manifold (M^n, g) has a C^k embedding into $\mathbb{R}^m$ (globally) if

$$\begin{aligned} m &= n(3n+11)/2 && \textit{compact case} \\ &= n(n+1)(3n+11)/2 && \textit{non-compact case.} \end{aligned}$$

Nash's theorem has been improved over the intervening years but the main point here is that global embedding results are always for the undetermined system.

For the determined case which conceptually are more familiar in applied mathematics where the number of equations equals the number of unknowns global embedding (smoothly) is in general not possible.

I quote the paper of **S.-T. Yau [10]**

"**Section 3.13 Isometric embedding**. Given a metric tensor on a manifold, the problem of isometric embedding is equivalent to find enough functions $f_1, \ldots, f_N$ so that the metric can be written as $\Sigma(df_i)^2$. Much work was accomplished for two dimensional surfaces as was mentioned in section 2.1.2. Isometric embedding for the general dimension was solved in the famous work of J. Nash. Nash used his famous implicit function theorem which depends on various smoothing operators to gain derivatives. In a remarkable work Gunther was able to avoid the Nash procedure. He used only standard Hölder regularity estimate for the Laplacian to reproduce the Nash isometric embedding with the same regularity result. In his book Gromov was able to lower the codimension of the work of Nash. He called his method the h-principle.

When the dimension of the manifold is n, the expected dimension of the Euclidean space for the manifold to be isometrically embedded is $\frac{n(n+1)}{2}$. It is important to understand manifolds isometrically embedded into Euclidean space with this optimal dimension. Only in such a dimension does it make sense to talk about rigidity questions. **It remains a major open problem whether one can find a nontrivial family of isometric embeddings of a closed manifold into Euclidean space with an optimal dimension...** .

Chern told me that he and H. Lewy studied local isometric embedding of a three manifold into six dimensional Euclidean space, but they didn't write any manuscript on it. The major work in this subject was done by E. Berger, Bryant, Griffiths and Yang. They showed that a generic three dimensional embedding system is strictly hyperbolic, and the generic four dimensional system is of real principle type. Local existence is true for a generic metric using a hyperbolic operator and the Nash-Moser implicit function theorem... .

Comment. The theory of isometric embedding is a classical subject, but our knowledge is still rather limited, especially in dimension greater than three. Many difficult problems are related to **nonlinear mixed type equations** or **hyperbolic differential systems**, over a closed manifold."

Some preliminary lemmas

Lemma 1.1. *Let $X = X' \times I \subset \mathbb{R}^n$ where $X' \subset \mathbb{R}^{n-1}$ be an open domain and I a connected open interval. Given smooth functions $f : X \times \mathbb{R}^m \to \mathbb{R}^m$ and $A_0 : X' \to \mathbb{R}^m$ and $t \in I$, there exists a unique solution $A : X \to \mathbb{R}^m$ to*

$$\begin{aligned} \partial_n A &= f(x', x_n, A) \\ A|_{x_n=t} &= A_0(x') \quad \text{for} \quad x' \in X', \end{aligned}$$

where $\partial_n := \partial_{x_n}$.

Proof. This is just the standard existence-uniqueness theorem for ordinary differential equations. Here x_n is "time" the independent variable, t is the initial time where the data $A_0(x')$ is specified, x' are parameters for which the data $A_0(x')$ and prescribed $f(x', x_n, A)$ may depend, and A is the unknown function (dependent variable) that we wish to find.

Lemma 1.2. *Let $\mathbf{X} \subset \mathbb{R}^n$ be an open contractable domain and let $f_i : X \times \mathbb{R}^m \to \mathbb{R}^m$ satisfy*

$$\frac{\partial f_i^a}{\partial x^j} + \frac{\partial f_i^a}{\partial A^b} f_j^b = \frac{\partial f_j^a}{\partial x^i} + \frac{\partial f_j^a}{\partial A^b} f_i^b$$

for each $(x, A) \in \mathbf{X} \times \mathbb{R}^m$, where the Einstein summation convention is used here and forever after unless otherwise specified. Then given $x_0 \in \mathbf{X}$ and $A_0 \in \mathbb{R}^m$, there exists a unique solution $A : \mathbf{X} \to \mathbb{R}^m$ to

$$\partial_i A = f_i(x, A) \qquad A(x_0) = A_0$$

where $\partial_i = \partial_{x_i}$ *and* $x = (x_1, \ldots, x_n)$.

Proof. Existence and uniqueness will follow from Lemma 1.1 as long as the system of ordinary differential equations is consistent:

$$\begin{aligned} \partial_i \partial_j A &= \partial_i \quad f_j(x, A), \\ \partial_j \partial_i A &= \partial_j \quad f_i(x, A), \end{aligned}$$

hence we need

$$\partial_i f_j(x, A) = \partial_j f_i(x, A)$$

$$\begin{aligned} \frac{\partial f_j}{\partial x^i} + \frac{\partial f_j}{\partial A^b} \frac{\partial A^b}{\partial x^i} &= \frac{\partial f_i}{\partial x^j} + \frac{\partial f_i}{\partial A^b} \frac{\partial A^b}{\partial x^j}, \quad \text{i.e.} \\ \frac{\partial f_j}{\partial x^i} + \frac{\partial f_j}{\partial A^b} f_i^b &= \frac{\partial f_i}{\partial x^j} + \frac{\partial f_i}{\partial A^b} f_j^b \end{aligned}$$

and this was the hypothesis of the lemma.

Remark 1. *Lemma 2 is a nonlinear version of the Poincaré Lemma but here the existence, uniqueness theorem for ordinary differential equations is used instead of the fundamental theorem of calculus. In the standard Poincaré Lemma f_i does not depend on A and the statement*

$$\frac{\partial f_i^a}{\partial x_j} = \frac{\partial f_j^a}{\partial x_i}$$

implies the existence of a "potential" A with

$$\frac{\partial A^a}{\partial x_i} = f_i^a,$$

and of course

$$\frac{\partial}{\partial x_j} \frac{A^a}{\partial x_i} = \frac{\partial A^a}{\partial x_i \, \partial x_j}\,.$$

Riemannian structure in local coordinates

Let $(\mathbf{X}, g)$ be an n-dimensional Riemannian manifold. The **covariant derivative** allows us to differentiate along the manifold

$$\underline{\nabla_k \varphi = \partial_k \varphi}$$

for a scalar φ,

$$\underline{\nabla_k \varphi_j = \partial_x \varphi_j - \Gamma^\ell_{jk} \varphi_\ell}$$

for a vector φ_ℓ,

$$\underline{\nabla_k \varphi_{ij} = \partial_k \varphi_{ij} - \Gamma^\ell_{ik} \varphi_{\ell j} - \Gamma^\ell_{jk} \varphi_{i\ell}}$$

for a second order tensor φ_{ij}.

The **Christoffel symbols** are calculated from the metric g

$$\underline{\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} (\partial_i g_{ij} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).}$$

Here $g^{k\ell}$ (upper indices) is the inverse matrix of g_{ij} (lower indices).

The **Riemann convecture tensor** is calculated as

$$R^\ell_{ijk} = \partial j \Gamma^\ell_{ki} - \partial_k \Gamma^\ell_{ji} + \Gamma^\ell_{jp} \Gamma^p_{ki} - \Gamma^\ell_{kp} \Gamma^p_{ji}\ .$$

As usual we have

$$\begin{aligned}
\Gamma^k_{ij} &= \Gamma^k_{ji}, \\
\partial_k g_{ij} &= g_{ip} \Gamma^p_{kj} + g_{jp} \Gamma^p_{ik}, \\
\nabla_i \partial_j &= \Gamma^\ell_{ij} \partial_\ell, \\
R^\ell_{ijk} \partial_\ell &= -\nabla_j \nabla_k \partial_i + \nabla_k \nabla_j \partial_i
\end{aligned}$$

We denote

$$\underline{R_{ijk\ell} = g_{iq}\, R^q_{jk\ell}}$$

so that

$$R_{ijk\ell} = g_{iq}(+\partial_k \Gamma^q_{pj} - \partial_\ell \Gamma^q_{kj} + \Gamma^q_{kp}\Gamma^p_{\ell j} - \Gamma^q_{\ell p}\Gamma^p_{kj}).$$

Identities:
Skew symmetry: $R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k}$
Interchange symmetry: $R_{ijk\ell} = R_{k\ell ij}$

First Bianchi

$$\underline{R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0}$$

Second Bianchi

$$\underline{R_{ijk\ell;\mu} + R_{ij\ell\mu;k} + R_{ij\mu k;\ell} = 0}$$

where " ; " denotes the covariant derivative.
Special case : $n = 2$

$$\underline{R_{ijk\ell} = K(g_{ik}g_{\ell i} - g_{i\ell}g_{jk})}$$

where K is the Gauss curvature.

Since from the definition of $R^\ell_{ijk}, R_{ijk\ell}$ is made up of first derivatives of the Christoffel symbols and hence second derivatives of the metric g we see the Gauss curvature is given in terms of and first and second derivatives of the metric. This is **Gauss's theorem egregium**.

Covariant derivatives do not commute.

$$\varphi_{i;jk} - \varphi_{i;kj} = R^\ell_{ijk}\varphi_\ell, \text{ i.e.}$$

$$\nabla_k\nabla_j\varphi_i - \nabla_j\nabla_k\varphi_i = R^\ell_{ijk}\varphi_\ell.$$

This was noted earlier.

Also

$$\underline{R^\ell_{ijk} = -\nabla_j\Gamma^\ell_{ik} + \nabla_k\Gamma^\ell_{ij}}$$

since

$$\begin{aligned}
\nabla_j\Gamma^\ell_{ik} &= \partial_j\Gamma^\ell_{ik} - \Gamma^p_{ij}\Gamma^\ell_{pk} - \underline{\underline{\Gamma^p_{kj}\Gamma^\ell_{ip}}}\ , \\
\nabla_k\Gamma^\ell_{ij} &= \partial_k\Gamma^\ell_{ij} - \Gamma^p_{ik}\Gamma^\ell_{pj} - \underline{\underline{\Gamma^p_{jk}\Gamma^\ell_{ip}}}
\end{aligned}$$

and when we subtract the last terms cancel so that

$$-\nabla_j\Gamma^{\ell}_{ik} + \nabla_k\Gamma^{\ell}_{ij} =$$

$$-\partial_j\Gamma^{\ell}_{ik} + \partial_k\Gamma^{\ell}_{ij} + \Gamma^{p}_{ij}\Gamma^{\ell}_{pk} - \Gamma^{p}_{ik}\Gamma^{\ell}_{pj}$$

$$= R^{\ell}_{ijk}$$

Isometric immersion

We use " . " to denote the Canonical Euclidean metric in a coordinate patch $(y^1,\ .\ , y^m)$ in $\mathbb{R}$. An $\mathbb{R}^m-$ valued function $y : (X, g) \rightarrow (\mathbb{R}^m,\ .)$ is called an **isometric immersion** of X into $\mathbb{R}^m$. If the induced metric is the same as the original, that is, written locally using coordinates $(x^1, .., x^n)$

$$\partial_i y.\partial_j y = g_{ij} \text{ for each } 1 \leq i, j \leq n.$$

where " . " stands for the canonical metric in $\mathbb{R}^m$.

In other words:

Pythagoras gives us

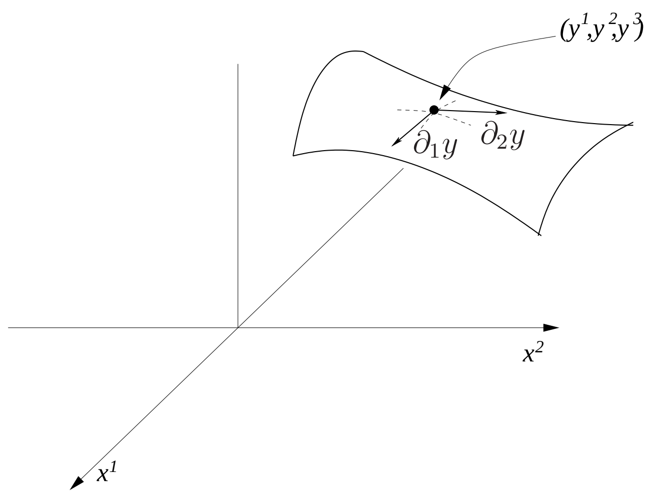
$$ds^2 = \partial_i y.\partial_j y\ dx_i dx_j$$

if we knew y.

Riemann gives us

$$ds^2 = g_{ij} dx_i dx_j$$

as the general distance formula for abstract manifold (X, g). Can we equate the two expressions and find the **realization** of the manifold?



$\partial_1 y, \partial_2 y$ are tangent to a surface in the case $n = 2, m = 3$. They span the tangent space at the point $y(x)\Big(= y^1(x^1, x^2), y^2(x^1, x^2), y^3(x^1, x^2)\Big)$. In this case the unit normal vector is defined (up to a sign) by the usual vector cross product

$$N = \frac{\partial_1 y \times \partial_2 y}{|\partial_1 y \times \partial_2 y|}.$$

In higher dimensions we have no cross product but the same ideas makes sense.

For (X, g) we have the coordinate patch $y = (y^1, ., y^m)$, we form the collection of tangents

$$\Big\{\partial_1 y(x), ..., \partial_n y(x)\Big\}$$

which span the tangent space to the manifold defines as $\underbrace{T_x X}_{n \text{ dimensional}}$. Let $\underbrace{N_x X}_{m-n \text{ dimensional}}$ denote $m - n$ dimensional subspace orthogonal and complementary to $T_x X$. Fix an orthogonal basis $\Big\{N_{n+1}(x), \ldots, N_m(x)\Big\}$ of $N_x X$ for each x, and assume they depend smoothly on x.

The second derivative of an immersion

For each x, the vectors $\{\partial_1 y(x), ., \partial_n y(x), N_{n+1}(x), ..., N_m(x)\}$ comprise a basis of $\mathbb{R}^m$. Therefore, for each pair of indices $1 \le i, j \le n$ the vector $\partial^2_{ij} y(x)$ can be written as a linear combination of these vectors. In other words there exist unique coefficients $\Gamma^k_{ij}, 1 \le k \le n$ and $H^\mu_{ij},\ n + 1 \le \mu \le m$ such that

$$\underline{\partial^2_{ij} y(x) \ = \ \Gamma^k_{ij}(x)\ \partial_k y(x) \ + \ H^\mu_{ij}(x)\ N_\mu(x)} \tag{1.3}$$

or component-wise

$$\underline{\partial^2_{ij} y^p(x) \ = \ \Gamma^k_{ij}(x)\ \partial_k y^p(x) + H^\mu_{ij}(x)\ N^p_\mu(x),}$$

$p = 1, \ldots, m,$

or

$\underline{\nabla_i\ \partial_j y(x) \ = \ H^\mu_{ij}(x)\ N_\mu(x)}$

Remark. Notice that the coefficients in the tangent direction for the second derivatives $\partial^2_{ij} y(x)$ are precisely the Christoffel symbols defined earlier. Let us see why this is true.

We ask

$$\partial^2_{ij} y(x) \underline{\underline{?}}\ \Gamma^k_{ij}(x)\ \partial_k y(x) \ + \ H^\mu_{ij}(x)\ N_\mu(x).$$

Take the " . " product of each side with the tangent vector $\partial_q y(x)$.

$$\partial^2_{ij} y(x) \cdot \partial_q y(x) \underline{\underline{?}}\, \Gamma^k_{ij}\ \partial_k y(x) \cdot \partial_q y(x)$$

since $\partial_q y(x) \cdot N_\mu(x) = 0$.

Since y is an immersion

$$\partial_k y(x) \cdot \partial_q y(x) \ = \ g_{kq}(x)$$

and our question is

$$\partial^2_{ij} y(x) \cdot \partial_q y(x) \underline{\underline{?}}\, \Gamma^k_{ij}(x)\ g_{kq}(x).$$

But

$\partial_i(\partial_j y\ .\ \partial_q y) \ = \ \partial^2_{ij} y\ .\ \partial_q y \ + \ \partial_j y\ .\ \partial_{iq} y,$ or $\partial_i\ g_{jq} \ = \ \partial^2_{ij}\ .\ \partial_q y \ + \partial_j y\ .\ \partial_{iq} y.$

So our question may be written as

$$\partial_i g_{jq} \underline{\underline{?}}\, \Gamma^k_{ij}(x)\ g_{kq}(x) \ + \ \Gamma^k_{iq}\ g_{kj}(x).$$

Now plug in the definition of the Christoffel symbol:

$$\begin{aligned}
\partial_i g_{jq} \quad &\underline{\underline{?}} \quad g_{kq}(x)\frac{1}{2}\ g^{k\ell}(x)\ (\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) \\
&\quad + g_{k\ell}(x)\frac{1}{2}\ g^{k\ell}(x)\ (\partial_i g_{\ell q} + \partial_q g_{i\ell} - \partial_\ell g_{iq}) \\
&= \quad \frac{1}{2}\ \delta^\ell_q\ (\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) \\
&\quad + \frac{1}{2}\ \delta^\ell_j\ (\partial_i g_{\ell q} + \partial_q g_{i\ell} - \partial_\ell g_{iq})
\end{aligned}$$

where δ^ℓ_q is the Kronecker delta ($\delta^j_\ell = 1,\ j = \ell,\ \delta^j_\ell = 0,\ j \neq \ell$).

So finally our question is

$$\begin{aligned}
\partial_i g_{jq} \quad &\underline{\underline{?}} \quad \frac{1}{2}\ (\partial_i g_{qj} + \underline{\partial_j g_{iq}} - \underline{\underline{\partial_q g_{ij}}}) \\
&\quad + \frac{1}{2}\ (\partial_i g_{jq} + \underline{\underline{\partial_q g_{ij}}} - \underline{\partial_j g_{iq}})
\end{aligned}$$

and the indicated cancellations show the answer to our question is "yes".

In addition from (1.2) we see

$$\partial^2_{ij} y(x) \cdot N_\nu(x) = H^\mu_{ij}(x) N_\mu(x)\ \cdot\ N_\nu(x)$$

and since $\{N_\mu^{m+1}(x),\ \dots\ ,\delta_\mu^m\}$ is an orthonormal set

$$N_\mu(x) \cdot N_\nu(x) = \delta_{\mu\nu}$$

we have

$$\underline{H_{ij}^\mu(x) = \partial_{ij}^2 y(x) \cdot N_\mu(x)}$$

The tensors $H_{ij}^\mu(x),\ \mu = n+1,\ \dots\ , m$ are called the <u>second fundamental form</u>. (g is the first fundamental form). They are <u>symmetric in (i,j)</u>.

Just as we decomposed the first derivative of the tangent vectors to obtain (2.2) we now decompose the first derivatives of the normals $N_\mu(x)$.

Lemma 1.3. There exist functions $A_{\mu i}^\nu = -A_{\nu i}^\mu$ such that

$$\partial_i N_\mu = -g^{jk}\ H_{ik}^\mu\ \partial_j y + A_{\mu i}^\nu N_\nu \tag{1.4}$$

or component $-$ wise

$$\partial_i N_\mu^p = -g^{jk}\ H_{ik}^\mu\ \partial_j y^p + A_{\mu i}^\nu N_\nu^p\ ,\ p = 1,\ \dots\ , m.$$

Proof. First, since as before we know the set of tangents and normals span $\mathbb{R}^m$, we decompose

$$\partial_i N_\mu \ = \ B_{i\mu}^j\ \partial_j y \ + \ A_{\mu i}^\nu N_\nu. \tag{1.5}$$

Since

$$0 = \partial_i(N_\mu \cdot N_\nu) = N_\nu \cdot \partial_i N_\mu + N_\mu \cdot \partial_i N_\nu$$

we see (1.3) implies

$$\begin{aligned}
N_\nu \cdot \partial_i N_\mu &= A_{\mu i}^\alpha N_\alpha \cdot N_\nu, \\
N_\mu \cdot \partial_i N_\nu &= A_{\nu i}^\alpha N_\alpha \cdot N_\mu; \\
N_\nu \cdot \partial_i N_\mu &= A_{\mu i}^\alpha \delta_{\alpha\nu} = A_{\mu i}^\nu \\
N_\mu \cdot \partial_i N_\nu &= A_{\nu i}^\alpha \delta_{\alpha\mu} = A_{\nu i}^\mu
\end{aligned}$$

and so addition yields

$$\underline{A^{\nu}_{\mu i} + A^{\mu}_{\nu i} = 0}.$$

On the other hand since

$$N_\mu \cdot \partial_k y = 0$$

we have

$$\begin{aligned}
0 &= g^{jk}\partial_i(N_\mu \cdot \partial_k y) \\
&= g^{jk}(\partial_i N_\mu \cdot \partial_k y + N_\mu \cdot \partial^2_{ik} y) \\
&= g^{jk}(\partial_i N_\mu \cdot \partial_k y + H^{\mu}_{ik}) && \text{by (1.4)} \\
&= g^{jk}(\partial_k y \cdot \partial_p y B^{B}_{i\mu} + H^{\mu}_{ik}) && \text{by (1.5)} \\
&= g^{jk}(g_{kp} B^{\gamma}_{i\mu} + H^{\mu}_{ik}) && \text{by (1.1)} \\
&= B^{j}_{i\mu} + g^{jk} H^{\mu}_{ik}, (\text{since } g^{jk} g_{kp} = \delta^j_p).
\end{aligned}$$

Hence

$$B^{j}_{i\mu} = -g^{jk} H^{\mu}_{ik}.$$

Substitution of this relation into (1.4), together with the equality $A^{\nu}_{\mu i} + A^{\mu}_{\nu i} = 0$, proves the lemma.

Commutation of second partials of the normal vectors

Differentiate (1.4) one more time.

$$\begin{aligned}
\partial_i N_\mu &= -g^{jk}\ H^{\mu}_{ik}\ \partial_j y + A^{\nu}_{\mu i} N_\nu \\
\partial_j(\partial_i N_\mu) &= -\partial_j(g^{qp}\ H^{\mu}_{ip}\ \partial_q y) + \partial_j(A^{\nu}_{\mu i} N_\nu) \\
&= -\partial_j(g^{qp}\ H^{\mu}_{ip})\ \partial_q y \\
&\quad -g^{qp}\ H^{\mu}_{ip}\ \partial_{jq} y) + \partial_j(A^{\nu}_{\mu i} N_\nu) \\
&= -\partial_j(g^{qp} H^{\mu}_{ip})\underline{\partial_q y} \\
&\quad -g^{qp} H^{\mu}_{ip}(\Gamma^{k}_{jq}\underline{\partial_k y} + H^{\nu}_{jq}\underline{\boldsymbol{N_\nu}}) \\
&\quad -(\partial_j A^{\nu}_{\mu i})\underline{\boldsymbol{N_\nu}} + \\
&\quad A^{\nu}_{\mu i}(-g^{pq} H^{\nu}_{pj}\underline{\partial_q y} + A^{\ell}_{\nu j}\underline{\boldsymbol{N_\ell}}
\end{aligned}$$

(where we have used both (1.3) and (1.4)).

Now let us collect terms in the **tangential** and **normal** directions to write

$$\partial_j(\partial_i N_\mu) = -\Big(\partial_j(g^{pq}H^\mu_{ip}) + g^{pk}\Gamma^q_{jk}H^\mu_{ip} + g^{pq}A^\nu_{\mu i}H^\nu_{pj}\Big)\ \underline{\partial_q y}$$
$$+\Big(\partial_j A^\nu_{\mu i} - g^{pq}H^\mu_{ip}H^\nu_{jq} + A^\ell_{\mu i}A^\nu_{\ell j}\Big)\underline{\boldsymbol{N_\nu}}.$$

But since

$$\partial_j(\partial_i N_\mu) = \partial_i(\partial_j N_\mu)$$

we can read off the equations from the tangent direction (**Codazzi equations**)

$$\partial_j(g^{pq}H^\mu_{ip}) + g^{pk}\Gamma^q_{jk}H^\mu_{ip} + g^{pq}A^\nu_{\mu i}H^\nu_{pj} = \partial_i(g^{pq}H^\mu_{jp}) + g^{pk}\Gamma^q_{ik}H^\mu_{jp} + g^{pq}A^\nu_{\mu j}H^\nu_{pi},$$

and from the normal direction (**Ricci equations**)

$$\partial_j A^\nu_{\mu i} - g^{pq}H^\mu_{ip}H^\nu_{jq} + A^\ell_{\mu i}A^\nu_{\ell j} = \partial_i A^\nu_{\mu j} - g^{pq}H^\mu_{jp}H^\nu_{iq} + A^\ell_{\mu j}A^\nu_{\ell i}.$$

We can rewrite the Codazzi equations in a more traditional form via simple computation:

$$g^{pq}g_{pr} = \delta^q_r, \text{ so}$$
$$\partial_j(g^{pq}g_{pr}) = 0 \text{ and}$$
$$\partial_j(p^{pq})g_{pr} + g^{pq}\partial_j(g_{pr}) = 0,$$
$$\partial_j(g^{pq})g_{pr}g^{rs} + g^{rs}g^{pq}\partial_j(g_{pr}) = 0,$$
$$\partial_j(g^{pq})\delta^s_p + g^{rs}g^{pq}\partial_j(g_{pr}) = 0,$$
$$\partial_j(g^{sq}) = -g^{rs}g^{pq}\partial_j(\partial_p y\cdot\partial_r y),$$

(by (1.1))

$$\partial_j(g^{sq}) = -g^{rs}g^{pq}(\partial_{jp}y.\partial_r y + \partial_p y.\partial_{jr}y)$$
$$= -g^{rs}g^{pq}(\Gamma^k_{jp}\partial_k y.\partial_r y + \partial_p y\cdot\Gamma^k_{jr}\partial_k y)$$

(by (1.3)). Thus

$$\partial_j(g^{sq}) = -g^{rs}g^{pq}(\Gamma^k_{jp}g_{rk} + \Gamma^k_{jr}g_{kp})$$
$$= -g^{pq}(\Gamma^k_{jp}\delta^s_k) - g^{rs}(\Gamma^k_{jr}\delta^q_k)$$
$$= -g^{pq}(\Gamma^s_{jp}) - g^{rs}(\Gamma^q_{jr});$$

i.e.

$$\partial_j(g^{sq}) = -g^{pq}\Gamma^s_{jp} - g^{rs}\Gamma^q_{jr}.$$

Let us use this relation in the above Codazzi equations via the rewriting

$$
\begin{aligned}
\partial_j(g^{pq}) &= -g^{sq}\Gamma^p_{js} - g^{rp}\Gamma^q_{jr} \ , \\
\partial_i(g^{pq}) &= -g^{sq}\Gamma^p_{is} - g^{rp}\Gamma^q_{ir} \ .
\end{aligned}
$$

Hence from the Codazzi equations we have

$$
g^{pq}\partial_j H^\mu_{ip} + (-g^{sq}\Gamma^p_{js} - g^{rp}\Gamma^q_{jr})\ H^\mu_{ip} + g^{pk}\Gamma^q_{jk}H^\mu_{ip} + g^{pq}A^\nu_{\mu i}H^\nu_{pj} =
$$

$$
g^{pq}\partial_i H^\mu_{jp} + (-g^{sq}\Gamma^p_{is} - g^{rp}\Gamma^q_{ir})\ H^\mu_{jp} + g^{pk}\Gamma^q_{ik}H^\mu_{jp} + g^{pq}A^\nu_{\mu j}H^\nu_{pi}
$$

Apply $g_{q\alpha}$ to both sides

$$
\delta^p_\alpha\partial_j H^\mu_{ip} + (-\delta^s_\alpha\Gamma^p_{js} - g_{q\alpha}g^{rp}\Gamma^q_{jr})\ H^\mu_{ip} \ + \ g_{q\alpha}g^{pk}\Gamma^q_{jk}H^\mu_{ip} + \delta^p_\alpha A^\nu_{\mu i}H^\nu_{pj} =
$$

$$
\delta^p_\alpha\partial_i H^\mu_{jp} + (-\delta^s_\alpha\Gamma^p_{is} - g_{q\alpha}g^{rp}\Gamma^q_{ir})\ H^\mu_{jp} \ + \ g_{qa}g^{pk}\Gamma^q_{ik}H^\mu_{jk} \ + \ \delta^p_\alpha A^\nu_{\mu j}H^\nu_{pi}
$$

and hence

$$
\partial_j H^\mu_{i\alpha} + (-\Gamma^p_{j\alpha}H^\mu_{ip}) \ - \ \underline{g_{q\alpha}g^{rp}\ \Gamma^q_{jr}H^\mu_{ip}} + \boldsymbol{g_{q\alpha}g^{pk}\ \Gamma^q_{jk}H^\mu_{ip}} \ + \ A^\nu_{\mu i}H^\nu_{\alpha j} \ =
$$

$$
\partial_i H^\mu_{j\alpha} + (-\Gamma^p_{i\alpha}H^\mu_{jp}) \ - \ \underline{g_{q\alpha}g^{rp}\ \Gamma^q_{ir}H^\mu_{jp}} + \boldsymbol{g_{q\alpha}g^{pk}\ \Gamma^q_{ik}H^\mu_{jk}} \ + \ A^\nu_{\mu j}H^\nu_{\alpha},
$$

$$
\partial_j H^\mu_{i\alpha} - \Gamma^p_{j\alpha}H^\mu_{ip} + A^\nu_{\mu i}H^\nu_{\alpha j} = \partial_i H^\mu_{j\alpha} - \Gamma^p_{i\alpha}H^\mu_{jp} + A^\nu_{\mu j}H^\nu_{\alpha i}.
$$

Now substract $\Gamma^p_{ij}H^\nu_{\alpha p}$ from both sides of the equation to see the usual form of **Codazzi** equations

$$
\partial_j H^\mu_{i\alpha} - \Gamma^p_{j\alpha}H^\mu_{ip} - \Gamma^p_{ij}H^\mu_{\alpha p} + A^\nu_{\mu i}H^\nu_{\alpha j} = \partial_i H^\mu_{j\alpha} - \Gamma^p_{i\alpha}H^\mu_{jp} - \Gamma^p_{ij}H^\mu_{\alpha p} + A^\nu_{\mu j}H^\nu_{\alpha i}.
$$

Also since

$$
\begin{aligned}
\nabla_j H^\mu_{i\alpha} &= \partial_j H^\mu_{i\alpha} - \Gamma^p_{ij}H^\mu_{p\alpha} - \Gamma^p_{\alpha j}H^\mu_{ip}, \\
\nabla_i H^\mu_{j\alpha} &= \partial_i H^\mu_{j\alpha} - \Gamma^p_{ji}H^\mu_{p\alpha} - \Gamma^p_{\alpha i}H^\mu_{jp},
\end{aligned}
$$

the symmetry of the Christoffel symbols and H^μ_{ij} yields the **Codazzi equations** as

$$
\boldsymbol{\nabla_j H^\mu_{i\alpha} - \nabla_i \cdot H^\mu_{j\alpha} + A^\nu_{\mu i}H^\nu_{\alpha j} - A^\nu_{\mu j}H^\nu_{\alpha i} = 0}
$$

A simple but important remark

Recall equation (1.4).

$$\partial_i N_\mu = -g^{jk}\ H^\mu_{ik}\ \partial_j y + A^\nu_{\mu i} N_\nu.$$

In the case of a hypersurface $m = n+1$ and there is only one normal since $n+1 \le \nu \le m = n+1$, i.e. N_{n+1}. But N_{n+1} is a unit vector so that

$$N_{n+1} \cdot N_{n+1} = 1$$

and

$$\partial_i N_{n+1} \cdot N_{n+1} = 0$$

Thus (1.4) says

$$\begin{aligned} \partial_i N_{n+1} &= -g^{pq}\ H^{n+1}_{ip}\ \partial_q y + A^{n+1}_{(n+1)i} N_{n+1} \\ 0 &= N_{n+1} \cdot \partial_i N_{n+1} = A^{n+1}_{(n+1)i} \end{aligned}$$

and the $A^{n+1}_{(n+1)i} = 0$. Of course this is the same as applying the skew symmetry of $A^\nu_{\mu i}$ in μ, ν.

Conclusion: In the case of hypersurfaces the Codazzi system simplifies to $\nabla_j H^\mu_{i\alpha} - \nabla_i H^\mu_{j\alpha} = 0.$

But if we are interested in the **determined case for hypersurfaces**

$m = n(\frac{n}{2} + 1)$ and $m = n+1$,

hence $n + 1 = \frac{n}{2}(n+1)$ i.e., $n = 2, m = 3$ which is the classical case of (M^2, g) embedded into $\mathbb{R}^3$.

The Gauss and Codazzi equations

We now return to equation (1.3) and once again commute partial derivatives.

$$\begin{aligned} 0 &= \partial_k(\partial^2_{ij} y) - \partial_j(\partial^2_{ik} y) \\ &= \partial_k(\Gamma^\ell_{ij} \partial_{\ell y} + H^\mu_{ij} N_\mu) \\ &\quad -\partial_j(\Gamma^\ell_{ik} \partial_{\ell y} + H^\mu_{ik} N_\mu) \\ &= (\partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik}) \partial_\ell y + \Gamma^\ell_{ij} \underbrace{\partial_{k\ell} y - \Gamma^\ell_{ik}}_{\text{use (1.3)}} \partial_{j\ell} y \\ &\quad +(\partial_k H^\mu_{ij} - \partial_j H^\mu_{ik}) N_\mu \\ &\quad +H^\mu_{ij}\ \underbrace{\partial_k n_\mu}_{\text{(use 1.4)}}\ -H^\mu_{ik}\ \underbrace{\partial_j n_\mu}_{\text{(use 1.4)}} \end{aligned}$$

$$
\begin{aligned}
&= (\partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik})\partial_\ell y \\
&\quad +\Gamma^\ell_{ij}(\Gamma^p_{k\ell}\partial_p y + H^\mu_{k\ell} N_\mu) \\
&\quad -\Gamma^\ell_{ik}(\Gamma^p_{j\ell}\partial_p y + H^\mu_{j\ell} N_\mu) \\
&\quad +(\partial_k H^\mu_{ij} - \partial_j H^\mu_{ik}) N_\mu \\
&+ \quad H^\mu_{ij}(-g^{\ell p} H^\mu_{kp}\partial_\ell y + A^\nu_{\mu k} N_\nu) \\
&- \quad H^\mu_{ik}(-g^{\ell p} H^\mu_{jp}\partial_\ell y + A^\nu_{\mu j} N_\nu)
\end{aligned}
$$

Again associate the above expression into tangential and normal contributions to see

$$
\begin{aligned}
0 \;=\; & \left[\partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik} + \Gamma^p_{ij}\Gamma^\ell_{kp} - \Gamma^p_{ik}\Gamma^\ell_{jp} \right. \\
& \left. -g^{\ell p}(H_{ij}\cdot H_{kp} - H_{ik}\cdot H_{jp})\right] \partial_\ell y \\
+ \; & \left[\Gamma^\ell_{ij} H^\mu_{k\ell} - \Gamma^\ell_{ik} H^\mu_{ji} + \partial_\ell H^\mu_{ij} - \partial_j H^\mu_{ik} \right. \\
& \left. +H^\nu_{ij} A^\mu_{\nu k} - H^\nu_{ik} A^\mu_{\nu j}\right] N_\mu
\end{aligned}
$$

which implies via the normal component

$$
\begin{gathered}
\underline{\partial_k H^\mu_{ij} - \partial_j H^\mu_{ik} + \Gamma^\ell_{ij} H^\mu_{k\ell} - \Gamma^\ell_{ik} H^\mu_{j\ell}} \\
+\underline{H^\nu_{ij} A^\mu_{\nu k} - H^\nu_{ik} A^\mu_{\nu j} = 0}
\end{gathered}
$$

which with $A^\mu_{\nu k} = -A^\nu_{\mu k}$ gives

$$
\begin{gathered}
\underline{\partial_k H^\mu_{ij} - \partial_j H^\mu_{ik} + \Gamma^\ell_{ij} H^\mu_{kl\ell} - \Gamma^\ell_{ik} H^\mu_{j\ell}} \\
+\underline{H^\nu_{ij} A^\nu_{\mu\ell} - H^\nu_{ik} A^\nu_{\mu j} = 0}
\end{gathered}
$$

which give the **Codazzi equations** we have derived earlier.

From the tangential component we have

$$
\partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik} + \Gamma^p_{ij}\Gamma^\ell_{kp} - \Gamma^p_{ik}\Gamma^\ell_{jp} \; - g^{\ell p}(H_{ij}\cdot H_{kp} - H_{ik}\cdot H_{jp}) = 0.
$$

But recall from the definition of Riemann curvature tensor

$$
\partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik} + \Gamma^p_{ij}\Gamma^\ell_{kp} - \Gamma^p_{ik}\Gamma^\ell_{jp} = -R^\ell_{ijk}
$$

and hence

$$g^{\ell p}(-R_{pijk} - H_{ij} \cdot H_{kp} + H_{ik} \cdot H_{jp}) = 0$$

$$H_{ij}H_{pk} - H_{ik}H_{jp} = -R_{pijk} = R_{ipjk}$$

or

$$\underline{H_{ij} \cdot H_{pk} - H_{ik} \cdot H_{jp} = R_{ipjk}}$$

This is the Gauss relation.

We summarise below: for $(M^n, g) \to (\mathbb{R}^m, \cdot)$.

A **necessary** condition for the existence of an isometric embedding is there exist functions $H^\mu_{ij} = H^\mu_{ji}, A^\nu_{\mu i} = -A^\mu_{\nu i}$, $1 \leq i, j \leq n, n+1 \leq \mu, \nu \leq m$ such that

$$\sum_{\mu=n+1}^{m} H^\mu_{ik}H^\mu_{j\ell} - H^\mu_{i\ell}H^\mu_{jk} = R_{ijk\ell} \tag{1.6}$$

(GAUSS)

$$\begin{aligned} &\partial_k H^\mu_{ij} + A^\mu_{\nu p}H^\nu_{ij} - \Gamma^p_{ki}H^\mu_{pj} - \Gamma^p_{kj}H^\mu_{ip} = \\ &\partial_j H^\mu_{\ell k} + A^\mu_{\nu j}H^\nu_{ik} - \Gamma^p_{ji}H^\mu_{pk} - \Gamma^p_{jk}H^\mu_{ip} \end{aligned} \tag{1.7}$$

(CODAZZI)

$$\begin{aligned} &\partial_i A^\nu_{\mu j} - \partial_j A^\nu_{\mu i} + A^\nu_{\eta i}A^\eta_{\mu j} \\ &-A^\nu_{\eta j}A^\eta_{\mu i} = g^{pq}(H^\mu_{ip}H^\nu_{jq} - H^\mu_{jp}H^\nu_{iq}) \end{aligned} \tag{1.8}$$

(RICCI)

The Ricci system (1.8) can be put in covariant form by adding and subtracting

$$\Gamma^q_{ij}A^\nu_{\mu q}.$$

Then we have

$$\begin{gathered} \nabla_i A^\nu_{\mu j} - \nabla_j A^\nu_{\mu i} + A^\nu_{\eta i}A^\nu_{\mu j} \\ -A^\nu_{\eta j}A^\eta_{\mu i} = \\ g^{pa}(H^\mu_{ip}H^\nu_{jq} - H^\mu_{jp}H^\nu_{iq}). \end{gathered} \tag{1.7$'$}$$

(RICCI)

Notice the $A^\nu_{\mu j}$ are **vectors** $j = 1, 2, 3$ and the ν, μ are only accounting for dimensions $n + 1 \leq \mu, \nu \leq m$.

Reconstructing an isometric embedding

Theorem 2. *Given a connected and simply connected Riemannian manifold X with coordinates $(x^1, \ldots, x^n)$ and Riemannian metric $g (= g_{ij})$ if there exist functions $H^\mu_{ij} = H^\mu_{ji}$ and*

$$A^\nu_{\mu i} = -A^\mu_{\nu i} \quad 1 \le \ell, j \le m, \; n+1 \le \mu, \nu \le m$$

such that (1.5), (1.6), (1.7) hold then there exist functions $N_{n+1}, \ldots, N_m : X \to \mathbb{R}^m$ and a function $y : X \to \mathbb{R}^m$ such that the following hold:

$$\begin{aligned} N_\mu \cdot N_\nu &= \delta_{\mu\nu}, && (1.9)\\ N_\mu \cdot \partial_{iy} &= 0, && (1.10)\\ \partial_i y \cdot \partial_j y &= g_{ij} && (1.11) \end{aligned}$$

and

$$\begin{aligned} \partial^2_{ij} y &= \Gamma^k_{ij} \partial_k y + H^\mu_{ij} N_\mu, && (1.12)\\ \partial_i N_\mu &= -g^{jk} H^\mu_{ik} \partial_j y + A^\nu_{\mu i} N_\nu. && (1.13) \end{aligned}$$

The theorem shows that the conditions on $H^\mu_{ij}, A^\nu_{\mu\ell}$ and (1.8)–(1.10) are both necessary and sufficient for embedding $(M^n, g) \to (\mathbb{R}^m, \cdot), X = M^n$.

Sketch of proof. Let $\{e_n, \ldots e_m\}$ denote the standard basis of $\mathbb{R}^m$. Fix a point $x_0 \in X$. Set $\{\partial_1 y(x_0) \ldots, \partial_n(x_0), N_{n+1}(x_0), \ldots, N_m(x_0)\}$ so that (1.8)–(1.10) hold. One possibility is to set $N_\mu(x_0) = e_\mu$ and $y(x_0) = 0$ and choose $\{\partial_1 y(x_0) \ldots, \partial_n y(x_0)\}$ to be a linear combination of $\{e_1, \ldots, e_n\}$ such that (1.10) holds at x_0.

[If $g_{ij}(x_0) = \delta_{ij}$ then

$$\begin{gathered} N_\mu(x_0) = e_\mu, \quad n+1 \le \mu \le m, \\ \partial_1 y(x_0) = e_1, \ldots, \partial_n y(x_0) = e_n \end{gathered}$$

works.]

If we denote $\phi_i = \partial_1 y$ then (1.11)–(1.12) form a total differential system for the unknown $\mathbb{R}^m$ valued function $\{\phi_1, \ldots, \phi_n, N_{n+1}, \ldots, N_m\}$. Check by differentiating these equations that the compatibility conditions obtained by constructing partial derivatives are consequences of the Gauss equations (1.5), Codazzi equations (1.6), Ricci equations (1.7) and the original equations (1.11), (1.12). Therefore by Lemma 1.2, there exists a unique solution (the "potential" ϕ_i) for extending the initial data specified at x_0.

Also the differentials of equations (1.8)–(1.10) are consequences of (1.11)–(1.12). Therefore they hold not only at x_0 but also on all of X.

Finally (1.11) implies $\partial_i \phi_j = \partial_j \phi_i$ because the right side is symmetric in i and j. Therefore by Lemma 1.2 there exists a unique $\mathbb{R}^m$-valued function y on X so that

$$y(x_0) = 0 \quad \text{and} \quad \partial_i y = \phi_i, \quad 1 \leq i \leq n.$$

Example: $(H^2, g) \to (\mathbb{R}^3, \cdot)$

(1.6) is $H^4_{ik} H^4_{jk} - H^4_{i\ell} H^\mu_{jk} = R_{ijk\ell}$,

(1.7) is $\partial_k H^4_{ij} - \partial_j H^4_{ik} = \Gamma^p_{ki} H^4_{pj} + \Gamma^p_{kj} H^4_{ip} - \Gamma^p_{j\ell} H^4_{pk} - \Gamma^p_{jk} H^4_{ip}$,

$R_{ijk\ell} = K(g_{ik} g_{\ell j} - g_{i\ell} g_{jk})$,

$K =$ Gauss curvature,

$1 \leq i, j \leq 2$.

$$H^4 = \begin{bmatrix} H^4_{11} & H^4_{12} \\ H^4_{21} & H^4_{22} \end{bmatrix} \qquad H^4_{12} = H^4_{21}$$

$$R_{1212} = K(g_{11} g_{22} - g^2_{12}) = K \det g, \quad \det g > 0$$

(1.6) is $H^4_{11} H^4_{22} - H^4_{12} H^4_{12} = K \det g.$ (1.13)

(1.7) is

$$\partial_2 H^4_{11} - \partial_1 H^4_{12} = \cdots,$$
$$\partial_2 H^4_{12} - \partial_1 H^4_{22} = \cdots \ ,$$

and we have three equations in three unknowns: $H^4_{11}, H^4_{12}, H^4_{22}$.

Quasi-linear system if we eliminate one of the unknowns via Gauss (1.6).

The Gauss relation becomes a "constitutive relation".

Example: $(H^3, g) \to (\mathbb{R}^6, \cdot)$

(1.6) is $\sum_{\mu=4}^{6} H^\mu_{ik} H^\mu_{j\ell} - H^\mu_{i\ell} H^4_{jk} = R_{ijk\ell}$ (1.14)

Non-zero components of $R_{ijk\ell}$:

$$\underbrace{R_{1212},\ R_{1313},\ R_{2323},\ R_{1223},\ R_{1332},\ R_{1231}}$$

6 non-zero components

6 Gauss relations are given by (1.6).

The second fundamentals form is

$$\begin{bmatrix} H^\mu_{11} & H^\mu_{12} & H^\mu_{13} \\ H^\mu_{21} & H^\mu_{22} & H^\mu_{23} \\ H^\mu_{31} & H^\mu_{32} & H^\mu_{33} \end{bmatrix}$$

and the Codazzi equations are just a statement about cross derivatives along rows (or columns since H^μ_{ij} is symmetric). There appears to be 3 equations across each row. Notice however the coupling

$$\begin{aligned} \partial_1 H^\mu_{23} - \partial^\mu_3 H_{21} &= \cdots \\ \partial_1 H^\mu_{32} - \partial_2 H^\mu_{31} &= \cdots \end{aligned}$$

yields upon subtraction

$$\partial_2 H^\mu_{31} - \partial_3 H^\mu_{21} = \cdots$$

Thus instead of 9 couplings for each μ, there are only 8. Since $\mu = 4, 5, 6$ we have **24 Codazzi equations**. We count equations and unknowns.

Equations: 6 Gauss + 24 Codazzi + 9 Ricci = 39 equations.

Unknowns: 18 H^μ_{ij} + 9 $A^\nu_{\mu k}$ = 27 unknowns.

Thus even though the problem of embedding $(H^3, g) \to (\mathbb{R}^6, \cdot)$ is determined $\left(m = \dfrac{n(n+1)}{2}; 6 = \dfrac{3(4)}{2}\right)$ using the Gauss, Codazzi, Ricci system it gives more equations than unknowns.

Of course this means that for the Gauss, Codazzi, Ricci system not all the equations are independent.

This rather painful counting process was clarified in a sequence of papers by R. Blum [11, 12, 13] in the 1940's and 1950's but an excellent survey is found in the paper of H.F. Goenner [14].

Here is Blum's counting result for embedding $(M^n, g) \to (\mathbb{R}^m, \cdot)$ as given on p.143 of Goenner's paper with some paraphrasing by me.

Theorem 3. *If the Gauss equation is satisfied by a set of second fundamental forms of maximal rank, then (i) for* $0 \le p = m - n \le \frac{1}{8} n(n-2)$ *all Codazzi and Ricci equations are consequences of the Gauss equation; (ii) for* $\frac{1}{8} n(n-2) < p = m - n \le \frac{1}{2} n(n-1)$ *a system of* $\frac{1}{3} n(n^2 - 1)\left[p - \frac{1}{8} n(n-2)\right]$ *Codazzi equations are independent. The remainder of the Codazzi equations and all Ricci equations are a consequence of the independent system and of the Gauss equations.*

For the case $m = 6, n = 3, p = 3$ we are in category (ii) of the above theorem:

$$\frac{1}{8}\,3(1) \leq 3 \leq 3$$

and the theorem asserts that

$$\frac{1}{3}\,3(8)\,\left[\frac{24}{8} - \frac{3}{8}\right] = 21$$

Codazzi equations are independent and all the Ricci equations follow from these independent Codazzi equations and the Gauss equations. So our elementary count gave 24 Codazzi equations and Blum's count gave 21 independent Codazzi equations.

So where is the discrepancy? The answer is that in our elementary counting **we did not use the three equations in Bianchi's second identity**. If we substitute the Gauss relations into these three equations we have three more equations relating derivatives of the second fundamental forms and hence only 21 not 24 Codazzi equations are independent.

But even with the above count which says that 21 Codazzi and 6 Gauss suffice we see the $A^{\nu}_{\mu\ell}$ (connections on the normal bundle) still enter the Codazzi equations (1.6). So while we have 21 Codazzi + 6 Gauss = 27 equations and 18 $H^{\mu}_{ij} + 9\,A^{\nu}_{\mu k} = 27$ unknowns it is not clear immediately how to prove even a local existence theorem for this system.

For the general case when $m = \frac{n}{2}\,(n+1)$, Blum's theorem case (ii) again applies and we have $p = \frac{1}{2}\,n(n-1)$ and there are $\frac{1}{24}\,n^2(n^2-1)(3n-2)$ independent Codazzi equations and under the maximal rank conditions the Codazzi and Gauss equations imply the Ricci equations.

A sketch of the proof of Blum's theorem when $n = 3, m = 6$

1. From the Codazzi equations

$$\nabla_j H^{\mu}_{i\alpha} - \nabla_i H^{\mu}_{ja} + A^{\nu}_{\mu i} H^{\nu}_{\alpha j} - A^{\nu}_{\mu j} H^{\nu}_{\alpha i} = 0$$

we have in particular

$$\begin{aligned} \nabla_1 H^{\mu}_{23} - \nabla_3 H^{\mu}_{21} + A^{\nu}_{\mu 3} H^{\nu}_{21} - A^{\nu}_{\mu 1} H^{\nu}_{23} &= 0, \\ \nabla_1 H^{\mu}_{32} - \nabla_2 H^{\mu}_{31} + A^{\nu}_{\mu 2} H^{\nu}_{31} - A^{\nu}_{\mu 1} H^{\nu}_{32} &= 0. \end{aligned} \tag{1.15}$$

Subtract to see

$$\nabla_2 H^{\mu}_{31} - \nabla_3 H^{\mu}_{21} - A^{\nu}_{\mu 2} H^{\nu}_{31} + A^{\nu}_{\mu 3} H^{\nu}_{21} = 0. \tag{1.16}$$

Thus the Codazzi relation (1.14) is implied by the first two (1.13). For $n = 3, m = 6$ this reduces the number of independent Codazzi relations by 3.

2. We can rewrite the Codazzi equations as

$$\varepsilon_{\ell ji} \nabla_j H^{\mu}_{1a} - \varepsilon_{\ell ji} A^{\nu}_{\mu i} H^{\nu}_{\alpha j} = 0$$

Recall the relation

$$\operatorname{cof} H^\mu_{i\ell} = \varepsilon_{ijk}\varepsilon_{\ell mn} H^\mu_{kn} H^\mu_{jm}.$$

Hence

$$\begin{aligned}\nabla_\ell \operatorname{cof} H^\mu_{i\ell} &= \varepsilon_{ijk}\varepsilon_{\ell mn}(\nabla_\ell H^\mu_{kn})H^\mu_{jm} \\ &\quad + \varepsilon_{ijk}\varepsilon_{\ell mn}H^\mu_{kn}(\nabla_\ell H^\mu_{jm}) \\ &= 2\varepsilon_{ijk}\varepsilon_{\ell mn}\, H^\mu_{jm}\nabla_\ell H^\mu_{kn} \text{ (no sum on } \mu) \qquad (1.17)\end{aligned}$$

From the Codazzi equations

$$\varepsilon_{\ell mn}\nabla_\ell H^\mu_{kn} - \varepsilon_{\ell mn}A^\nu_{\mu n}H^\nu_{k\ell} = 0, \qquad (1.18)$$

$$\varepsilon_{\ell mn}\nabla_\ell H^\mu_{jm} - \varepsilon_{\ell mn}A^\nu_{\mu m}H^\nu_{j\ell} = 0 \qquad (1.19)$$

and substitution of these relations into (1.15) yields

$$\begin{aligned}\nabla_\ell(\operatorname{cof} H^\mu_{i\ell}) &- \varepsilon_{ijk}\varepsilon_{\ell mn}A^\nu_{\mu n}H^\nu_{k\ell}H^\mu_{jm} \\ &- \varepsilon_{ijk}\varepsilon_{\ell mn}A^\mu_{\mu m}H^\nu_{j\ell}H^\mu_{kn} = 0 \text{ (no sum on } \mu).\end{aligned}$$

Interchange m, n and j, k in the above expression to see

$$\nabla_\ell(\operatorname{cof} H^\mu_{i\ell}) - 2\varepsilon_{ijk}\varepsilon_{\ell mn}\, A^\nu_{\mu m}H^\nu_{j\ell}H^\mu_{kn} = 0, \text{ (no sum on } \mu).$$

Now sum on μ to find

$$\sum_{\mu=4}^{6}\nabla_\ell(\operatorname{cof} H^\mu_{i\ell}) - 2\varepsilon_{ijk}\varepsilon_{\ell mn}\sum_{\mu=4}^{6}A^\nu_{\mu m}H^\nu_{j\ell}H^\mu_{kn} = 0.$$

The Gauss equations may be written as

$$\sum_{\mu=4}^{6}\operatorname{cof} H^\mu_{i\ell} = R_{i\ell}$$

where

$$R = \begin{bmatrix} R_{2323} & R_{2331} & R_{2312} \\ R_{2331} & R_{3131} & R_{3112} \\ R_{2312} & R_{3112} & R_{1212} \end{bmatrix}$$

Thus our equation for divergence of the cofactors becomes for $n = 3, m = 6$:

$$\nabla_\ell R_{i\ell} - 2\varepsilon_{ijk}\varepsilon_{mn\ell}\sum_{\mu=4}^{6}A^\nu_{\mu m}H^\nu_{j\ell}H^\mu_{kn} = 0 \quad i = 1, 2, 3$$

The first term on the left hand side is zero by the second Bianchi identity, i.e.

$$\nabla_\ell(R_{1\ell}) = 0,\ \nabla_\ell(R_{2\ell}) = 0,\ \nabla_\ell(R_{3\ell}) = 0.$$

The second term on the left hand side is zero from the skew symmetry of $A^{\nu}_{\mu k}$ in μ, ν. Hence we have shown a combination of the Codazzi equations combined with the Gauss relation gives three trivial relations $0 = 0$, and thus we have reduced the number of independent Codazzi equations by an additional 3.

3. Now write the Codazzi and Ricci equations as

$$\begin{aligned} C^{\mu}_{k\alpha} &\stackrel{\text{def}}{=} \varepsilon_{ijk}\nabla_j H^{\mu}_{i\alpha} + \varepsilon_{ijk}A^{\nu}_{\mu i}H^{\nu}_{\alpha j} &= 0, \\ K^{\mu}_{k\mu} &\stackrel{\text{def}}{=} \varepsilon_{ijk}\nabla_i A^{\nu}_{\mu j} + \varepsilon_{ijk}A^{\nu}_{\eta i}A^{\eta}_{\mu j} - g^{pq}\varepsilon_{ijk}H^{\mu}_{ip}H^{\nu}_{jq} &= 0. \end{aligned}$$

Apply ∇_k to the Codazzi system

$$\varepsilon_{ijk}\nabla_k\nabla_j H^{\mu}_{i\alpha} + \varepsilon_{ijk}(\nabla_k A^{\nu}_{\mu i})H^{\nu}_{\alpha j} + \varepsilon_{ijk}A^{\nu}_{\mu i}(\nabla_k H^{\nu}_{\alpha j}) = 0$$

The last term can be rewritten using Codazzi as

$$\varepsilon_{ijk}\nabla_k H^{\nu}_{\alpha j} = -\varepsilon_{ijk}A^{\eta}_{\nu j}H^{\eta}_{\alpha k}.$$

Thus

$$\varepsilon_{ijk}\nabla_k\nabla_j H^{\mu}_{i\alpha} + \varepsilon_{ijk}\nabla_k A^{\nu}_{\mu i}H^{\nu}_{\alpha j} - \varepsilon_{ijk}A^{\eta}_{\nu j}H^{\eta}_{\alpha k}A^{\nu}_{\mu i} = 0$$

or interchanging $i \to j \to k \to i$ in the last term

$$\varepsilon_{ijk}\nabla_k\nabla_j H^{\mu}_{i\alpha} + \varepsilon_{ijk}H^{\nu}_{\alpha j}(\nabla_k A^{\nu}_{\mu i} - A^{\eta}_{\nu k}A^{\eta}_{\mu i}) = 0$$

Finally use the formula for the commutation $\varepsilon_{ijk}\nabla_k\nabla_j H^{\mu}_{ia}$ and the Gauss equations and we recover $H^{\nu}_{\alpha j}$ multiplying the Ricci equations, i.e.

$$H^{\nu}_{aj}\, K^{\nu}_{i\mu} = 0$$

where $K^{\mu}_{i\mu} = 0$. Since $1 \leq \alpha, j \leq 3$ this gives 9 equations in the nine unknowns $K^{\nu}_{i\mu}$. Blum's rank condition asserts that this system has the unique solution $K^{\nu}_{i\mu} = 0$ and the Ricci equations are satisfied.

2 Symmetrization of the Codazzi equations

A subset of the Codazzi equations is given by

$$\nabla_1 H^{\mu}_{i\ell} - \nabla_\ell H^{\mu}_{i1} + A^{\mu}_{\nu 1}H^{\nu}_{i\ell} - A^{\mu}_{\nu\ell}H^{\nu}_{i1} = 0. \tag{2.1}$$

On the other hand the full set of Codazzi equations can be written as

$$\varepsilon_{\ell ji}\nabla_j H^{\mu}_{ip} + \varepsilon_{\ell ji}A^{\mu}_{\nu i}H^{\nu}_{jp} = 0. \tag{2.2}$$

Recall the relation

$$\operatorname{cof} H^\mu_{i\ell} = \varepsilon_{ijk}\varepsilon_{\ell mn} H^\mu_{kn} H^\mu_{jm}. \tag{2.3}$$

Hence we have

$$\begin{aligned} \nabla_\ell \operatorname{cof} H^\mu_{i\ell} &= \varepsilon_{ijk}\varepsilon_{\ell mn} \nabla_\ell H^\mu_{kn} H^\mu_{jm} \\ &\quad + \varepsilon_{ijk}\varepsilon_{\ell mn} H^\mu_{kn} \nabla_\ell H^\mu_{jm}. \end{aligned} \tag{2.4}$$

But from the Codazzi relations we have

$$\begin{aligned} \varepsilon_{\ell mn}\nabla_\ell H^\mu_{kn} + \varepsilon_{\ell mn} A^\mu_{\nu n} H^\nu_{\ell k} &= 0, \\ \varepsilon_{\ell mn}\nabla_\ell H^\mu_{jm} + \varepsilon_{\ell mn} A^\mu_{\nu m} H^\nu_{\ell j} &= 0, \end{aligned}$$

and substitution into (2.4) yields

$$\begin{aligned} \nabla_\ell \operatorname{cof} H^\mu_{i\ell} &= -\varepsilon_{ijk}\varepsilon_{\ell mn} A^\mu_{\nu n} H^\mu_{jm} H^\nu_{\ell k} \\ &\quad - \varepsilon_{ijk}\varepsilon_{\ell mn} A^\mu_{\nu m} H^\mu_{kn} H^\nu_{\ell j}, \end{aligned}$$

i.e.

$$\nabla_\ell \operatorname{cof} H^\mu_{i\ell} = -2\varepsilon_{ijk}\varepsilon_{\ell mn} A^\mu_{\nu n} H^\mu_{jm} H^\nu_{\ell k}. \tag{2.5}$$

Since

$$\frac{\partial^2 W}{\partial H^\mu_{jk}\partial H^\mu_{i\ell}} = 2\varepsilon_{jim}\varepsilon_{k\ell n} H^\mu_{mk}$$

and

$$\operatorname{cof} H^\mu_{i\ell} = \varepsilon_{jim}\varepsilon_{k\ell n} H^\mu_{mn} H^\mu_{kj}$$

we see (2.5) can be written as

$$\begin{aligned} \varepsilon_{jim}\varepsilon_{k\ell n}\nabla_\ell H^\mu_{mn} H^\mu_{kj} + \varepsilon_{jim}\varepsilon_{k\ell n} H^\mu_{mn}\nabla_\ell H^\mu_{jk} &= -\frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{kn}}\, A^\mu_{\nu n} H^\nu_{\ell k}, \\ \frac{1}{2}\,\frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{mn}}\,\nabla_\ell H^\mu_{mn} + \frac{1}{2}\,\frac{\partial^2 W}{\partial H^\mu_{jk}\partial H^\mu_{i\ell}}\,\nabla_\ell H^\mu_{jk} &= -\frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{kn}}\, A^\mu_{\nu n} H^\nu_{\ell k}, \end{aligned}$$

and hence

$$\frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}}\,\nabla_\ell H^\mu_{jk} = -\frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jm}}\, A^\mu_{\nu m} H^\nu_{\ell k}. \tag{2.6}$$

Next multiply (2.1) by

$$-\frac{\partial^2 W}{\partial H^\mu_{i\ell}\,\partial H^\mu_{jk}}$$

to obtain

$$\begin{aligned} &-\frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}}\,\nabla_1 H^\mu_{i\ell} + \frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}}\nabla_\ell H^\mu_{i1} - \frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}}\, A^\mu_{\nu 1} H^\nu_{i\ell} \\ &+ \frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}}\, A^\mu_{\nu\ell} H^\nu_{i1} = 0. \end{aligned} \tag{2.7}$$

We can write (2.6), (2.7) as

$$
\begin{aligned}
& \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}} \end{bmatrix} \nabla_1 \begin{bmatrix} H^\mu_{i1} \\ H^\mu_{i\ell} \end{bmatrix} \\
+ & \begin{bmatrix} 0 & \dfrac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}} \\ \dfrac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}} & 0 \end{bmatrix} \nabla_\ell \begin{bmatrix} H^\mu_{i1} \\ H^\mu_{jk} \end{bmatrix} \\
+ & \begin{bmatrix} \dfrac{\partial W}{\partial H^\mu_{i\ell}\partial H^\mu_{kn}} & A^\mu_{\nu n} H^\nu_{\ell k} \\ \dfrac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}} & (-A^\mu_{\nu 1} H^\mu_{i\ell} + A^\mu_{\nu\ell} H^\nu_{i1}) \end{bmatrix} \\
= & \ 0
\end{aligned}
\tag{2.8}
$$

Explicitly we have, say for coefficient of ∇_2:

$$
\begin{bmatrix} \dfrac{\partial^2 W}{\partial H^\mu_{11}\partial H^\mu_{12}} & \dfrac{\partial^2 W}{\partial H^\mu_{12}\partial H^\mu_{12}} & \cdots & \dfrac{\partial^2 W}{\partial H^\mu_{33} H^\mu_{12}} \\ \dfrac{\partial^2 W}{\partial H^\mu_{11}\partial H^\mu_{22}} & \dfrac{\partial^2 W}{\partial H^\mu_{12}\partial H^\mu_{22}} & \cdots & \dfrac{\partial^2 W}{\partial H^\mu_{33}\partial H^\mu_{22}} \\ \dfrac{\partial^2 W}{\partial H^\mu_{11}\partial^\mu_{32}} & \cdots & & \dfrac{\partial^2 W}{\partial H^\mu_{33}\partial H^\mu_{32}} \end{bmatrix}_{3\times 9} \nabla_2 \begin{bmatrix} H^\mu_{11} \\ H^\mu_{12} \\ \vdots \\ H^\mu_{33} \end{bmatrix}
$$

$$
\begin{bmatrix} \dfrac{\partial^2 W}{\partial H^\mu_{12}\partial H^\mu_{11}} & \dfrac{\partial^2 W}{\partial H^\mu_{22}\partial H^\mu_{11}} & \dfrac{\partial^2 W}{\partial H_{32} H^\mu_{11}} \\ \dfrac{\partial^2 W}{\partial H^\mu_{12}\partial H^\mu_{12}} & \dfrac{\partial^2 W}{\partial H_{22} H^\mu_{12}} & \dfrac{\partial^2 W}{\partial H^\mu_{32}\partial H^\mu_{12}} \\ \vdots & & \\ \dfrac{\partial^2 W}{\partial H^\mu_{12}\partial H^\mu_{33}} & \dfrac{\partial^2 W}{\partial H^\mu_{22}\partial H^\mu_{33}} & \dfrac{\partial^2 W}{\partial H^\mu_{32}\partial H^\mu_{33}} \end{bmatrix}_{9\times 3} \nabla_2 \begin{bmatrix} H^\mu_{11} \\ H^\mu_{21} \\ H^\mu_{31} \end{bmatrix}
$$

We set the

$$
\begin{aligned}
3\times 9 \quad \text{coefficient matrix} &= L^\mu_2 \\
9\times 3 \quad \text{coefficient matrix} &= (L^\mu_2)^T
\end{aligned}
$$

and the ∇_2 term is

$$12\left\{\left|\begin{array}{c|c} 0 & L_2^\mu \\ & \qquad 3\times 9 \\ \hline (L_2^\mu)^T & 0 \\ \qquad 9\times 3 & \end{array}\right|\right. \quad \nabla_2 \left|\begin{array}{c} H_{11}^\mu \\ H_{21}^\mu \\ H_{31}^\mu \\ H_{11}^\mu \\ H_{12}^\mu \\ \vdots \\ H_{33}^\mu \end{array}\right|$$

$$\underbrace{\qquad\qquad\qquad}_{12}$$

Hence if we define the ∇_2 coefficient matrix as

$$B_2^\mu = \begin{bmatrix} 0 & L_2^\mu \\ (L_2^\mu)^T & 0 \end{bmatrix}$$

we see B_2^μ is symmetric.

This symmetry holds for every coefficient matrix in (2.8) including $\ell = 1$. We check the coefficient matrix in the first term in (2.8) by a separate argument. Note

$$\begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{-\partial^2 W}{\partial H_{i\ell}^\mu \partial H_{jk}^\mu} \end{bmatrix} \quad \text{is}$$

$$\begin{bmatrix} 0^{3\times 3} & 0^{9\times 3} \\ 0^{3\times 9} & -L^\mu_{0_{3\times 9}} \end{bmatrix} = B_0^\mu$$

where

$$L_0^\mu = \begin{bmatrix} \dfrac{\partial^2 W}{\partial H_{11}^\mu \partial H_{11}^\mu} & \dfrac{\partial^2 W}{\partial H_{12}^\mu \partial H_{11}^\mu} & \dfrac{\partial^2 W}{\partial H_{33}^\mu \partial_{11}^\mu} \\ \dfrac{\partial^2 W}{\partial H_{11}^\mu \partial H_{12}^\mu} & \dfrac{\partial^2 W}{\partial H_{12}^\mu \partial H_{12}^\mu} & \dfrac{\partial^2 W}{\partial H_{33}^\mu \partial H_{12}^\mu} \\ \vdots & & \\ \dfrac{\partial^2 W}{\partial H_{11}^\mu \partial H_{33}^\mu} & & \dfrac{\partial^2 W}{\partial H_{33}^\mu \partial H_{33}^\mu} \end{bmatrix}$$

Hence our Codazzi system (2.8) can be written as

$$B_0^\mu \nabla_1 U^\mu + B_\ell^\mu \nabla_\ell U^\mu + Q^\mu = 0 \tag{2.9}$$

where

$$(U^\mu)^T = (H^\mu_{11}, H^\mu_{21}, H^\mu_1, H^\mu_{11}, H^\mu_2, H^\mu_{13}, \ldots\ldots H^\mu_{33})$$

and

$$Q^\mu = \left[\begin{array}{cc} \dfrac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{kn}} & A^\mu_{\nu m} H^\nu_{\ell k} \\ \\ \dfrac{\partial^2 W}{\partial H^\mu_{u}\partial H^\mu_{jn}} & (-A^\mu_{u1} H^\nu_{i\ell} + A^\mu_{\nu\ell} H^\nu_{21}) \end{array}\right] \begin{array}{l} \}3 \\ \\ \}9 \end{array} \tag{2.10}$$

3 Symmetrization of the linearized Codazzi equations

The linearization of the relations (2.1), (2.2) is

$$\begin{aligned} \nabla_1 \dot H^\mu_{i\ell} \ &- \ \nabla_\ell\, \dot H^\mu_{i1} h + \dot A^\mu_{\nu 1} H^\nu_{i\ell} + A^\mu_{\nu 1} \dot H^\nu_{i\ell} \\ &- \ \dot A^\mu_{\nu\ell} H^\nu_{i1} - \dot A^\mu_{\nu\ell} \dot H^\nu_{i1} \\ &- \ \dot\Gamma^q_{i1} H^\mu_{\ell q} - \dot\Gamma^q_{\ell 1} H^\mu_{iq} \\ &+ \ \dot\Gamma^q_{i\ell} H^\mu_{1q} + \dot\Gamma^q_{1\ell} H^\mu_{iq} = 0. \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\varepsilon_{\ell j i}(\nabla_j \dot H^\mu_{ip} - \dot\Gamma^q_{jp} H^\mu_{iq} - \dot\Gamma^q_{ji} H^\mu_{pq}) \\ &+ \varepsilon_{\ell j i}(\dot A^\mu_{\nu i} H^\nu_{jp} + A^\nu_{\nu i} \dot H^\nu_{jp}) = 0 \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} &\varepsilon_{k\ell n}(\nabla_\ell \dot H^\mu_{mn} - \dot\Gamma^q_{\ell m} H^\mu_{nq} - \dot\Gamma^q_{\ell n} H^\mu_{mq}) \\ &+ \varepsilon_{k\ell n}(\dot A^\mu_{\nu n} H^\nu_{\ell m} + A^\mu_{\nu n} \dot H^\nu_{\ell m}) = 0 \end{aligned} \tag{3.3}$$

Multiply (3.3) by $\varepsilon_{jim} H^\mu_{kj}$ and we have

$$\begin{aligned} &\varepsilon_{jim}\varepsilon_{k\ell n} H^\mu_{kj}(\nabla_\ell \dot H^\mu_{mn} - \dot\Gamma^q_{\ell m} H^\mu_{nq} - \dot\Gamma^q_{\ell n} H^\mu_{mq}) \\ &+ \varepsilon_{jim}\varepsilon_{k\ell n} H^\mu_{jk}(\dot A^\mu_{\nu n} H^\nu_{\ell m} + A^\mu_{\nu n} \dot H^\nu_{\ell m}) = 0, \end{aligned} \tag{3.4}$$

and hence

$$\begin{aligned} &\frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{mn}}\,(\nabla_\ell \dot H^\mu_{mn} - \dot\Gamma^q_{\ell m} H^\mu_{nq} - \dot\Gamma^q_{\ell n} H^\mu_{mq}) \\ &+ \frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{mn}}\,(\dot A^\mu_{\nu n} H^\nu_{\ell m} + A^\mu_{\nu n} \dot H^\nu_{\ell m}) = 0. \end{aligned} \tag{3.5}$$

Next multiply (3.1) by

$$-\frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}}$$

to obtain

$$
\begin{aligned}
- \quad & \frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}} \nabla_\ell \dot{H}^\mu_{i\ell} + \frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jn}} \nabla_\ell \dot{H}^\mu_{i1} \\
& + \frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}} (\dot{A}_{\nu 1} H^\nu_{i\ell} + A_{\nu 1}\dot{H}^\nu_{i\ell} \\
& - \dot{A}^\mu_{\nu\ell} H^\nu_{i1} - A^\mu_{\nu\ell}\dot{H}^\nu_{i1} \\
& - \dot{\Gamma}^q_{i1} H^\mu_{\ell q} - \dot{\Gamma}^q_{\ell 1} H^\mu_{iq} \\
& - \dot{\Gamma}^q_{i\ell} H^\mu_{1q} - \dot{\Gamma}^q_{\mathbf{1}\ell} H^\mu_{iq})
\end{aligned}
\tag{3.6}
$$

Define

$$
\dot{Q}^\mu = \begin{bmatrix} \frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{kn}} (\dot{A}^\mu_{\nu n} H^\nu_{\ell k} + A^\mu_{\nu n}\dot{H}^\nu_{\ell k}) \\ \\ \frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}} (-\dot{A}^\mu_{\nu 1} H^\nu_{i\ell} - A_{\nu 1}\dot{H}^\nu_{i\ell} + \dot{A}^\mu_{\nu\ell} H^\nu_{i1} + A_{\nu\ell}\dot{H}^\nu_{i1}) \end{bmatrix} \tag{3.7}
$$

$$
\dot{S}^\mu = \begin{bmatrix} \frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{mn}} (-\dot{\Gamma}^q_{\ell m} H^\mu_{nq} - \dot{\Gamma}^q_{\ell n} H^\mu_{mq}) \\ \\ \frac{\partial^2 W}{\partial H^\mu_{i\ell}\partial H^\mu_{jk}} (-\dot{\Gamma}^q_{i1} H^\mu_{\ell q} - \dot{\Gamma}^q_{\ell 1} H^\mu_{iq} - \dot{\Gamma}^q_{i\ell} H^\mu_{1q} - \dot{\Gamma}^q_{1\ell} H^\mu_{iq}) \end{bmatrix} \tag{3.8}
$$

Then the linearized Codazzi system (3.4), (3.5) is now

$$
B^\mu_\nu \nabla_1 \dot{U}^\mu + B^\mu_\ell \nabla_\ell \dot{U}^\mu + \dot{Q}^\mu + \dot{S}^\mu = 0. \tag{3.9}
$$

4 The Ricci equations

Without loss of generality[1] we can take

$$
A^\nu_{\mu 1} = 0 \tag{4.1}
$$

and hence the Ricci equations imply

$$
\partial_1 A^\nu_{\mu 2} = g^{pq}(H^\mu_{1p} H^\nu_{2q} - H^\mu_{2p} H^\nu_{1q}), \tag{4.2}
$$

$$
\partial_1 A^\nu_{\mu 3} = g^{pq}(H^\mu_{1p} H^\nu_{3q} - H^\mu_{3p} H^\nu_{1q}). \tag{4.3}
$$

[1]Deane Yang pointed out this equality to me and called it a "gauge condition". An analogy with continuum mechanics might be setting the pressure equal to zero on the surface of a water wave.

Thus $A^{\nu}_{\mu 2}, A^{\nu}_{\mu 3}$ are completely determined by their data on a plane $x_1 = \text{const} = -L$ and the set of H^{ν}_{jk}. Thus we may substitute

$$\begin{aligned} A^{\nu}_{\mu 2}(x_1, x_2, x_3) &= A^{\nu}_{\mu 2}(-L, x_2, x_3) + \int_{-L}^{x_1} g^{pq}(H^{\mu}_{1p}H^{\nu}_{2q} - H^{\mu}_{2p}H^{\nu}_{1q})dx'_1 \qquad (4.4) \\ A^{\nu}_{\mu 3}(x_1, x_2, x_3) &= A^{\nu}_{\mu 3}(-L, x_2, x_3) + \int_{-L}^{x_1} g^{pq}(H^{\mu}_{1p}H^{\nu}_{3q} - H^{\mu}_{3p}H^{\nu}_{1q})dx'_1, \qquad (4.5) \end{aligned}$$

into (2.10). Hence the dependence on $A^{\mu}_{\nu\ell}$ is now reduced to dependence on the data provided on $x_1 = -L$. Of course the data must be consistent with the additional Ricci equations.

5 The full nonlinear system

In analogy with **continuum mechanics** we restate our derivation of the full nonlinear system.

The **balance laws** are given by the quasi-linear Codazzi equations (2.9):

$$B^{\mu}_0 \nabla_1 U^{\mu} + B^{\mu}_{\ell} \nabla_{\ell} U^{\mu} + Q^{\mu} = 0$$

where $U^{\mu} \in \mathbb{R}^{12}$ for each $\mu = 4, 5, 6$, Q_{μ} given by (2.10).

The **constitutive relations** are given by the Gauss equations

$$\sum_{\mu} H^{\mu}_{ik}H^{\mu}_{j\ell} - H^{\mu}_{i\ell}H^{\mu}_{jk} = R_{ijk\ell}$$

and constitutive relation for $A^{\mu}_{\nu\ell}$ given by (4.1), (4.4).

According to the theorem of Blum [11] if H^{μ}_{jk} are of full rank there are 27 independent equations in the 27 unknowns $H^{\mu}_{ij}, A^{\mu}_{\nu\ell}$ since the Ricci equations follow from the Gauss and Codazzi equations. Notice relations (4.4), (4.5) from the Ricci equations do not completely eliminate the $A^{\mu}_{\nu\ell}$ in favour of H^{μ}_{ij}, the initial data on $x_1 = -L$ still enter into values of $A^{\mu}_{\nu\ell}$.

6 The linearized Ricci equations

We linearize (4.1), (4.4), (4.5) to obtain the linearized Ricci equations

$$\dot{A}^{\nu}_{\mu 1} = 0, \qquad (6.1)$$

$$
\begin{aligned}
\dot{A}^{\nu}_{\mu 2}(x_1,x_2,x_3) &= \dot{A}^{\nu}_{\mu 2}(-L,x_2,x_3) \\
&+ \int_{-L}^{x_1} \dot{g}^{pq}(H^{\mu}_{1p}H^{\nu}_{2q} - H^{\mu}_{2p}H^{\nu}_{1q}) \\
&+ g^{pq}(\dot{H}^{\mu}_{1p}H^{\nu}_{2q} + H^{\mu}_{1p}\dot{H}^{\nu}_{2q} \\
&- \dot{H}^{\mu}_{1p}H^{\nu}_{2q} - H^{\mu}_{1p}\dot{H}^{\nu}_{2q})dx_1',
\end{aligned} \tag{6.2}
$$

$$
\begin{aligned}
\dot{A}^{\nu}_{\mu 3}(x_1,x_2,x_3) &= \dot{A}^{\nu}_{\mu 3}(-L,x_2,x_3) \\
&+ \int_{-L}^{x_1} \dot{g}^{pq}(H^{\mu}_{1p}H^{\nu}_{3q} - H^{\mu}_{3p}H^{\nu}_{1q}) \\
&+ g^{pq}(\dot{H}^{\mu}_{1p}H^{\nu}_{3q} + H^{\mu}_{1p}\dot{H}^{\nu}_{3q} \\
&- \dot{H}^{\mu}_{1p}H^{\nu}_{3q} - H^{\mu}_{1p}\dot{H}^{\nu}_{3q})dx_1'.
\end{aligned} \tag{6.3}
$$

If on the boundary of our domain $\dot{A}_{\mu 2}(-L,x_2,x_3), \dot{A}_{\mu 3}(-L,x_2,x_3)$ are zero then their contribution to (6.2), (6.3) is zero. Similarly the integral terms in (6.2), (6.3) are bounded by $K\,\mathrm{vol}(\Omega)$ where

$$
\begin{aligned}
K &= \|\dot{g}^{pq}\|_{L^2(\Omega)} \sup_{\substack{n\\ \mu\\ j,k}} |H^{\mu}_{jk}|^2 \\
&+ \sup_{\Omega} \|g^{pq}\| \sup_{\substack{\Omega\\ \mu\\ j,k}} |H^{\mu}_{jk}|\, \|\dot{H}^{\mu}_{jk}\|_{L^2(\Omega;\mathbb{R}_{27})}
\end{aligned}
$$

i.e.

$$
|\dot{A}^{\nu}_{\mu 2}| \le K\ \mathrm{vol}(\Omega)^{1/3}, \tag{6.4}
$$

$$
|\dot{A}^{\nu}_{\mu 3}| \le K\ \mathrm{vol}(\Omega)^{1/3}. \tag{6.5}
$$

Proof of 6.4

Let us consider typical terms in (6.2).

$$
\begin{aligned}
a(x_1,x_2,x_3) &= \int_{-L}^{x_1} \dot{g}^{pq}(H^{\mu}_{1p}H^{\nu}_{2q})dx_1' \\
b(x_1,x_2,x_3) &= \int_{-L}^{x_1} g^{pq}(\dot{H}^{\mu}_{1p}H^{\nu}_{2q})dx_1'
\end{aligned}
$$

where we do not sum on p,q.

From Cauchy-Schwarz we have

$$
\begin{aligned}
|a(x_1,x_2,x_3)| &\le \sup_{\Omega} |H^{\mu}_{1p}H^{\nu}_{2q}| \left(\int_{-L}^{x_1} dx_1'\right)^{1/2} \left(\int_{-L}^{x_1} |\dot{g}^{pq}|^2 dx_1'\right)^{1/2} \\
&\le \sup_{\Omega} |H^{\mu}_{1p}H^{\nu}_{2q}|(2L)^{1/2} \left(\int_{-L}^{L} |\dot{g}^{pq}|^2 dx_1'\right)^{1/2}.
\end{aligned}
$$

Since the right hand side is independent of x_1 we have

$$\int_{-L}^{L}\int_{-L}^{L}\int_{-L}^{L} |a(x_1,x_2,x_3)|^2 dx_1\, dx_2\, dx_3$$
$$\le (\sup_{\Omega} |H^\mu_{1p}H^\nu_{2q}|)^2 4L^2 \int_{-L}^{L}\int_{-L}^{L}\int_{-L}^{L} |\dot{g}^{pq}|^2 dx_1' dx_2 dx_3$$

and

$$\|a\|_{L^2(\Omega)} \le \sup_{\Omega} |H^\mu_{1p}H^\nu_{2q}| 2L \|\dot{g}^{pq}\|_{L^2(\Omega)}.$$

Similarly we have

$$|b(x_1,x_2,x_3)| \le \sup_{\Omega} |g^{pq}H^\nu_{2q}| \int_{-L}^{L} |\dot{H}^\mu_{1p}(x_1',x_2,x_3)| dx_1'$$

where again the right hand side is independent of x_1.

Thus we see

$$|b(x_1,x_2,x_3)|^2 \le (\sup_{\Omega} |g^{pq}H^\nu_{pq}|)^2 2L \int_{-L}^{L} |\dot{H}^\mu_{1p}(x_1',x_2,x_3)|^2 dx_1'$$

and

$$\int_{-L}^{L}\int_{-L}^{L}\int_{-L}^{L} |b(x_1,x_2,x_3)|^2 \le$$
$$\sup_{\Omega} |g^{pq}H^\nu_{2q}|^2 4L^2 \int_{-L}^{L}\int_{-L}^{L}\int_{-L}^{L} |\dot{H}^\mu_{1p}(x_1',x_2,x_3)|^2 dx_1', dx_2, dx_3.$$

This yields

$$\|b\|^2_{L^2(\Omega)} \le \sup_{\Omega} |g^{pq}H^\nu_{2q}|^2\, 2L \|\dot{H}^\mu_{1p}\|_{L^2(\Omega)}$$

7 The linearized Gauss equations

The linearized Gauss equations are

$$\begin{aligned} \sum_\mu & \dot{H}^\mu_{ik}H^\mu_{j\ell} + H^\mu_{ik}\dot{H}^\mu_{j\ell} \\ & - \dot{H}^\mu_{i\ell}H^\mu_{jk} - H^\mu_{i\ell}\dot{H}^\mu_{jk} \\ & = \dot{R}_{ijk\ell}. \end{aligned} \tag{7.1}$$

These are 6 equations in the 18 components $\dot{H}^\mu_{ij}$. We say H^μ_{ij} is **non-degenerate** in the neighbourhood of $x = 0$ if 6 of the components of $\dot{H}^\mu_{ij}$ can be solved in terms of the remaining 12 components and $\dot{R}_{ijk\ell}$. A **sufficient condition** for non-degeneracy is provided in Theorem F of the paper of Bryant, Griffiths, Yang [2] which establishes non-degeneracy if at least one component of the Riemann curvature tensor $R_{ijk\ell}$ is non-zero.

We assume our set H^μ_{ij} is non-degenerate in a neighbourhood of $x = 0$. This means that the vector

$$\dot{U} = \begin{bmatrix} \dot{U}^4 \\ \dot{U}^5 \\ \dot{U}^6 \end{bmatrix}$$

can be written as

$$\dot{U} = C\hat{H} + D\dot{R} \tag{7.2}$$

where $\hat{H}$ denotes the distinguished 12 components of the set $\dot{H}^\mu_{ij}$ and $\dot{R}$ denotes the 6 non-trivial elements of the perturbed Riemann curvature tensor. Hence $\dot{U} \in \mathbb{R}^{36}, \hat{H} \in \mathbb{R}^{12}, \dot{R} \in \mathbb{R}^6$, C is a 36×12 matrix, D is a 36×6 matrix.

8 The closed symmetric system for the linearized problem and quasi-linear problem

Recall the symmetrized Codazzi equations for the linearized problem are

$$B^\mu_0 \nabla_1 \dot{U}^\mu + B^\mu_\ell \nabla_\ell \dot{U}^\mu + \dot{Q}^\mu + \dot{S}^\mu = 0.$$

Set

$$B_0 = \begin{vmatrix} B^4_0 & 0 & 0 \\ 0 & B^5_0 & 0 \\ 0 & 0 & B^6_0 \end{vmatrix}$$

$$B_\ell = \begin{vmatrix} B^4_\ell & 0 & 0 \\ 0 & B^5_\ell & 0 \\ 0 & 0 & B^6_\ell \end{vmatrix}$$

$$\dot{Q} = \begin{vmatrix} \dot{Q}^4 \\ \dot{Q}^5 \\ \dot{Q}^6 \end{vmatrix} \qquad \dot{S} = \begin{vmatrix} \dot{S}^4 \\ \dot{S}^5 \\ \dot{S}^6 \end{vmatrix}$$

so that the system becomes

$$B_0 \nabla_1 \dot{U} + B_\ell \nabla_\ell \dot{U} + \dot{Q} + \dot{S} = 0. \tag{8.1}$$

$\dot{Q}$ depends linearly on $\dot{H}^\mu_{ij}$ and $\dot{A}^\mu_{\nu m}$ as given in (3.6). We represent this dependence by

$$\dot{Q} = E\dot{U} + F\dot{A} \tag{8.2}$$

$\dot{U} \in \mathbb{R}^{36}, \dot{A} \in \mathbb{R}^6, E$ a 36×36 matrix, F a 36×6 matrix.

Substitute (6.2) into (7.2) to obtain

$$\begin{aligned}\dot{Q} &= E(C\hat{H} + D\dot{R}) + F\dot{A} \\ &= G\hat{H} + J\dot{R} + F\dot{A} \qquad (8.3)\end{aligned}$$

where $G = EC$ is a 36×12 matrix, $J = ED$ is a 36×36 matrix. Hence via (7.2) the system (8.1) has the form

$$\begin{aligned}&B_0\nabla_1(C\hat{H} + D\dot{R}) + B_\ell\nabla_\ell(C\hat{H} + D\dot{R}) \\ &\quad +G\hat{H} + F\dot{R} + F\dot{A} + \dot{S} = 0. \qquad (8.4)\end{aligned}$$

We rewrite this in the form

$$\begin{aligned}&B_0C\nabla_1\hat{H} + B_\ell C\nabla_\ell\hat{H} \\ &+ (B_0\nabla_1 C + B_\ell\nabla_\ell C + G)\hat{H} \\ &+ B_0\nabla_1 D\dot{R} + B_\ell\nabla_\ell D\dot{R} \qquad (8.5) \\ &+ G\dot{R} + F\dot{A} + \dot{S} = 0.\end{aligned}$$

Next multiply (7.5) from the left by the 12×36 matrix C^T to obtain

$$\begin{aligned}\mathcal{A}_0\nabla_1\hat{H} &+ \mathcal{A}_\ell\nabla_\ell\hat{H} + \mathcal{B}\hat{H} \\ &+ C^TF\dot{A} + \Lambda = 0 \qquad (8.6)\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}_0 &= C^TB_0C, \\ \mathcal{A}_\ell &= C^TB_\ell C, \\ \mathcal{B} &= C^T(B_0\nabla_1 C + B_\ell\nabla_\ell C + G), \\ \Lambda &= C^T(B_0\nabla_1 D\dot{R} + B_\ell\nabla_\ell D\dot{R} + G\dot{R} + \dot{S}).\end{aligned}$$

To system (7.6) we amend the linearized Ricci equations (6.2), (6.3) with $\dot{A}^\nu_{p2}(-L, x_2, x_3) = \dot{A}^\nu_{\mu 3}(-L, x_2, x_3) = 0$

$$\begin{aligned}\dot{A}^\nu_{\mu\ell}(x_1, x_2, x_3) &= \int_{-L}^{x_1} \dot{g}^{pq}(H^\mu_{1p}H^\nu_{\ell q} - H^\mu_{\ell p}H^\nu_{1q}) \\ &\quad + g^{pq}(\dot{H}^\mu_{1p}H^\nu_{\ell q} + H^\mu_{1p}\dot{H}^\nu_{\ell q} - \dot{H}^\mu_{1p}H^\nu_{\ell q} - H^\mu_{1p}\dot{H}^\nu_{\ell q})dx_1' \\ \ell &= 2, 3, \qquad (8.7) \\ \dot{A}^\nu_{\mu 1} &= 0. \qquad (8.8)\end{aligned}$$

If we substitute (8.7), (8.8) into (8.6) we see that this system is a symmetric system of 12 equations in the 12 unknowns $\hat{H}$ which are weakly non-local due to (8.7). However the non-locality is very weak as seen by relations (6.4), (6.5).

The above derivation was of course for the linearized system, but examination of the steps used shows the same procedure will yield a quasi-linear system of 12 equations for the non-linear problem as well.

9 The weak form of the closed system

Define the linear operator $\mathcal{L}$ by

$$\mathcal{L}\hat{H} = \mathcal{A}_0\nabla_1\hat{H} + \mathcal{A}_\ell\nabla_\ell\hat{H} + \mathcal{B}\hat{H} + C^T F\dot{A} \tag{9.1}$$

where $\dot{A}$ is defined by (8.7), (8.8).

Let $(\cdot,\cdot)$ denote the inner product on $L^2(\Omega;\mathbb{R}^{12})$ and $V \in C_0^\infty(\Omega;\mathbb{R}^{12})$. Then the weak form of

$$\mathcal{L}\hat{H} = -\Lambda$$

is

$$(\mathcal{L}^*V, \hat{H}) = -(V, \Lambda). \tag{9.2}$$

Hence $(\mathcal{L}^*V, \hat{H})$ defines a bilinear form on $H_0^1(\Omega;\mathbb{R}^{12})$.

Recall the Lax-Milgram Theorem [15].

Theorem 4. *Let X be a Hilbert space and $\mathcal{C}(\chi,\psi)$ a (possibly complex) bilinear functional defined on the product space $X \times X$. Let $\|\cdot\|_X$ and $(\cdot,\cdot)_X$ denote the norm and inner product of X. If*

(i) (boundedness) $|\mathcal{C}(\chi,\psi)| \leq \gamma\|\chi\|_X\|\psi\|_X$

(ii) (coerciveness) $\mathcal{C}(\chi,\chi) \geq \delta\|\chi\|_X^2$

for δ, γ positive constants, then there exists a uniquely determined bounded linear operator T with bounded inverse T^{-1} such that

$$\mathcal{C}(\chi, T\psi) = (\chi,\psi)_X,$$

$\|T\|_X \leq \delta^{-1}, \|T^{-1}\|_X \leq \gamma$ *whenever $\chi,\psi \in X$.*

Let us choose $X = H_0^1(\Omega;\mathbb{R}^{12})$ for (9.2). Condition (i) of the Lax-Milgram Theorem holds but condition (ii) does not. Hence we regularize (9.2) and write the regularized problem as

$$(\mathcal{L}^*V, \hat{H}) + \varepsilon(\partial V, \partial\hat{H}) = -(V, \Lambda) - \varepsilon(\partial V, \partial\Lambda). \tag{9.3}$$

Now our bilinear form $\mathcal{C}_\varepsilon$ is defined by the left hand side of (9.3). We assume the weaker coerciveness estimate

$$(\mathcal{L}^*\hat{H}, \hat{H}) \geq \delta_1\|\hat{H}\|^2_{L^2(\Omega;\mathbb{R}^{12})} \tag{9.4}$$

for some positive constant δ_1. Then we have

(i) $|\mathcal{C}_\varepsilon(V,\hat{H})| \leq \gamma\|V\|_{\mathbf{x}}\|\hat{H}\|_{\mathbf{x}}$

and

(ii) $\mathcal{C}_\varepsilon(\hat{H},\hat{H}) \geq \delta_1 \|\hat{H}\|^2_{L^2(\Omega,\mathbb{R}^{12})} + \varepsilon(\partial\hat{H},\partial\hat{H})$.

Thus the Lax-Milgram Theorem applies and we get a solution $\hat{H}_\varepsilon = T_\varepsilon \Lambda$ of (8.3):

$$(L^*V,\hat{H}_\varepsilon) + \varepsilon(\partial V,\partial\hat{H}_\varepsilon) = -(V,\Lambda) - \varepsilon(\partial V,\partial\Lambda) \tag{9.5}$$

or alternatively

$$(\mathcal{L}^*V,\hat{H}_\varepsilon) - \varepsilon(\partial^2 V,\hat{H}_\varepsilon) = -(V,\Lambda) - \varepsilon(\partial V,\partial\Lambda) \tag{9.6}$$

for all $V \in H^1_0(\Omega;\mathbb{R}^{12})$.

Since (9.5) holds for all $V \in H^1_0(\Omega)$ take $V = \hat{H}_\varepsilon$ and we have

$$\begin{aligned}(\mathcal{L}^*\hat{H}_\varepsilon,\hat{H}_\varepsilon) \quad &+ \quad \varepsilon(\partial\hat{H}_\varepsilon,\partial\hat{H}_\varepsilon) \\ &= \quad (-H^\varepsilon,\Lambda) - \varepsilon(\partial\hat{H}_\varepsilon,\partial\Lambda).\end{aligned} \tag{9.7}$$

We estimate the left hand side of (8.7) from below and the right hand side from above

$$\begin{aligned}\delta_1\|\hat{H}_\varepsilon\|^2_{L^2(\Lambda;\mathbb{R}^{12})} \quad &+ \quad \varepsilon\|\partial\hat{H}_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^{12})} \\ &\leq \quad \|\hat{H}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{12})}\|\Lambda\|_{L^2(\Omega;\mathbb{R}^{12})} + \ \varepsilon\|\partial\hat{H}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{12})}\partial\Lambda\|_{L^2(\Omega;\mathbb{R}^{12})}\end{aligned} \tag{9.8}$$

or

$$\begin{aligned}\delta_1\|\hat{H}_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^{12})} \quad &+ \quad \varepsilon(\|\partial\hat{H}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{12})} - \frac{1}{2}\|\partial\Lambda\|^2_{L^2(\Omega;\mathbb{R}^{12})}) \\ &\leq \quad \|\hat{H}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{12})}\|\Lambda\|_{L^2(\Omega,\mathbb{R}^{12})} + \frac{\varepsilon}{4}\|\partial\Lambda\|^2_{L^2(\Omega;\mathbb{R}^{12})} \\ &\leq \quad \frac{\delta_1}{2}\,\|\hat{H}_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^{12})} + \frac{1}{2\delta_1}\,\|\Lambda\|_{L^2(\Omega;\mathbb{R}^{12})} + \ \frac{\varepsilon}{4}\,\|\partial\Lambda\|^2_{L^2(\Omega;\mathbb{R}^{12})}\end{aligned}$$

and hence we have

$$\frac{\delta_1}{2}\,\|\hat{H}_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^{12})} \leq \frac{1}{2\delta_1}\,\|\Lambda\|_{L^2(\Omega;\mathbb{R}^{12})} + \frac{\varepsilon}{4}\,\|\partial\Lambda\|^2_{L^2(\Omega;\mathbb{R}^{12})}. \tag{9.9}$$

Thus for $\Lambda \in H^1_0(\Omega)$ we have $\hat{H}_\varepsilon$ bounded independently of ε in $L^2(\Omega;\mathbb{R}^{12})$. Thus $\hat{H}_\varepsilon$ has a weakly convergent subsequence (also denoted by $\hat{H}_\varepsilon$) so that

$$\hat{H}_\varepsilon \rightharpoonup \hat{H} \quad \text{in} \quad L^2(\Omega;\mathbb{R}^{12}).$$

Thus for all $V \in C^\infty_0(\Omega)$ we may pass to the limit as $\varepsilon \to 0$ in (8.6) to find

$$(\mathcal{L}^*V,\hat{H}) = -(V,\Lambda).$$

The weak solution $\hat{H}$ is unique via coercivity (9.4).

We summarize this result in the following theorem.

Theorem 5. *If the operator $\mathcal{L}$ defined by (9.1) satisfies the coercivity condition*

$$(\mathcal{L}^*\hat{H}, \hat{H}) \geq \delta_1 \|\hat{H}\|_{L^2(\Omega;\mathbb{R}^{12})}$$

for some $\delta > 0$ the weak form of the linearized isometric embedding problem (9.2) has a unique solution for all $\Lambda \in H_0^1(\Omega)$.

Now let us apply Theorem 2 to our system (9.1), (9.7), (9.8). First we assume that we only perturb from our here (undotted) embedding in a small neighbourhood of $x = 0$. The point $x = 0$ is chosen to be the origin of a system of normal coordinates where the Christoffel symbols Γ^q_{ij} vanish at $x = 0$. We take our small neighbourhood to be the box, $-L \leq x_i \leq L,\ i = 1, 2, 3$. Hence on this box $\dot{A}$ defined by (8.7), (8.8) and satisfying (6.4), (5.5) will be negligible and not enter into the coercivity computation. Thus we can state

Theorem 6. *If the quadratic form*

$$\hat{H}^T(-\partial_1\mathcal{A}_0 - \partial_\ell\mathcal{A}_\ell + \mathcal{B})\hat{H} \tag{9.10}$$

is positive (or negative) definite at $x = 0$ there exists a unique weak solution to our linearized isometric embedding equations (9.1), (8.7), (8.8).

The parameters entering our 12×12 coefficient symmetric matrix

$$-\partial_1\mathcal{A}_0 - \partial_\ell\mathcal{A}_\ell + \frac{1}{2}(\mathcal{B}^T + \mathcal{B}) \tag{9.11}$$

are $H^\mu_{ij}, \partial_1\mathcal{A}_0, \partial_1\mathcal{A}_1, \partial_2\mathcal{A}_2, \partial_3\mathcal{A}_3, A^\nu_{\mu 2}, A^\nu_{\mu 3}$ all evaluated at $x = 0$. Hence via the classical chain rule applied to $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ the parameters entering into the coefficient matrix are $H^\mu_{ij}, \partial_\ell H^\mu_{i\ell}, A^\nu_{\mu 2}, A^\nu_{\mu 3}$ all evaluated at $x = 0$

(i) From the Gauss relations this gives 12 independent H^μ_{ij}.

(ii) From the differentiated Gauss relations (see e.g. Poole [6]) this gives 15 independent $\partial_\ell H^\mu_{ij}$.

(ii) There are 6 independent $A^\nu_{\mu 2}, A^\nu_{\mu 3}$.

Hence there are 12 + 15 + 6 = 33 free parameters entering into the 12×12 matrix.

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References

[1] Q. Han, J.-X. Hong, *Isometric Embedding of Riemannian Manifolds in Euclidean Spaces* (AMS, Providence, 2006).

[2] R.L. Bryant, P.A. Griffiths, D. Yang, Characteristics and existence of isometric embeddings. Duke Math. J. **50**, 893–994 (1983).

[3] G. Nakamura, Y. Maeda, Local isometric embedding problem of Riemannian 3-manifold into R^6. Proc. Jpn. Acad. Ser. A Math. Sci. **62**, 257–259 (1986).

[4] G. Nakamura, Y. Maeda, Local smooth isometric embeddings of low-dimensional Riemannian manifolds into Euclidean spaces. Trans. Am. Math. Soc. **313**, 1–51 (1989).

[5] J. Goodman, D. Yang, Local solvability of nonlinear partial differential equations of real principal type, unpublished, 1988, www.deaneyang.com/paper, goodman-yang.pdf.

[6] T.E. Poole, The local isometric embedding problem for 3-dimensional Riemannian manifolds with cleanly vanishing curvature. Commun. Partial Differ. Equ. **35**, 1802–1826 (2010).

[7] K.O. Friedrichs, Symmetric positive linear differential equations. Commun. Pure Appl. Math. **11**, 333–418 (1956).

[8] Q. Han, M. Khuri, The linearized system for isometric embeddings and its characteristic variety, Adv. Math. **23**, 263–293 (2012).

[9] J.F. Nash, Jr, The embedding problem for Riemannian manifolds. Annals of Math, **63**, 20–63, (1956).

[10] S.-T. Yau, Perspectives on Geometric Analysis [ar Xiv: math/0602363 v2, 16 Feb 2006], also Proceedings of the International Conference on Complex Geometry and Related Fields, 289–378, AMS/IP Stud. Adv. Math. **39**, American Mathematical Society, Providence, R.I., (2007).

[11] R. Blum, Subspaces of Riemannian spaces. Can. J. Math. **7**, 445–452 (1955).

[12] R. Blum, Ueber die Bedingungsgleichungen einer Riemann'schen Mannigfaltigkeit, die in einer Euklidischen Mannigfaltigkeit eingebetter ist (in German). Bull. Math. Soc. Roum. Sci **47**, 144–201 (1946).

[13] R. Blum, Sur les tenseurs dérivés de Gauss et Codazzi. C.R. Acad. Sci. Paris **244**, 708–709 (1947).

[14] H.F. Goenner, On the interdependency of the Gauss-Codazzi-Ricci equations of local isometric embedding. Gen. Relativ. Gravitation **8**, 139–145 (1977).

[15] K. Yosida, Functional Analysis. Springer-Verlag, Berlin-Heidelberg-New York (1965).