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CHANNELED SAMPLING IN SHIFT INVARIANT SPACES

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We develop sampling expansion formulas on the shift invariant closed subspace $V(\phi)$ of $L^2(\mathbb{R})$ generated by a frame or a Riesz generator $\phi(t)$. We find necessary and sufficient conditions under which a regular shifted sampling expansion to hold on $V(\phi)$ and also introduce a single channel sampling on $V(\phi)$ together with some illustrating examples.

Keywords: Sampling; shift invariant spaces; frames; Riesz basis; channeling.

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1. Introduction

The celebrated WSK (Whittaker–Shannon–Kotel'nikov)-sampling theorem[7,9] says that any signal $f(t)$ of finite energy with band-width π, that is, $f \in PW_\pi$ can be reconstructed via its regularly spaced discrete sample values $\{f(n) : n \in \mathbb{Z}\}$ as

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinc}(t-n),$$

which converges both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$, where, $\mathrm{sinc}t = \frac{\sin \pi t}{\pi t}$ is the cardinal sinc function and PW_π is the Paley–Wiener space:

$$PW_\pi := \{f \in L^2(\mathbb{R}) : \mathrm{supp}\hat{f}(\xi) \subseteq [-\pi, \pi]\}.$$

Here $\mathcal{F}[f](\xi) = \hat{f}(\xi)$ is the Fourier transform of $f(t)$, which is normalized as

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-it\xi}dt \quad \text{for } f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$$

so that $\mathcal{F}[\cdot]$ is a unitary operator from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

As a natural generalization of the WSK-sampling theorem, many authors have developed sampling theory on general shift invariant spaces.[1–3,8,11–14] For any $\phi(t)$

in $L^2(\mathbb{R})$, we let

$$V(\phi) := \overline{\mathrm{span}}\{\phi(t-n) : n \in \mathbb{Z}\}$$

be the closed subspace of $L^2(\mathbb{R})$ generated by integer translates $\{\phi(t-n) : n \in \mathbb{Z}\}$ of $\phi(t)$ and call $V(\phi)$ the shift invariant space generated by $\phi(t)$. Then PW_π is the shift invariant space generated by sinc$t$, of which $\{\mathrm{sinc}(t-n) : n \in \mathbb{Z}\}$ is an orthonormal basis. For example, Walter[11] developed a regular sampling theorem on a shift invariant space $V(\phi)$, where $\phi(t)$ is a continuous real valued orthonormal generator (in fact, a scaling function of an MRA) with decaying property $\phi(t) = O(|t|^{-1-\epsilon})$ $(\epsilon > 0)$ for $|t|$ large. Following Ref. 11, Janssen[8] used Zak transform to generalize Walter's result to regular shifted sampling. Zhou and Sun[13] found a necessary and sufficient condition for a regular sampling expansion to hold on $V(\phi)$ when $V(\phi)$ is a space of continuous functions generated by a frame generator $\phi(t)$. Later noting that sinc$t$ does not satisfy the Walter's decaying condition, Chen and Itoh[3] extended Walter's work by removing too much restrictive conditions in Ref. 11 like continuity and the decaying property on $\phi(t)$ when $\phi(t)$ is a Riesz generator. Zhao, Liu, and Zhao[12] extended further results in Ref. 3 by considering frame generators. However, there are some gaps in the arguments of the proofs of results in Refs. 3 and 12. In this work, we first find necessary and sufficient conditions under which a regular and a regular shifted sampling expansion to hold on $V(\phi)$ and then extend them into a single channeled sampling expansion.

In the following, we assume that $\phi(t)$ is a frame or a Riesz (stable) generator of $V(\phi)$, that is, $\{\phi(t-n) : n \in \mathbb{Z}\}$ is a frame or a Riesz basis of $V(\phi)$ so that

$$V(\phi) = \left\{ f(t) = \sum_{n\in\mathbb{Z}} c(n)\phi(t-n) : \mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}} \in l^2 \right\},$$

where $f(t)$ is the L^2-limit of $\sum_{n\in\mathbb{Z}} c(n)\phi(t-n)$. We are then concerned on the problem: When is there a function $S(t)$, called an interpolation generating function of $V(\phi)$ for which the sampling expansion formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n), \quad f \in V(\phi)$$

holds in $L^2(\mathbb{R})$-sense?

For any $\phi(t)$ in $L^2(\mathbb{R})$ and $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ in l^2, let

$$\hat{\mathbf{c}}^*(\xi) := \sum_{n\in\mathbb{Z}} c(n)\, e^{-in\xi} : \text{discrete Fourier transform of } \mathbf{c};$$

$$(\mathbf{c} * \phi)(t) := \sum_{n\in\mathbb{Z}} c(n)\phi(t-n) : \text{discrete-continuous convolution product of } \mathbf{c} \text{ and } \phi(t);$$

$$G_\phi(\xi) := \sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi + 2n\pi)|^2, \quad \xi \in \mathbb{R}.$$

Then

$$\hat{\mathbf{c}}^*(\xi) = \hat{\mathbf{c}}^*(\xi + 2\pi) \in L^2[0, 2\pi] \quad \text{and} \quad \|\hat{\mathbf{c}}^*(\xi)\|^2_{L^2[0,2\pi]} = 2\pi\|\mathbf{c}\|^2 = 2\pi \sum_{n\in\mathbb{Z}} |c(n)|^2;$$

$$G_\phi(\xi) = G_\phi(\xi + 2\pi) \in L^1[0, 2\pi] \quad \text{and} \quad \|G_\phi(\xi)\|_{L^1[0,2\pi]} = \|\phi(t)\|^2_{L^2(\mathbb{R})}.$$

Moreover, we have (cf. Ref. 4) that $\{\phi(t - n) : n \in \mathbb{Z}\}$ is

- a Bessel sequence with a Bessel bound $B > 0$, i.e.

  $$\sum_{n\in\mathbb{Z}} |\langle\psi(t), \phi(t - n)\rangle|^2 \le B\|\psi\|^2, \quad \psi \in L^2(\mathbb{R}) \quad (\|\psi\| = \|\psi\|_{L^2(\mathbb{R})})$$

  if and only if

  $$2\pi G_\phi(\xi) \le B \quad \text{a.e. on } \mathbb{R}; \tag{1.1}$$

- a frame of $V(\phi)$ with frame bounds $B \ge A > 0$, i.e.,

  $$A\|\psi\|^2 \le \sum_{n\in\mathbb{Z}} |\langle\psi(t), \phi(t - n)\rangle|^2 \le B\|\psi\|^2, \quad \psi \in V(\phi)$$

  if and only if

  $$A \le 2\pi G_\phi(\xi) \le B \quad \text{a.e. on } \operatorname{supp} G_\phi; \tag{1.2}$$

- a Riesz basis of $V(\phi)$ with Riesz bounds $B \ge A > 0$, i.e.,

  $$A\|\mathbf{c}\|^2 \le \|(\mathbf{c} * \phi)(t)\|^2 \le B\|\mathbf{c}\|^2, \quad \mathbf{c} \in l^2 \tag{1.3}$$

  if and only if

  $$A \le 2\pi G_\phi(\xi) \le B \quad \text{a.e. on } \mathbb{R}; \quad \text{and} \tag{1.4}$$

- an orthonormal basis of $V(\phi)$, i.e., $\|(\mathbf{c} * \phi)(t)\|^2 = \|\mathbf{c}\|^2$, $\mathbf{c} \in l^2$ if and only if

  $$2\pi G_\phi(\xi) = 1 \quad \text{a.e. on } \mathbb{R}.$$

Here we use $\operatorname{supp} f$ for any $f(\xi)$ in $L^1_{loc}(\mathbb{R})$ to denote the support of f viewing f as a distribution on $\mathbb{R}$, that is,

$$\mathbb{R}\backslash\operatorname{supp} f = \{\xi \in \mathbb{R} : f(\cdot) = 0 \text{ a.e. on some neighborhood of } \xi\}.$$

2. Main Results

We begin with two simple lemmas, which play key roles in the following.

Lemma 2.1 (cf. Lemma 2 in Ref. 13). *For any* $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ *and* $\mathbf{d} = \{d(n)\}_{n\in\mathbb{Z}}$ *in* l^2, *let*

$$\mathbf{c} * \mathbf{d} := \left\{(\mathbf{c} * \mathbf{d})(n) := \sum_{k\in\mathbb{Z}} c(k)d(n - k)\right\}_{n\in\mathbb{Z}}$$

be the discrete convolution product of $\mathbf{c}$ *and* $\mathbf{d}$*. Then*

$$\hat{\mathbf{c}}^*(\xi)\hat{\mathbf{d}}^*(\xi) \sim \sum_{n\in\mathbb{Z}}(\mathbf{c}*\mathbf{d})(n)e^{-in\xi}, \tag{2.1}$$

which means that $\sum_{n\in\mathbb{Z}}(\mathbf{c}*\mathbf{d})(n)e^{-in\xi}$ *is the Fourier series expansion of* $\hat{\mathbf{c}}^*(\xi)\hat{\mathbf{d}}^*(\xi) \in L^1[0,2\pi]$*. Moreover,* $\mathbf{c}*\mathbf{d} \in c_0$ *and*

$$\int_0^{2\pi} |\hat{\mathbf{c}}^*(\xi)\hat{\mathbf{d}}^*(\xi)|^2 d\xi = 2\pi \sum_{n\in\mathbb{Z}} |(\mathbf{c}*\mathbf{d})(n)|^2. \tag{2.2}$$

Proof. Since $\hat{\mathbf{c}}^*(\xi)$ and $\hat{\mathbf{d}}^*(\xi) \in L^2[0,2\pi], \hat{\mathbf{c}}^*(\xi)\hat{\mathbf{d}}^*(\xi) \in L^1[0,2\pi]$ of which the Fourier series is

$$\hat{\mathbf{c}}^*(\xi)\hat{\mathbf{d}}^*(\xi) \sim \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\langle \hat{\mathbf{c}}^*(\xi)\hat{\mathbf{d}}^*(\xi), e^{-in\xi}\rangle_{L^2[0,2\pi]} e^{-in\xi}$$

from which (2.1) follows. Then $\mathbf{c}*\mathbf{d} \in c_0$ by Riemann–Lebesgue lemma and (2.2) is an immediate consequence of the Parseval's identity. □

In particular, (2.2) implies that $\hat{\mathbf{c}}^*(\xi)\hat{\mathbf{d}}^*(\xi) \in L^2[0,2\pi]$ if and only if $\mathbf{c}*\mathbf{d} \in l^2$.

Lemma 2.2. *Let* $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}} \in l^2$, $\phi(t) \in L^2(\mathbb{R})$, *and assume that* $(\mathbf{c}*\phi)(t)$ *converges in* $L^2(\mathbb{R})$*. If either* $\mathbf{c} \in l^1$ *or* $\{\phi(t-n) : n \in \mathbb{Z}\}$ *is a Bessel sequence, then*

$$\mathcal{F}[\mathbf{c}*\phi](\xi) = \hat{\mathbf{c}}^*(\xi)\hat{\phi}(\xi). \tag{2.3}$$

Proof. Since $(\mathbf{c}*\phi)(t) = \sum_{n\in\mathbb{Z}} c(n)\phi(t-n)$ converges in $L^2(\mathbb{R}), \mathcal{F}[\mathbf{c}*\phi](\xi) = \sum_{n\in\mathbb{Z}}(c(n)e^{-in\xi}\hat{\phi}(\xi))$ converges in $L^2(\mathbb{R})$, that is, $\hat{\mathbf{c}}_n^*(\xi)\hat{\phi}(\xi) := \sum_{|k|\le n} c(k)e^{-ik\xi}\hat{\phi}(\xi)$ converges to $\mathcal{F}[\mathbf{c}*\phi](\xi)$ in $L^2(\mathbb{R})$. Hence to show (2.3), it is enough to show that $\hat{\mathbf{c}}_n^*(\xi)\hat{\phi}(\xi)$ converges to $\hat{\mathbf{c}}^*(\xi)\hat{\phi}(\xi)$ in $L^2(\mathbb{R})$ when either $\mathbf{c} \in l^1$ or $\{\phi(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence. Now

$$\begin{aligned}
\|\hat{\mathbf{c}}_n^*(\xi)\hat{\phi}(\xi) - \hat{\mathbf{c}}^*(\xi)\hat{\phi}(\xi)\|^2 &= \int_{-\infty}^{\infty} |\hat{\mathbf{c}}_n^*(\xi) - \hat{\mathbf{c}}^*(\xi)|^2 |\hat{\phi}(\xi)|^2 d\xi \\
&= \int_0^{2\pi} |\hat{\mathbf{c}}_n^*(\xi) - \hat{\mathbf{c}}^*(\xi)|^2 G_\phi(\xi) d\xi \\
&\le \begin{cases} \|\hat{\mathbf{c}}_n^*(\xi) - \hat{\mathbf{c}}^*(\xi)\|^2_{L^\infty[0,2\pi]} \displaystyle\int_0^{2\pi} G_\phi(\xi)d\xi \\[2ex] \|G_\phi(\xi)\|_{L^\infty[0,2\pi]} \displaystyle\int_0^{2\pi} |\hat{\mathbf{c}}_n^*(\xi) - \hat{\mathbf{c}}^*(\xi)|^2 d\xi \end{cases}
\end{aligned}$$

so that $\lim_{n\to\infty} \|\hat{\mathbf{c}}_n^*(\xi)\hat{\phi}(\xi) - \hat{\mathbf{c}}^*(\xi)\hat{\phi}(\xi)\| = 0$ provided that either $\mathbf{c} \in l^1$ so that $\hat{\mathbf{c}}_n^*(\xi)$ converges to $\hat{\mathbf{c}}^*(\xi)$ uniformly on $[0,2\pi]$ or $\{\phi(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence so that $G_\phi(\xi) \in L^\infty[0,2\pi]$ by (1.1). □

In the following, we let $\phi(t)$ be a complex valued square integrable function on $\mathbb{R}$ such that $\phi(t)$ is a frame or a Riesz generator of $V(\phi)$, that is, $\{\phi(t-n) : n \in \mathbb{Z}\}$ is a frame or a Riesz basis of $V(\phi)$. We also assume $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^2$ and set $\hat{\phi}^*(\xi) := \sum_{n\in\mathbb{Z}} \phi(n)e^{-in\xi} \in L^2[0, 2\pi]$. Then

$$V(\phi) = \left\{(\mathbf{c} * \phi)(t) = \sum_{k\in\mathbb{Z}} c(k)\phi(t-k) : \mathbf{c} \in l^2\right\},$$

where each $f(t) := (\mathbf{c}*\phi)(t) = \sum_{k\in\mathbb{Z}} c(k)\phi(t-k)$ converges in $L^2(\mathbb{R})$. In particular, for each $n \in \mathbb{Z}$, $\sum_{k\in\mathbb{Z}} c(k)\phi(n-k)$ converges absolutely, which we may set to be

$$f(n) := \sum_{k\in\mathbb{Z}} c(k)\phi(n-k).$$

Note that as a shift invariant space, $V(\phi)$ contains $S(t-n)$ for any n in $\mathbb{Z}$ if $S(t)$ is in $V(\phi)$. For a measurable set E in $\mathbb{R}$, we let $|E|$ be the Lebesgue measure of E and $\chi_E(\xi)$ the characteristic function of E. For a measurable function $f(t)$ on $\mathbb{R}$, let

$$\|f\|_0 := \sup_{|E|=0} \inf_{\mathbb{R}\setminus E} |f(t)| \quad \text{and} \quad \|f\|_\infty := \inf_{|E|=0} \sup_{\mathbb{R}\setminus E} |f(t)|$$

be the essential infimum and essential supremum of $f(t)$, respectively.

Theorem 2.1. *Assume that $\phi(t)$ is a frame generator of $V(\phi)$ and $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^2$.*

(a) *If there is $S(t)$ in $V(\phi)$ such that $\{S(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence (respectively, a frame) of $V(\phi)$ for which the sampling expansion formula*

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n), \quad f \in V(\phi) \tag{2.4}$$

holds in the L^2 sense, then

$$\operatorname{supp}\hat{\phi} = \operatorname{supp}\hat{S} \subset \operatorname{supp} G_\phi = \operatorname{supp} G_S \subset \operatorname{supp}\hat{\phi}^* \tag{2.5}$$

and there is a constant $\alpha > 0$ (respectively, there are constants $\beta \geq \alpha > 0$) such that

$$\alpha \leq |\hat{\phi}^*(\xi)| \quad (\textit{respectively}, \ \alpha \leq |\hat{\phi}^*(\xi)| \leq \beta) \quad \textit{a.e. on} \operatorname{supp} G_\phi. \tag{2.6}$$

Moreover

$$\hat{S}(\xi) = \frac{\hat{\phi}(\xi)}{\hat{\phi}^*(\xi)}\chi_{\operatorname{supp} G_\phi}(\xi) \quad \textit{a.e. on } \mathbb{R}. \tag{2.7}$$

(b) *If $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^1$ and there is $S(t) \in V(\phi)$ such that (2.4) holds, then (2.5), (2.7) hold and*

$$\frac{1}{\hat{\phi}^*(\xi)}\chi_{\operatorname{supp} G_\phi}(\xi) \in L^2[0, 2\pi]. \tag{2.8}$$

Proof. (a) Assume that $\{S(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence of $V(\phi)$ with a Bessel bound B_S for which the sampling expansion formula (2.4) holds. Then

$$S(t) = \sum_{n\in\mathbb{Z}} a(n)\phi(t-n) \quad \text{and} \quad \phi(t) = \sum_{n\in\mathbb{Z}} \phi(n)S(t-n)$$

for some $\mathbf{a} = \{a(n)\}_{n\in\mathbb{Z}}$ in l^2. Then by Lemma 2.2,

$$\hat{S}(\xi) = \hat{\mathbf{a}}^*(\xi)\hat{\phi}(\xi) \quad \text{and} \quad \hat{\phi}(\xi) = \hat{\phi}^*(\xi)\hat{S}(\xi) \tag{2.9}$$

and so

$$G_S(\xi) = |\hat{\mathbf{a}}^*(\xi)|^2 G_\phi(\xi) \quad \text{and} \quad G_\phi(\xi) = |\hat{\phi}^*(\xi)|^2 G_S(\xi), \tag{2.10}$$

from which (2.5) follows immediately. We also have from (2.9)

$$\hat{S}(\xi) = 0 \quad \text{a.e. on } (\operatorname{supp}\hat{\phi})^c \quad \text{and} \quad \hat{S}(\xi) = \frac{\hat{\phi}(\xi)}{\hat{\phi}^*(\xi)} \quad \text{a.e. on } \operatorname{supp}\hat{\phi}^*(\xi)$$

so that (2.7) holds by (2.5). Now (2.10) implies

$$|\hat{\phi}^*(\xi)|^2 = \frac{G_\phi(\xi)}{G_S(\xi)} \quad \text{a.e. on } \operatorname{supp} G_\phi \tag{2.11}$$

so that $\frac{A_\phi}{B_S} \le |\hat{\phi}^*(\xi)|^2$ a.e. on $\operatorname{supp} G_\phi$, where (A_ϕ, B_ϕ) are frame bounds of $\{\phi(t-n) : n \in \mathbb{Z}\}$ [cf. (1.2)]. If $\{S(t-n) : n \in \mathbb{Z}\}$ is also a frame of $V(\phi)$ with frame bounds (A_S, B_S), then (2.11) implies

$$\frac{A_\phi}{B_S} \le |\hat{\phi}^*(\xi)|^2 \le \frac{B_\phi}{A_S} \quad \text{a.e. on } \operatorname{supp} G_\phi.$$

Hence (2.6) holds.

(b) Assume $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^1$ and (2.4) holds on $V(\phi)$ for some $S(t) \in V(\phi)$. Then (2.5) and (2.7) hold by the same arguments as in the proof of (a). We now have from (2.7) and $\chi_{\operatorname{supp} G_\phi}(\xi) = \chi_{\operatorname{supp} G_\phi}(\xi + 2\pi)$,

$$\begin{aligned}\infty > \int_{-\infty}^{\infty} |\hat{S}(\xi)|^2 d\xi &= \int_{-\infty}^{\infty} \left|\frac{\hat{\phi}(\xi)}{\hat{\phi}^*(\xi)}\right|^2 \chi_{\operatorname{supp} G_\phi}(\xi) d\xi \\ &= \int_0^{2\pi} \frac{G_\phi(\xi)}{|\hat{\phi}^*(\xi)|^2} \chi_{\operatorname{supp} G_\phi}(\xi) d\xi \\ &\ge \frac{A_\phi}{2\pi} \int_0^{2\pi} \frac{1}{|\hat{\phi}^*(\xi)|^2} \chi_{\operatorname{supp} G_\phi}(\xi) d\xi\end{aligned}$$

so that (2.8) holds. □

Theorem 2.1 gives some necessary conditions for the sampling expansion formula (2.4) to hold. Conversely, we have:

Theorem 2.2. *Assume that $\phi(t)$ is a frame generator of $V(\phi)$ and $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^2$. If there are constants $\beta \geq \alpha > 0$ such that*

$$\alpha \leq |\hat{\phi}^*(\xi)| \leq \beta \quad \text{a.e. on } \operatorname{supp} G_\phi, \tag{2.12}$$

then there is a frame generator $S(t)$ of $V(\phi)$ for which

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n) \tag{2.13}$$

holds for any $f(t) = (\mathbf{c} * \phi)(t) \in V(\phi)$ satisfying

$$\hat{\mathbf{c}}^*(\xi)\hat{\phi}^*(\xi) \in L^2[0, 2\pi]. \tag{2.14}$$

If moreover $|\hat{\phi}^*(\xi)| \leq \beta$ a.e. on $\mathbb{R}$, then (2.4)–(2.7) hold and $\{f(n)\}_{n\in\mathbb{Z}} \in l^2$ for any $f \in V(\phi)$.

Proof. Inequality (2.12) implies that $\frac{1}{\hat{\phi}^*(\xi)}\chi_{\operatorname{supp} G_\phi}(\xi) \in L^\infty[0, 2\pi] \subset L^2[0, 2\pi]$ so that

$$\frac{1}{\hat{\phi}^*(\xi)}\chi_{\operatorname{supp} G_\phi}(\xi) = \sum_{n\in\mathbb{Z}} a(n)e^{-in\xi} = \hat{\mathbf{a}}^*(\xi)$$

for some $\mathbf{a} = \{a(n)\}_{n\in\mathbb{Z}}$ in l^2.

Define $\hat{S}(\xi)$ by (2.7), that is,

$$\hat{S}(\xi) = \frac{\hat{\phi}(\xi)}{\hat{\phi}^*(\xi)}\chi_{\operatorname{supp} G_\phi}(\xi) = \hat{\mathbf{a}}^*(\xi)\hat{\phi}(\xi).$$

Then

$$\int_{-\infty}^{\infty} |\hat{S}(\xi)|^2 d\xi = \int_0^{2\pi} |\hat{\mathbf{a}}^*(\xi)|^2 G_\phi(\xi) d\xi \leq \|G_\phi(\xi)\|_\infty \int_0^{2\pi} |\hat{\mathbf{a}}^*(\xi)|^2 d\xi < \infty$$

so that $\hat{S}(\xi) \in L^2(\mathbb{R})$. Since

$$\hat{S}(\xi) = \hat{\mathbf{a}}^*(\xi)\hat{\phi}(\xi) = \sum_{n\in\mathbb{Z}} (a(n)e^{-in\xi}\hat{\phi}(\xi)) \tag{2.15}$$

by Lemma 2.2, we have by Fourier inversion

$$S(t) = \sum_{n\in\mathbb{Z}} a(n)\phi(t-n) \in V(\phi).$$

Now (2.15) implies $\operatorname{supp}\hat{S} \subset \operatorname{supp}\hat{\phi} \subset \operatorname{supp} G_\phi$ so that

$$\hat{\phi}(\xi) = \hat{\phi}^*(\xi)\hat{S}(\xi) \quad \text{a.e. on } \mathbb{R} \tag{2.16}$$

since (2.16) holds on $\operatorname{supp} G_\phi$ by (2.7) and $\hat{\phi}(\xi) = \hat{S}(\xi) = 0$ a.e. on $(\operatorname{supp} G_\phi)^c$. Then as in the proof of Theorem 2.1, (2.10) holds so that

$$G_S(\xi) = \frac{G_\phi(\xi)}{|\hat{\phi}^*(\xi)|^2} \text{ on } \operatorname{supp}\hat{\phi}^* \supset \operatorname{supp} G_\phi = \operatorname{supp} G_S. \tag{2.17}$$

Hence, we have by (2.12) and (2.17)

$$\frac{A_\phi}{2\pi\beta^2} \le G_S(\xi) \le \frac{B_\phi}{2\pi\alpha^2} \quad \text{a.e. on } \operatorname{supp} G_S \tag{2.18}$$

so that $\{S(t-n) : n \in \mathbb{Z}\}$ is at least a Bessel sequence of $V(\phi)$. Now for any $f(t) = (\mathbf{c} * \phi)(t)$ in $V(\phi)$ with $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ in l^2,

$$\hat{f}(\xi) = \hat{\mathbf{c}}^*(\xi)\hat{\phi}(\xi) = \hat{\mathbf{c}}^*(\xi)\hat{\phi}^*(\xi)\hat{S}(\xi) \tag{2.19}$$

by (2.16). If $\hat{\mathbf{c}}^*(\xi)\hat{\phi}^*(\xi) \in L^2[0,2\pi]$, then $\{f(n)\}_{n\in\mathbb{Z}} \in l^2$ and

$$\hat{\mathbf{c}}^*(\xi)\hat{\phi}^*(\xi) = \hat{f}^*(\xi) = \sum_{n\in\mathbb{Z}} f(n)e^{-in\xi}$$

in $L^2[0,2\pi]$ by Lemma 2.1. Hence, we get

$$\hat{f}(\xi) = \hat{f}^*(\xi)\hat{S}(\xi) = \sum_{n\in\mathbb{Z}} (f(n)e^{-in\xi}\hat{S}(\xi)) \tag{2.20}$$

by Lemma 2.2 since $\{S(t-n) : n \in Z\}$ is a Bessel sequence. Then we have (2.13) by taking Fourier inversion on (2.20). On the other hand, we also have from (2.19)

$$\hat{f}(\xi) = \hat{\mathbf{c}}^*(\xi)\hat{\phi}^*(\xi)\hat{S}(\xi) = \hat{\mathbf{c}}^*(\xi)\hat{\phi}^*(\xi)\chi_{\operatorname{supp} G_\phi}(\xi)\hat{S}(\xi)$$

since $\operatorname{supp}\hat{S} \subset \operatorname{supp} G_\phi$. Let $\hat{\phi}^*(\xi)\chi_{\operatorname{supp} G_\phi}(\xi) = \hat{\mathbf{d}}^*(\xi) = \sum_{n\in\mathbb{Z}} d(n)e^{-in\xi}$ be the Fourier series expansion of $\hat{\phi}^*(\xi)\chi_{\operatorname{supp} G_\phi}(\xi) \in L^\infty[0,2\pi] \subset L^2[0,2\pi]$. Then

$$\hat{\mathbf{c}}^*(\xi)\hat{\phi}^*(\xi)\chi_{\operatorname{supp} G_\phi}(\xi) = \hat{\mathbf{c}}^*(\xi)\hat{\mathbf{d}}^*(\xi) = \sum_{n\in\mathbb{Z}} (\mathbf{c} * \mathbf{d})(n)e^{-in\xi}$$

so that

$$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} (\mathbf{c} * \mathbf{d})(n)e^{-in\xi}\hat{S}(\xi) \quad \text{and so} \quad f(t) = \sum_{n\in\mathbb{Z}} (\mathbf{c} * \mathbf{d})(n)S(t-n), \quad f \in V(\phi).$$

Hence $V(S) = V(\phi)$ so that (2.18) implies $\{S(t-n) : n \in \mathbb{Z}\}$ is a frame of $V(\phi)$. Finally, assume

$$\alpha\chi_{\operatorname{supp} G_\phi}(\xi) \le |\hat{\phi}^*(\xi)| \le \beta \quad \text{a.e. on } \mathbb{R}. \tag{2.21}$$

Then (2.14) holds for any $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ in l^2 since $\hat{\phi}^*(\xi) \in L^\infty[0, 2\pi]$. Hence $\{f(n)\}_{n\in\mathbb{Z}} \in l^2$ for any $f \in V(\phi)$ and (2.13) holds on $V(\phi)$, that is, (2.4) holds. (2.5)–(2.7) then follows from (2.4) by Theorem 2.1. □

Corollary 2.1. *If* $\phi(t)$ *is a frame generator of* $V(\phi)$, $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^1$ *and* $\hat{\phi}^*(\xi) \neq 0$ *on* $\operatorname{supp} G_\phi$, *then there is a frame generator* $S(t)$ *of* $V(\phi)$ *for which* (2.4)–(2.7) *hold.*

Proof. If $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^1$ and $\hat{\phi}^*(\xi) \neq 0$ on $\operatorname{supp} G_\phi$, then $\hat{\phi}^*(\xi) \in C(\mathbb{R})$ satisfies the condition (2.21) so that the conclusion follows from Theorem 2.2. □

In Ref. 12, the authors assumed that $\phi(t)$ is a frame generator of $V(\phi)$ and $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^2$ and then claimed (see Theorems 1 and 2 in Ref. 12) that there is $S(t)$ in $V(\phi)$ for which the sampling expansion formula (2.4) holds if and only if the condition (2.8) is satisfied. In particular, in Ref. 12, the authors assumed nothing on the sequence $\{S(t-n) : n \in \mathbb{Z}\}$. However arguments in the proof of Theorems 1 and 2 in Ref. 12 have some gaps. Assume first that (2.4) holds. Then $\phi(t) = \sum_{n\in\mathbb{Z}} \phi(n)S(t-n)$, which needs not imply $\hat{\phi}(\xi) = \hat{\phi}^*(\xi)\hat{S}(\xi)$ [see Eq. (2) or (7) in Ref. 12] in general unless either $\{S(t-n) : n \in \mathbb{Z}\}$ is at least a Bessel sequence or $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^1$ [cf. Lemma 2.2]. Conversely if the condition (2.8), instead of the condition (2.12), holds in Theorem 2.2, then we still have (2.15)–(2.17) and (2.19). However $\hat{\mathbf{c}}^*(\xi)\hat{\phi}^*(\xi)$ may not be in $L^2[0, 2\pi]$ so that (2.20) may not hold and $\{S(t-n) : n \in \mathbb{Z}\}$ may not be a Bessel sequence in general. Hence, contrary to the claim in (Theorem 2, Ref. 12), we cannot be sure if (2.4) holds assuming only the condition (2.8).

Remark 2.1. Assume that $\phi(t)$ is a continuous frame generator satisfying $\sup_{\mathbb{R}} \sum_{n\in\mathbb{Z}} |\phi(t-n)|^2 < \infty$. Then Zhou and Sun show (see Theorem 1 in Ref. 13) that there is a frame $\{S(t-n) : n \in \mathbb{Z}\}$ of $V(\phi)$ for which the sampling expansion formula (2.4) holds on $V(\phi)$ if and only if there are constants $\beta \geq \alpha > 0$ such that

$$\alpha\chi_{\operatorname{supp} G_\phi}(\xi) \leq |\hat{\phi}^*(\xi)| \leq \beta\chi_{\operatorname{supp} G_\phi}(\xi) \quad \text{a.e. on } \mathbb{R}.$$

Note that in this case, $V(\phi)$ is a reproducing kernel Hilbert space, $V(\phi) \subset C(\mathbb{R})$, and the sampling series $\sum_{n\in\mathbb{Z}} f(n)S(t-n)$ converges not only in $L^2(\mathbb{R})$ but also uniformly on $\mathbb{R}$ to $f(t)$.

In the case of Riesz basis setting, we have:

Lemma 2.3. *Assume that* $\phi(t)$ *is a Riesz generator of* $V(\phi)$ *and* $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^2$. *Assume that there is* $S(t) \in V(\phi)$ *for which the sampling expansion formula* (2.4) *holds. If either* $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^1$ *or* $\{S(t-n) : n \in \mathbb{Z}\}$ *is a Bessel sequence, then* $\hat{\phi}^*(\xi)^{-1} \in L^2[0, 2\pi]$ *and*

$$\operatorname{supp}\hat{\phi} = \operatorname{supp}\hat{S} \subseteq \operatorname{supp} G_\phi = \operatorname{supp} G_S = \operatorname{supp}\hat{\phi}^* = \mathbb{R}; \tag{2.22}$$

$$\hat{S}(\xi) = \frac{\hat{\phi}(\xi)}{\hat{\phi}^*(\xi)} \quad \text{a.e. on } \mathbb{R}. \tag{2.23}$$

Proof. By the same arguments as in the proof of Theorem 2.1, (2.9) and (2.5) hold. Since $\phi(t)$ is a Riesz generator, $\operatorname{supp} G_\phi = \mathbb{R}$ (cf. (1.4)). Hence (2.22) comes from (2.5) and then (2.23) comes from (2.9) and (2.22). □

Theorem 2.3. *Assume that $\phi(t)$ is a frame generator of $V(\phi)$ and $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^2$. Then there is a Riesz generator $S(t)$ of $V(\phi)$ for which* (2.4) *holds if and only if $\phi(t)$ is also a Riesz generator of $V(\phi)$ and*

$$0 < \|\hat{\phi}^*(\xi)\|_0 \le \|\hat{\phi}^*(\xi)\|_\infty < \infty. \tag{2.24}$$

Furthermore in this case, we have, in addition to (2.22) *and* (2.23);

$$S(t) \text{ is cardinal}, \quad \text{i.e.} \quad S(n) = \delta_{0,n} \quad \text{for } n \in \mathbb{Z}. \tag{2.25}$$

Proof. First assume that (2.4) holds on $V(\phi)$ for some Riesz generator $S(t)$ of $V(\phi)$. Then we have (2.9), (2.10) and so (2.5). Since $\operatorname{supp} G_\phi = \operatorname{supp} G_S = \mathbb{R}$, $\{\phi(t-n) : n \in \mathbb{Z}\}$ must be a Riesz basis of $V(\phi)$ so that (2.22) and (2.23) hold by Lemma 2.3. Now (2.24) comes from (2.11): $|\hat{\phi}^*(\xi)|^2 = \frac{G_\phi(\xi)}{G_S(\xi)}$ a.e. on $\mathbb{R}$ and (2.25) comes immediately from $S(t) = \sum_{n\in\mathbb{Z}} S(n)S(t-n)$.

Conversely, assume that $\phi(t)$ is a Riesz generator of $V(\phi)$ and (2.24) hold. Define $\hat{S}(\xi)$ by (2.23). Then $\hat{S}(\xi) = \hat{\mathbf{a}}^*(\xi)\hat{\phi}(\xi) \in L^2(\mathbb{R})$, where $\hat{\mathbf{a}}^*(\xi) = \hat{\phi}^*(\xi)^{-1} \in L^\infty[0, 2\pi]$ so that $S(t) = (\mathbf{a} * \phi)(t) \in V(\phi)$. The rest of the proof is the same as the one in Theorem 2.2. □

Remark 2.2. Let $\phi(t)$ be a Riesz generator which is piecewise continuous on $\mathbb{R}$ and $\phi(t) = O(|t|^{-1/2-\epsilon})$ $(\epsilon > 0)$ for $|t|$ large. Then $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^2$ and $M := \sup_{\mathbb{R}} \sum_{n\in\mathbb{Z}} |\phi(t-n)|^2 < \infty$. Hence, for any $\mathbf{c}$ in l^2, $(\mathbf{c} * \phi)(t) = \sum_{n\in\mathbb{Z}} c(n)\phi(t-n)$ converges not only in $L^2(\mathbb{R})$ but also absolutely and locally uniformly on $\mathbb{R}$. Therefore, we have

$$V(\phi) = \{f(t) = (\mathbf{c} * \phi)(t) : \mathbf{c} \in l^2\},$$

where $f(t)$ is the pointwise (and also L^2-) limit of $(\mathbf{c} * \phi)(t)$. Then $V(\phi)$ is a closed subspace of $L^2(\mathbb{R})$ since if $f(t) = (\mathbf{c} * \phi)(t) \in V(\phi)$ is such that $\|f(t)\| = 0$, i.e. $\sum_{n\in\mathbb{Z}} c(n)\phi(t-n) = 0$ in $L^2(\mathbb{R})$, then $c(n) = 0$ for all n in $\mathbb{Z}$ so $f(t) = 0$ on $\mathbb{R}$. Moreover $V(\phi)$ is a reproducing kernel Hilbert space since for any $f(t) = (\mathbf{c}*\phi)(t) \in V(\phi)$ and any t in $\mathbb{R}$

$$|f(t)| \le \|\mathbf{c}\|\sqrt{M} \le \frac{M}{\sqrt{A}}\|f\| \quad [\text{cf. } (1.3)].$$

Let $k(t,s) = \sum_{n\in\mathbb{Z}} \overline{\phi(s-n)}\tilde{\phi}(t-n)$ be the reproducing kernel of $V(\phi)$, where $\{\tilde{\phi}(t-n) : n \in \mathbb{Z}\}$ is the dual Riesz basis of $\{\phi(t-n) : n \in \mathbb{Z}\}$. Then

$$\|k(\cdot,s)\|^2 = \left\|\sum_{n\in\mathbb{Z}} \overline{\phi(s-n)}\tilde{\phi}(\cdot-n)\right\|^2 \le \tilde{B}\sum_{n\in\mathbb{Z}} |\phi(s-n)|^2 \le \tilde{B}M,$$

where $\tilde{B} \geq \tilde{A} > 0$ are Riesz bounds of $\{\tilde{\phi}(t-n) : n \in \mathbb{Z}\}$. Therefore, assuming further that the condition (2.24) holds, the sampling series (2.4) converges not only in $L^2(\mathbb{R})$ but also absolutely and uniformly on $\mathbb{R}$.

Corollary 2.2. *Assume that $\phi(t)$ is a frame generator of $V(\phi)$ and $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^1$. Then there is a Riesz generator $S(t)$ of $V(\phi)$ for which* (2.4) *holds if and only if $\phi(t)$ is also a Riesz generator of $V(\phi)$ and $\hat{\phi}^*(\xi) \neq 0$ on* $[0, 2\pi]$.

Proof. If $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^1$, then $\hat{\phi}^*(\xi) = \hat{\phi}^*(\xi + 2\pi) \in C[0, 2\pi]$ so that

$$\|\hat{\phi}^*(\xi)\|_0 = \min_{[0,2\pi]} |\hat{\phi}^*(\xi)| \quad \text{and} \quad \|\hat{\phi}^*(\xi)\|_\infty = \max_{[0,2\pi]} |\hat{\phi}^*(\xi)|.$$

Hence the condition (2.24) is equivalent to $\hat{\phi}^*(\xi) \neq 0$ on $[0, 2\pi]$. Therefore, the conclusion follows from Theorem 2.3. □

In Ref. 11, Walter assumed that $\phi(t)$ is a continuous real-valued orthonormal generator with $\phi(t) = O(|t|^{-1-s})$ $(s > 0)$ for $|t|$ large. Then $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^1$ so that the main Theorem of Ref. 11 is a special case of Corollary 2.2.

In Ref. 3, Chen and Itoh claimed (cf. Theorem 1 in Ref. 3) that assuming $\phi(t)$ is a Riesz generator of $V(\phi)$ with $\{\phi(n)\}_{n\in\mathbb{Z}} \in l^2$, (2.4) holds for some $S(t)$ in $V(\phi)$ if and only if $\hat{\phi}^*(\xi)^{-1} \in L^2[0, 2\pi]$. However, as in Theorem 2 from Ref. 12, there are some gaps in the proof of Theorem 1 in Ref. 3, which are filled and extended by Theorem 2.3.

As it was done in Refs. 3 and 12 (see also Ref. 8), we can extend Theorems 2.1 and 2.2 to the regular shifted sampling, Theorem 2.4 below, which also corrects and extends Theorem 2 in Ref. 3 and Theorem 3 in Ref. 12.

We now assume that $\phi(t)$ is a complex valued square integrable function on $\mathbb{R}$ such that $\phi(t)$ is a frame generator and $\{\phi(\sigma + n)\}_{n\in\mathbb{Z}} \in l^2$ for some σ in $[0, 1)$. Then for any $f(t) = \sum_{n\in\mathbb{Z}} c(n)\phi(t - n)$ in $V(\phi)$ with $\mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}}$ in l^2,

$$f(\sigma + n) := \sum_{k\in\mathbb{Z}} c(k)\phi(\sigma + n - k)$$

converges absolutely for each n in $\mathbb{Z}$. Let

$$Z_\phi(t, \xi) := \sum_{n\in\mathbb{Z}} \phi(t + n)e^{-in\xi}$$

be the Zak transform of $\phi(t)$ (cf. Ref. 8).

Theorem 2.4. *Assume that $\phi(t)$ is a frame generator of $V(\phi)$ and $\{\phi(\sigma+n)\}_{n\in\mathbb{Z}} \in l^2$ for some σ in* $[0, 1)$.

(a) *If there is a frame generator* $S_\sigma(t)$ *of* $V(\phi)$ *for which the regular shifted sampling expansion formula*

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma+n)S_\sigma(t-n), \quad f \in V(\phi) \tag{2.26}$$

holds, then there are constants $\beta \geq \alpha > 0$ *such that*

$$\alpha \leq |Z_\phi(\sigma,\xi)| \leq \beta \quad a.e. \ on \ \operatorname{supp} G_\phi;$$
$$\operatorname{supp}\hat{\phi} = \operatorname{supp}\hat{S}_\sigma \subset \operatorname{supp} G_\phi = \operatorname{supp} G_{S_\sigma} \subset \operatorname{supp} Z_\phi(\sigma,\xi);$$

and

$$\hat{S}_\sigma(\xi) = \frac{\hat{\phi}(\xi)}{Z_\phi(\sigma,\xi)}\chi_{\operatorname{supp} G_\phi}(\xi). \tag{2.27}$$

(b) *Conversely, if there are constants* $\beta \geq \alpha > 0$ *such that*

$$\alpha\chi_{\operatorname{supp} G_\phi}(\xi) \leq |Z_\phi(\sigma,\xi)| \leq \beta \quad a.e. \ on \ \mathbb{R}$$

then there is a frame generator $S_\sigma(t)$ *of* $V(\phi)$ *for which* (2.26) *and* (2.27) *hold.*

(c) *There is a Riesz generator* $S_\sigma(t)$ *of* $V(\phi)$ *for which* (2.26) *holds if and only if* $\phi(t)$ *is a Riesz generator and*

$$0 < \|Z_\phi(\sigma,\xi)\|_0 \leq \|Z_\phi(\sigma,\xi)\|_\infty < \infty.$$

Furthermore, in this case, we have $S_\sigma(\sigma+n) = \delta_{0,n}$ *for* n *in* $\mathbb{Z}$ *and*

$$\hat{S}_\sigma(\xi) = \frac{\hat{\phi}(\xi)}{Z_\phi(\sigma,\xi)} \quad a.e. \ on \ \mathbb{R}.$$

Proof. Proofs of (a), (b) and (c) are essentially the same as the ones in Theorems 2.1 and 2.2 respectively. □

Corollary 2.1 can be extended similarly as:

Corollary 2.3. *If* $\phi(t)$ *is a frame generator of* $V(\phi)$, $\{\phi(\sigma+n)\}_{n\in\mathbb{Z}} \in l^1$, *and* $Z_\phi(\sigma,\xi) \neq 0$ *on* $\operatorname{supp} G_\phi$, *then there is a frame generator* $S_\sigma(t)$ *of* $V(\phi)$ *for which* (2.26) *and* (2.27) *hold.*

Example 2.1. The Shannon function $\phi(t) = \sin \pi t/\pi t$ is a continuous real-valued Riesz (in fact orthonormal) generator and $\{\phi(n)\}_{n\in\mathbb{Z}} = \{\delta_{0,n}\}_{n\in\mathbb{Z}}$. Since $\hat{\phi}^*(\xi) = 1$ on $[0, 2\pi]$ but $|\phi(t)| = O(|t|^{-1})$ for $|t|$ large so that $\phi(t)$ does not satisfy the Walter's decaying condition, the WSK sampling theorem is not covered by the sampling theorem in Ref. 11 but follows from Remark 2.2 and Corollary 2.2.

3. Single Channel Sampling

Channeled sampling expansion recovers a signal via discrete sample values taken from one or more channeled (output) signals, which are obtained by passing the original (input) signal through a linear time invariant system of pre-filters. Channeled sampling goes back to the work by Shannon,[10] where sample values are taken from the original signal and its derivative. For general discussion of channeled sampling on Paley–Wiener spaces, we refer to Refs. 6, 7 and references therein.

Here we consider a single channel sampling on shift invariant spaces. Let $\phi(t) \in L^2(\mathbb{R})$ be a frame generator and $H(\xi) \in L^\infty(\text{supp}(\hat{\phi}))$ a transfer function (or a pre-filter). Let

$$C(f)(t) := \mathcal{F}^{-1}(H(\xi)\hat{f}(\xi)), \quad f \in V(\phi).$$

Then $C(f)(t) \in L^2(\mathbb{R})$ for any $f \in V(\phi)$. Note that if $\hat{f}(\xi) \in L^1(\mathbb{R})$ or $H(\xi) \in L^2(\text{supp}\hat{\phi})$, then $C(f)(t) \in C(\mathbb{R}) \cap L^2(\mathbb{R})$.

Lemma 3.1. *If $\phi(t) \in L^2(\mathbb{R})$ is such that $\{\phi(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence and $H(\xi) \in L^\infty(\text{supp}(\hat{\phi}))$, then $\{C(\phi)(t-n) : n \in \mathbb{Z}\}$ is also a Bessel sequence.*

Proof. Let $B > 0$ be a Bessel bound of $\{\phi(t-n) : n \in \mathbb{Z}\}$. Then

$$\begin{aligned} 2\pi G_{C(\phi)}(\xi) &= 2\pi \sum_{n\in\mathbb{Z}} |H(\xi+2n\pi)\hat{\phi}(\xi+2n\pi)|^2 \\ &\leq \|H(\xi)\chi_{\text{supp}\hat{\phi}}(\xi)\|_\infty^2 2\pi G_\phi(\xi) \leq \|H(\xi)\chi_{\text{supp}\hat{\phi}}(\xi)\|_\infty^2 B \quad \text{a.e. on } \mathbb{R} \end{aligned}$$

so that $\{C(\phi)(t-n) : n \in \mathbb{Z}\}$ is also a Bessel sequence (cf. (1.1)). □

In the following, we assume that $\phi(t) \in L^2(\mathbb{R})$ is a frame generator and $H(\xi) \in L^\infty(\text{supp}(\hat{\phi}))$ is a transfer function such that either $H(\xi) \in L^2(\text{supp}(\hat{\phi}))$ or $\hat{\phi}(\xi) \in L^1(\mathbb{R})$. Then $\{C(\phi)(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence by Lemma 3.1 and $C(\phi)(t) \in C(\mathbb{R}) \cap L^2(\mathbb{R})$ since $H(\xi)\hat{\phi}(\xi) \in L^1(\mathbb{R})$. We assume further that $\{C(\phi)(n)\}_{n\in\mathbb{Z}} \in l^2$. Then for any $f(t) = (\mathbf{c} * \phi)(t) \in V(\phi)$ with $\mathbf{c} = \{c_n\}_{n\in Z} \in l^2$,

$$C(f)(t) = \mathcal{F}^{-1}(H(\xi)\hat{f}(\xi)) = \mathcal{F}^{-1}(\hat{\mathbf{c}}^*(\xi)H(\xi)\hat{\phi}(\xi)) = (\mathbf{c} * C(\phi))(t)$$

by Lemma 2.2 since $\{C(\phi)(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence. Moreover for any n in $\mathbb{Z}$

$$C(f)(n) := \sum_{k\in\mathbb{Z}} c(k)C(\phi)(n-k)$$

converges absolutely and $\lim_{|n|\to\infty}(\mathbf{c} * C(\phi))(n) = 0$ (cf. Lemma 2.1). We then have the following, whose proof is essentially the same as the one in Theorem 2.3.

Theorem 3.1. *Let $\phi(t) \in L^2(\mathbb{R})$ be a frame generator and $H(\xi) \in L^\infty(\mathrm{supp}(\hat{\phi}))$ a transfer function such that either $H(\xi) \in L^2(\mathrm{supp}(\hat{\phi}))$ or $\hat{\phi}(\xi) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Assume $\{C(\phi)(n)\}_{n\in\mathbb{Z}} \in l^2$. Then there is a Riesz generator $S(t)$ of $V(\phi)$ for which the channeled sampling expansion formula*

$$f(t) = \sum_{n\in\mathbb{Z}} C(f)(n)S(t-n), \quad f \in V(\phi)$$

holds if and only if $\phi(t)$ is a Riesz generator of $V(\phi)$ and

$$0 < \|\widehat{C(\phi)}^*(\xi)\|_0 \le \|\widehat{C(\phi)}^*(\xi)\|_\infty < \infty.$$

Furthermore in this case, $C(S)(t)$ is interpolatory, i.e. $C(S)(n) = \delta_{0,n}$ for $n \in \mathbb{Z}$ and

$$\hat{S}(\xi) = \frac{\hat{\phi}(\xi)}{\widehat{C(\phi)}^*(\xi)} \quad a.e.\ on\ \mathbb{R}.$$

Example 3.1. Let $\phi(t) = t\chi_{[0,1)}(t) + (2-t)\chi_{[1,2)}(t)$ be the cardinal B-spline of degree 1. Then $\phi(t)$ is a continuous Riesz generator (cf. Ref. 5) and

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\xi}}{i\xi}\right)^2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Take a transfer function $H(\xi) = e^{i\sigma\xi}$ with $0 \le \sigma < 1$. Then $C(\phi)(t) = \phi(t+\sigma)$ so that

$$C(\phi)(\sigma) = \sigma, \quad C(\phi)(\sigma+1) = 1-\sigma, \quad \text{and} \quad C(\phi)(\sigma+n) = 0 \quad \text{for } n = 0, 1.$$

Therefore $\widehat{C(\phi)}^*(\xi) = Z_\phi(\sigma,\xi) = \sigma + (1-\sigma)e^{-i\xi}$ so that $\|\widehat{C(\phi)}^*(\xi)\|_0 = |2\sigma - 1|$ and $\|\widehat{C(\phi)}^*(\xi)\|_\infty = 1$. Hence, by Theorem 3.1, for any $\sigma \in [0,1)\backslash\{\frac{1}{2}\}$, there is a Riesz generator $S(t)$ of $V(\phi)$ for which we have the sampling expansion $f(t) = \sum_n f(n+\sigma)S(t-n)$ on $V(\phi)$, which converges not only in $L^2(\mathbb{R})$ but also uniformly on $\mathbb{R}$ since $\sup_{\mathbb{R}} \sum_{n\in\mathbb{Z}} |\phi(t-n)|^2 < \infty$.

Example 3.2. Let $\phi(t) = \mathrm{sinc}\, t$ so that $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}}\chi_{[-\pi,\pi]}(\xi)$. Then $\phi(t)$ is an orthonormal generator of $V(\phi) = PW_\pi$. Take a measurable function $H(\xi)$ on $\mathbb{R}$ such that $H(\xi)$ and $H(\xi)^{-1}$ belong to $L^\infty[-\pi,\pi]$. Then $H(\xi) \in L^2[-\pi,\pi]$ and $C(\phi)(t) = \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}H(\xi)\chi_{[-\pi,\pi]}(\xi))(t) \in PW_\pi$ so that

$$\sum_{n\in\mathbb{Z}} |C(\phi)(n)|^2 = \|C(\phi)(t)\|^2 = \frac{1}{2\pi}\|H(\xi)\|^2_{L^\infty[-\pi,\pi]} < \infty,$$

that is, $\{C(\phi)(n)\}_{n\in\mathbb{Z}} \in l^2$. On the other hand, by the Poisson summation formula, $\widehat{C(\phi)}^*(\xi) = \sqrt{2\pi}\sum_{n\in\mathbb{Z}} \widehat{C(\phi)}(\xi + 2n\pi) = H(\xi)$ on $[-\pi,\pi]$. Hence by Theorem 3.1, there is a Riesz generator $S(t) = \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}H(\xi)}\chi_{[-\pi,\pi]}(\xi))$ of PW_π for which we have the sampling expansion $f(t) = \sum_n C(f)(n)S(t-n)$ on PW_π, which converges not only in $L^2(\mathbb{R})$ but also uniformly on $\mathbb{R}$. It is exactly the single channel sampling introduced in Ref. 6.

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