Implications

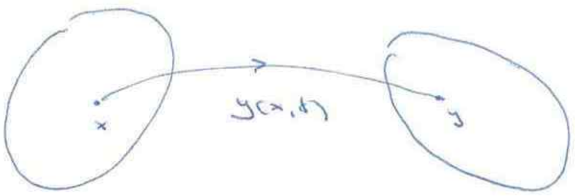
1) $$\partial_t^2 y_i = \partial_{x_\alpha} \frac{\partial W(F)}{\partial F_{i\alpha}}(F) + \varepsilon \, \partial^2_{x_i x_\alpha} v_i \qquad (1)$$

~~Solutions of (1) converge to smooth~~

Estimation between smooth solution of elasticity and solution of (1)

2) Variational approximation

$$\begin{cases} \dfrac{v - v_0}{h} = \partial_{x_\alpha}\left(\dfrac{\partial W}{\partial F_{i\alpha}}(F)\right) \\[2ex] \dfrac{y - y_0}{h} = v \end{cases}$$



Ω $\qquad\qquad\qquad$ $\Omega_t = y(\Omega)$

$\rho_0(x)$ $\qquad\qquad\qquad$ $\rho(y,t)$.

$v = \frac{\partial y}{\partial t}$ velocity

$F = \nabla_x y$ deformation gradient

balance of mass $\qquad$ $\rho_0(x) = \rho(y,t) \det F$

balance of linear mom. $\qquad$ $\rho_0 \frac{\partial^2 y}{\partial t^2} = \underbrace{\operatorname{div} S}_{\text{stress}} + \underbrace{\rho_0 b}_{\text{body force}}$

ELASTIC MATERIALS

$$S = S(F)$$

Hyperelastic $\qquad$ $S = \frac{\partial W(F)}{\partial F}$ $\qquad$ $W(F)$ stored energy function

consistency with 2^{nd} law of thermodynamics

existence of free energy

$$\rho_0 \frac{\partial^2 y}{\partial t^2} = \operatorname{div} \frac{\partial W}{\partial F}(\nabla y) + \rho_0 b$$

$$\boxed{\partial_t\left(\frac{1}{2}\rho_0\left|\frac{\partial y}{\partial t}\right|^2 + W(\nabla y)\right) = \operatorname{div}\left(\frac{\partial y}{\partial t}\cdot \underbrace{\frac{\partial W}{\partial F}(\nabla y)}\right) + \rho_0 b\cdot\frac{\partial y}{\partial t}}$$

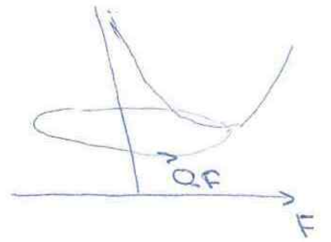
$S = \frac{\partial W}{\partial F}(F)$ Piola-Kirchhoff stress

Requirements on stored energy

1. Material frame indifference $\qquad W(QF) = W(F) \quad \forall\, Q$ orthogonal

2. Realizability of mechanical motion $\qquad \det F > 0$

$$W(F) \to \infty \quad \text{as } \det F \to 0$$



W can not be in general str. convex.

Calculus of variation

$$\min_{y \in W^{1,p}} \int_\Omega W(\nabla y)\, dx$$

Euler-Lagrange equation $\qquad \frac{\partial}{\partial x_\alpha}\left(\frac{\partial W}{\partial F_{i\alpha}}(\nabla y)\right) = 0$

$$\frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}(\nabla y)\,\frac{\partial^2 y_j}{\partial x_\alpha \partial x_\beta} = 0 \qquad \text{system of 3 equation}$$

strong ellipticity at $\bar F = \nabla \bar y$ if

$$\frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}(\bar F)\, \xi_i \xi_j \nu_\alpha \nu_\beta > 0 \qquad \forall\, \xi \neq 0,\ \nu \in S^2$$

$$\iff \quad Q(\bar F)_{\alpha\beta} = \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}(\bar F)\, \xi_i \xi_j \text{ is positive definite}$$

Dynamic elasticity

Find extrema of $I[y] = \int_0^T \int_\Omega \frac{1}{2}|y_t|^2 - W(\nabla y)\, dx\, dt$

E-L equations $\quad y_{tt} = \partial_{x_\alpha} \frac{\partial W}{\partial F}(\nabla y)$

System is hyperbolic $\iff \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}(\bar{F})\, \xi_i \xi_j \nu_\alpha \nu_\beta > 0 \quad \forall \xi \neq 0, \nu \in S^2$

rank-1 convexity

1st order system
$$\begin{cases} \partial_t F_{i\alpha} = \partial_\alpha v_i & 9 \text{ eqns} \\ \partial_t v_i = \partial_\alpha \frac{\partial W}{\partial F_{i\alpha}}(F) & 3 \text{ eqns} \end{cases}$$

Hyperbolicity $\iff$ $W(F)$ rank-1 convex

wave speeds. $\lambda_1 = \dots = \lambda_6 = 0$

$\lambda_6, \dots, \lambda_{12} = \pm\sqrt{\text{e.v. of accoustic tensor}}$

$$\partial_t \Big(\underbrace{\tfrac{1}{2}|v|^2 + W(F)}_{\substack{E(v,F) \\ \text{convex}}} \Big) = \partial_\alpha \left(v_i S_{i\alpha} \right)$$

Relative entropy - Stability

$\partial_t u + \operatorname{div} F(u) = 0$ system of c.l.

$\partial_t \eta(u) + \operatorname{div} q(u) = 0$ entropy - entropy flux pair

$(\eta(u), q_\alpha(u), \alpha = 1..., d)$ $\nabla q_\alpha(u) = \nabla\eta(u) \cdot \nabla F_\alpha(u)$

$\iff \nabla^2\eta \cdot \nabla F = \nabla F \cdot \nabla^2\eta = \nabla^2\eta \cdot \nabla F^T$

Suppose $\boxed{\eta(u) \text{ is convex}}$

(1) $\int_\Omega \eta(u(x,t))\,dx \leq \int_\Omega \eta(u_0(x))\,dx$ control of some norm

(2) $u(x,t)$ smooth solution $\partial_t \eta(u) + \partial_x q(u) = 0$

$v(x,t)$ approxim. solution $\partial_t \eta(v) + \partial_x q(v) = -\mu \leq 0$

$(= -\mu_\varepsilon \quad \mu_\varepsilon \to \mu, \mu \geq 0)$

Relative entropy $\eta(v|u) = \eta(v) - \eta(u) - \nabla\eta(u) \cdot (v-u)$

$\partial_t \eta(v) + \partial_x q(v) = -\mu$

$\partial_t \eta(u) + \partial_x q(u) = 0$

$\nabla\eta(u) \cdot \partial_t(v-u) + \partial_x(f(v) - f(u)) = 0$

$\partial_t \nabla\eta(u) \cdot (v-u) + \partial_x \nabla\eta(u) \cdot (f(v) - f(u))$

$= \partial_t \nabla\eta(u) \cdot (v-u) + \partial_x(\nabla\eta(u)) \cdot f(v) - f(u)$

$= -\nabla^2\eta(u) \nabla f(u) u_x \cdot (v-u) + \nabla^2\eta(u) u_x \cdot (f(v) - f(u))$

$= \nabla^2\eta(u) u_x \left[f(v) - f(u) - \nabla f(u)(v-u) \right]$

(5)

$$\partial_t \left[\eta(v) - \eta(u) - \nabla\eta(u)\cdot(v-u) \right] + \partial_x \left[q(v) - q(u) - \nabla\eta(u)\left(f(v)-f(u)\right)\right]$$

$$= -\mu - \nabla^2\eta(u)\, u_x \left[f(v) - f(u) - \nabla f(u)(v-u) \right]$$

$$\leq O(1)\; C(\|u\|_{L^\infty})\, |v-u|^2$$

$$\boxed{\text{Control of} \quad \int \eta(v|u)\,dx}$$

Gives
- uniqueness of smooth solution in the class of entropy weak solution
- stability approximation.

Defn

1. $W(F)$ is quasiconvex at F if

$$\int_\Omega W(F + \nabla\varphi(x))\,dx \geq \int_\Omega W(F)\,dx \qquad \forall\, \varphi \in C_c^\infty(\Omega)$$

2. $W(F)$ quasiconvex if $W(F)$ quasiconvex at any F.

$W(F)$ quasiconvex $\iff$ $I[y] = \int_\Omega W(\nabla y)\,dx$ is weakly lower semicontinuous in $W^{1,p}(\Omega)$

Defn $\Phi(F)$ is a null-Lagrangian, if

$$\int \Phi(\nabla y + \nabla\varphi(x))\,dx = \int \Phi(\nabla y)\,dx \qquad \forall\, \varphi \in C_c^\infty(\Omega)$$

Rmk Φ is null-Lagrangian

$\iff$ $\int_\Omega \Phi(F + \nabla\varphi(x))\,dx = \int_\Omega \Phi(F)\,dx \qquad \forall\, \varphi \in C_c^\infty(\Omega)\ \forall F$

$\iff$ Φ is rank-1 affine

$\iff$ $\Phi(F) = a F + b \operatorname{cof} F + c \det F$ (Rivlin-Ericksen)

$$\boxed{\frac{\partial}{\partial x_\alpha}\left(\frac{\partial \Phi}{\partial F_{i\alpha}}(\nabla y)\right) = 0 \qquad \forall\, y}$$

$\boxed{W(F) \text{ is polyconvex}}$

$$W(F) = G \circ \Phi(F) = G(F, \operatorname{cof} F, \det F) \qquad G \text{ convex.}$$

<u>Elastostatics</u>

$$\min_{y \in W^{1,p}} \int_\Omega W(\nabla y)\, dx = \min_{y \in W^{1,p}(\Omega)} \int_\Omega G(\Phi(\nabla y))\, dx$$

$$= \min_{\substack{\Xi = \Phi(F) \\ F = \nabla y \\ y \in W^{1,p}}} \int_\Omega G(\Xi)\, dx$$

$\Xi = \Phi(F)$ is a weakly continuous constraint

<u>Thm</u> (Ball)

$$A = \{ y \in W^{1,1}(\Omega) : I(y) < \infty,\ y|_{\partial\Omega_1} = \bar{y} \}$$

If (i) W polyconvex

(ii) $W(F) \geq c_0 \left(|F|^2 + |\operatorname{cof} F|^{3/2} \right) - c_1 \qquad \forall F$

and if A is nonempty then there exists a global minimizer $y^* \in A$.

Elastodynamics

$$\partial_t F = \nabla_x \cdot v$$

$$\partial_t \Phi(F) = \frac{\partial \Phi(F)}{\partial F_{i\alpha}} \partial_t F_{i\alpha} = \frac{\partial \Phi(F)}{\partial F_{i\alpha}} \partial_\alpha v_i$$

$$= \partial_\alpha \left(\frac{\partial \Phi(F)}{\partial F_{i\alpha}} v_i \right) - \cancel{\partial_\alpha \left(\frac{\partial \Phi(F)}{\partial F_{i\alpha}} \right)} v_i$$

$$(*) \quad \boxed{\begin{aligned} \partial_t v_i &= \partial_\alpha \frac{\partial W}{\partial F_{i\alpha}} = \partial_\alpha \left(\frac{\partial G}{\partial \Xi^A}(\Phi(F)) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right) \\ \partial_t \Phi^A(F) &= \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right) \end{aligned}}$$

enlargement of elasticity w. trivial eqns

embed to larger system

$$(**) \quad \begin{cases} \partial_t v_i = \partial_\alpha \left(\frac{\partial G}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right) \\ \partial_t \Xi^A = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right) \qquad (1) \end{cases}$$

equipped with convex entropy

$$\partial_t \Big(\underbrace{\tfrac{1}{2}|v|^2 + G(\Xi)}_{\text{convex}} \Big) = \partial_\alpha \left(\frac{\partial G}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right)$$

(*) can be viewed as constrained evolution of (**)

$$\partial_t \Phi(F) = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}} v_i \right) \qquad (2)$$

From (1) + (2)

$$\Xi = \Phi(F) \Big|_{t=0} \implies \Xi = \Phi(F) \quad \forall t$$