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Sampling Theory in Abstract Reproducing Kernel Hilbert Space

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Abstract

Let H be a separable Hilbert space and $k(t)$ an H-valued function on a subset Ω of the real line R such that $\{k(t) \,|\, t \in \Omega\}$ is total in H. Then

$$\{f_x := \langle x, k(t)\rangle_H \,|\, x \in H\}$$

becomes a reproducing kernel Hilbert space (RKHS) in a natural way. Here, we develop a sampling formula for functions in this RKHS, which generalizes the well-known celebrated Whittaker-Shannon-Kotel'nikov sampling formula in the Paley-Wiener space of band-limited signals. To be more precise, we develop a multi-channel sampling formula in which each channel is given a rather arbitrary sampling rate. We also discuss stability and oversampling.

Key words and phrases : Sampling, Reproducing Kernel Hilbert space, oversampling

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1 Introduction

Let $f(t)$ be a band-limited signal with band region $[-\pi, \pi]$, that is, a square-integrable function on R of which the Fourier transform $\hat{f}$ vanishes outside

$[-\pi, \pi]$. Then f can be recovered by its uniformly spaced discrete values as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)},$$

which converges absolutely and uniformly over R. This series is called the cardinal series or the Whittaker-Shannon-Kotel'nikov (WSK) sampling series. This formula tells us that once we know the values of a band-limited signal f at certain discrete points, we can recover f completely. In 1941 Hardy [4] recognized that this cardinal series is actually an orthogonal expansion.

WSK sampling series was generalized by Kramer [8] in 1957 as follows: Let $k(\xi, t)$ be a kernel on $I \times \Omega$, where I is a bounded interval and Ω is a subset of R. Assume that $k(\cdot, t) \in L^2(I)$ for each t in Ω and there are points $\{t_n\}_{n \in \mathrm{Z}}$ in Ω such that $\{k(\xi, t_n)\}_{n \in \mathrm{Z}}$ is an orthonormal basis of $L^2(I)$. Then any $f(t) = \int_I F(\xi) k(\xi, t) d\xi$ with $F(\xi) \in L^2(I)$ can be expressed as a sampling series

$$f(t) = \sum_{n \in \mathrm{Z}} f(t_n) \int_I k(\xi, t)\, \overline{k(\xi, t_n)}\, d\xi,$$

which converges absolutely and uniformly over the subset D on which $\|k(\cdot, t)\|_{L^2(I)}$ is bounded. While WSK sampling series treats sample values taken at uniformly spaced points, Kramer's series may take sample values at nonuniformly spaced points.

Recently, A. G. Garcia and A. Portal [3] extended the WSK and Kramer sampling formulas further to a more general setting using a suitable abstract Hilbert space valued kernel.

On the other hand, Papoulis [10] (see also [7]) introduced a multi-channel sampling formula for band-limited signals in which a signal is recovered from discrete sample values of several transformed versions of the signal.

In this work, following the setting introduced by Garcia and Portal [3], we first extend and modify Theorem 1 in [3] into a single channel sampling formula (see Theorem 3.2 below), which is more transparent. It is then easy to extend it to a multi-channel sampling formula in which each channel can be given rather arbitrary sampling rate. Comparing two-channel sampling formula, Theorem 3 in [3] and our multi-channel sampling formula, Theorem 3.3, reveals the advantage of modification made in Theorem 3.2. Finally, we also discuss the oversampling and recovery of missing samples in the single-channel sampling formula.

2 Preliminaries

For $f(t) \in L^2(\mathrm{R})$, we let

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\xi t}\, dt$$

be the Fourier transform of $f(t)$ and

$$f(t) = \mathcal{F}^{-1}(\hat{f})(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i\xi t}\, d\xi$$

the inverse Fourier transform of $\hat{f}(\xi)$.

Definition 2.1. *For any $w > 0$, the Paley-Wiener space, $PW_{\pi w}$, is defined to be*

$$PW_{\pi w} := \{f \mid f \in L^2(\mathrm{R}),\ \operatorname{supp} \widehat{f} \in [-\pi w, \pi w]\}.$$

Note that $PW_{\pi w}$ is isometrically isomorphic onto $L^2[-\pi w, \pi w]$ under the Fourier transform.

We call a basis $\{\varphi_n\}$ of a separable Hilbert space H to be an unconditional basis of H if for every $f \in H$ the expansion $f = \sum c_n(f)\varphi_n$ still converges to f after any permutation of its terms. We also call a basis $\{\varphi_n\}$ to be a Riesz basis of H if there is a linear isomorphism T from H onto H such that $T(e_n) = \varphi_n$ where $\{e_n\}$ is an orthonormal basis for H. Then any Riesz basis of H is an unconditional basis of H but not conversely in general.

Definition 2.2. *[12] A Hilbert space H consisting of complex-valued functions defined on a set $D(\neq \emptyset)$ is called a reproducing kernel Hilbert space (RKHS in short) if there exists a function $k(s,t)$ on $D \times D$ satisfying*

(1) $k(\cdot, t) \in H$ for each $t \in D$;

(2) $\langle f(s), k(s,t)\rangle_H = f(t)$ for all $f \in H$ and all $t \in D$.

Such a function $k(s,t)$ is called a reproducing kernel of H.

We need some properties of RKHS's.

Proposition 2.3. *[5] Let H be a Hilbert space as in Definition 2.2. Then we have:*

(a) H is an RKHS if and only if the point evaluation map $l_t(f) := f(t)$ is a bounded linear functional on H for each $t \in D$;

(b) an RKHS H has a unique reproducing kernel;

(c) the convergence of a sequence in an RKHS H implies its uniform convergence over any subset of D on which $k(t,t)$ is bounded.

For example, the Paley-Wiener space $PW_{\pi w}$ is an RKHS with the reproducing kernel $k(s,t) = w\, \dfrac{\sin \pi w(s-t)}{\pi w(s-t)}$.

3 Multi-channel sampling

Let H be a separable Hilbert space and $k : \Omega \longrightarrow H$ be an H-valued function on a subset Ω of the real line R. Define a linear operator T on H by

$$T(x) = f_x := \langle x, k(t)\rangle_H,\ t \in \Omega.$$

We call $k(t)$ the kernel of the linear operator T.

Lemma 3.1. *([3])*

(a) T is one-to-one if and only if $\{k(t) \mid t \in \Omega\}$ is total in H.

Assume $\{k(t) \mid t \in \Omega\}$ is total in H so that $T : H \longrightarrow T(H)$ is a bijection. Then

(b) $\langle T(x), T(y)\rangle_{T(H)} := \langle x, y\rangle_H$ defines an inner product on $T(H)$ with which $T(H)$ is a Hilbert space and $T : H \longrightarrow T(H)$ is unitary. Moreover, $T(H)$ becomes an RKHS with the reproducing kernel $k(s,t) := \langle k(t), k(s)\rangle_H$.

Proof. (a) T is one-to-one if and only if $\{k(t) \mid t \in \Omega\}^{\perp} = \{0\}$ if and only if $\overline{span}\{k(t) \,|\, t \in \Omega\} = H$, that is, $\{k(t) \mid t \in \Omega\}$ is total in H.

(b) It is trivial that $\langle T(x), T(y)\rangle_{T(H)} := \langle x, y\rangle_H$ defines an inner product on $T(H)$ with which $T : H \longrightarrow T(H)$ is unitary. Now for any $f(\cdot) = \langle x, k(\cdot)\rangle_H$ in $T(H)$ and $t \in \Omega$,

$$|f(t)| = |\langle x, k(t)\rangle_H| \le \|x\|_H \|k(t)\|_H = \|f\|_{T(H)} \|k(t)\|_H$$

so that $l_t(f) = f(t)$ is a bounded linear functional on $T(H)$. Hence, $T(H)$ is an RKHS by Proposition 2.3. Since

$$f(t) = \langle x, k(t)\rangle_H = \langle T(x)(s), T(k(t))(s)\rangle_{T(H)} = \langle f(s), \langle k(t), k(s)\rangle_H\rangle_{T(H)},$$

the reproducing kernel $k(s,t)$ of $T(H)$ is $\langle k(t), k(s)\rangle_H$. □

First, we develop a single-channel sampling formula. Let $\tilde{k} : \Omega \longrightarrow H$ be another H-valued function on Ω and $\widetilde{T}$ the linear operator on H defined by

$$\widetilde{T}(x)(t) = \tilde{f}_x(t) = \langle x, \tilde{k}(t)\rangle_H.$$

Theorem 3.2. *If $KerT \subseteq Ker\widetilde{T}$ and there exists a sequence $\{t_n\}_{n\in Z}$ in Ω such that $\{\tilde{k}(t_n)\}_{n\in Z}$ is a basis of H, then T is one-to-one so that $T(H)$ becomes an RKHS under the inner product $\langle T(x), T(y)\rangle_{T(H)} := \langle x, y\rangle_H$. Moreover, there is a basis $\{S_n\}_{n\in Z}$ of $T(H)$ such that, for any $f_x \in T(H)$ the sampling expansion*

$$f_x(t) = \sum_n \tilde{f}_x(t_n)\, S_n(t), \qquad t \in \Omega \tag{3.1}$$

holds. The convergence of the series is not only in $T(H)$ but also uniform over any subset on which $\|k(t)\|_H$ is bounded.

Proof. Assume $\widetilde{T}(x)(t) = \langle x, \tilde{k}(t)\rangle = 0$ on Ω. Then $\langle x, \tilde{k}(t_n)\rangle = 0$ for any $n \in Z$ so that $x = 0$ since $\{\tilde{k}(t_n)\}_{n\in Z}$ is a basis of H. Hence, $KerT = Ker\widetilde{T} = \{0\}$ and $T(H)$ becomes an RKHS as in Lemma 3.1 (b).

Let $\{x_n\}_{n\in Z} = \{\tilde{k}(t_n)\}_{n\in Z}$ and $\{x_n^*\}_{n\in Z}$ be its dual. Then $\{T(x_n)\}$ and $\{T(x_n^*)\}$ are bases of $T(H)$, which are dual each other since T is unitary.

Expanding any $f_x = T(x)$ in $T(H)$ via the basis $\{S_n\}_{n\in Z} = \{T(x_n^*)\}$ gives

$$\begin{aligned} f_x(t) &= \sum_{n\in Z}\langle T(x),\, T(x_n)\rangle_{T(H)} S_n(t) = \sum_{n\in Z}\langle x,\, x_n\rangle_H \ S_n(t) \\ &= \sum_{n\in Z}\langle x,\, \tilde{k}(t_n)\rangle_H \ S_n(t) = \sum_{n\in Z}\tilde{f}_x(t_n)\, S_n(t). \end{aligned}$$

Uniform convergence of the series (3.1) follows from Proposition 2.3 (c). □

The single channel sampling expansion (3.1) may not converge absolutely unless $\{x_n\}_{n\in Z}$ is an unconditional basis and may not be stable. However, if $\{x_n\}_{n\in Z}$ is an unconditional basis and $\sup_n \|x_n^*\| < \infty$, then (3.1) is a stable sampling expansion, which converges absolutely on Ω. In fact, if then, $\{S_n\}_{n\in Z}$ becomes an unconditional basis of $T(H)$ and $\sup_n \|S_n\| = \sup_n \|x_n^*\| < \infty$. Since $\{\frac{1}{\|S_n\|}S_n\}_{n\in Z}$ is a Riesz basis of $T(H)$ by the Köthe-Toeplitz Theorem [9], there is a constant $B > 0$ such that

$$\|f_x\|^2_{T(H)} \le B\sum_{n\in Z}|\tilde{f}_x(t_n)|^2\|S_n\|^2 \le (\sup_n \|S_n\|)^2 B\sum_{n\in Z}|\tilde{f}_x(t_n)|^2,\ \ f_x \in T(H).$$

Furthermore, the sampling series expansion (3.1) remains valid when $\{\tilde{k}(t_n)\}_{n\in Z}$ is not a basis but a frame of H. When $\tilde{k}(t) = k(t)$ on Ω so that $T = \tilde{T}$, Theorem 3.2 is essentially Theorem 1 in [3]. However, Theorem 3.2 might have some advantage over Theorem 1 in [3]. While Theorem 1 in [3] requires first the expansion of the kernel $k(t)$ in terms of a given basis of H and then the interpolatory condition for the expansion coefficients at some points in Ω, Theorem 3.2 simply requires points in Ω, whose values under $k(\cdot)$ form a basis of H.

Now, we can extend Theorem 3.2 naturally to a multi-channel setting. Let $\{k_i\}_{i=1}^N$ be N H-valued functions on Ω and $\{T_i\}_{i=1}^N$ linear operators on H defined by

$$T_i(x) = f_x^i := \langle x, k_i(t)\rangle_H, \quad x \in H.$$

Theorem 3.3. *(Asymmetric nonuniform multi-channel sampling formula) If* $KerT \subseteq \cap_{i=1}^N KerT_i$, *and there exist points* $\{t_{i,n} \mid 1 \le i \le N, n \in Z\} \subset \Omega$ *and constants* $\{\alpha_{i,n}^j \mid 1 \le i \le N,\ 1 \le j \le M \text{ and } n \in Z\}$ *for some integer* $M \ge 1$ *such that* $\{\sum_{i=1}^N \alpha_{i,n}^j k_i(t_{i,n}) \mid 1 \le j \le M \text{ and } n \in Z\}$ *is an unconditional basis* *of H, then there is a basis $\{S_{j,n} \,|\, 1 \le j \le M \text{ and } n \in \mathrm{Z}\}$ of $T(H)$ such that for any $f_x = T(x) \in T(H)$,*

$$f_x(t) = \sum_{n\in \mathrm{Z}} \sum_{j=1}^{M} \{\overline{\alpha_{1,n}^j}\, f_x^1(t_{1,n}) + \overline{\alpha_{2,n}^j}\, f_x^2(t_{2,n}) + \cdots + \overline{\alpha_{N,n}^j}\, f_x^N(t_{N,n})\} S_{j,n}(t) \tag{3.2}$$

which converges in $T(H)$. Moreover, the series (3.2) converges absolutely and uniformly on any subset of Ω over which $\|k(t)\|_H$ is bounded.

Proof. First, we prove that T is one-to-one. Suppose $T(x)(t) = \langle x, k(t)\rangle = 0$ for all $t \in \Omega$. Then, $\langle x, k_i(t)\rangle = 0$, $1 \le i \le N$ on Ω since $KerT \subseteq \cap_{i=1}^N KerT_i$. In particular, $\langle x, \sum_{i=1}^N \alpha_{i,n}^j k_i(t_{i,n})\rangle = 0$ for all $1 \le j \le M$ and $n \in \mathrm{Z}$ so that $x = 0$ since $\{\sum_{i=1}^N \alpha_{i,n}^j k_i(t_{i,n}) \,|\, 1 \le j \le M \text{ and } n \in \mathrm{Z}\}$ is a basis of H. Therefore, $T : H \longrightarrow T(H)$ is a bijection and $T(H)$ becomes an RKHS under the inner product $\langle T(x), T(y)\rangle_{T(H)} := \langle x, y\rangle_H$ by Lemma 3.1.

Let $x_n^j := \sum_{i=1}^N \alpha_{i,n}^j k_i(t_{i,n})$ for $1 \le j \le M$ and $n \in \mathrm{Z}$ and $\{x_n^{j*}\}_{j=1,n\in\mathrm{Z}}^M$ be the dual of $\{x_n^j\}$. Then, $\{T(x_n^j)\}_{j=1,n\in\mathrm{Z}}^M$ becomes an unconditional basis of $T(H)$ with the dual basis $\{T(x_n^{j*})\}_{j=1,n\in\mathrm{Z}}^M := \{S_{j,n}\}_{j=1,n\in\mathrm{Z}}^M$, which is also unconditional.

Expanding $f_x = T(x)$ in $T(H)$ with respect to $\{S_{j,n}\}_{j=1,n\in\mathrm{Z}}^M$, we have

$$\begin{aligned} f_x(t) &= \sum_{n\in \mathrm{Z}} \sum_{j=1}^{M} \langle T(x), T(x_n^j)\rangle_{T(H)} S_{j,n}(t) \\ &= \sum_{n\in \mathrm{Z}} \sum_{j=1}^{M} \langle x, x_n^j\rangle_H S_{j,n}(t) \\ &= \sum_{n\in \mathrm{Z}} \sum_{j=1}^{M} \{\overline{\alpha_{1,n}^j}\, f_x^1(t_{1,n}) + \cdots + \overline{\alpha_{N,n}^j}\, f_x^N(t_{N,n})\}\, S_{j,n}(t). \end{aligned}$$

Uniform convergence of the series (3.2) follows from Proposition 2.3 (c). Finally, the series (3.2) converges also absolutely since it is an unconditional basis expansion. □

If either $KerT = \{0\}$ or $k_i(t) = A_i(k(t)), 1 \le i \le N$, where A_i's are automorphisms of H, then the first assumption $KerT \subseteq \cap_{i=1}^N KerT_i$ of Theorem 3.3 is trivially satisfied. For example, it is so when $H = L^2[-\pi, \pi]$, $\Omega = \mathrm{R}$ and $k(t) = \dfrac{e^{-it\xi}}{\sqrt{2\pi}}$ so that $T = \mathcal{F}^{-1}$ is the inverse Fourier transform. In particular, if $N = M = 2$, $k_1(t) = k(t), t_{1,n} = t_{2,n} = t_n$, and $\{\alpha_{1,n}^j k(t_n) + \alpha_{2,n}^j k_2(t_n) \,|\, j =$

$1, 2$ and $n \in \mathrm{Z}\}$ is a Riesz basis of H, then Theorem 3.3 is essentially the same as Theorem 3 in [3].

When $H = L^2[-\pi w, \pi w](w > 0), \Omega = \mathrm{R}$ and

$$k(t) = \frac{1}{\sqrt{2\pi}}\, e^{-it\xi},\ k_i(t) = \frac{1}{\sqrt{2\pi}}\, \overline{A_i(\xi)}\, e^{-it\xi} \quad (1 \leq i \leq N)$$

for suitable bounded measurable functions $A_i(\xi)(1 \leq i \leq N)$ on $[-\pi w, \pi w]$, we have

$$T(\phi)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi w}^{\pi w} \phi(\xi)\, e^{it\xi}\, d\xi = \mathcal{F}^{-1}(\phi)(t)$$
$$T_i(\phi)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi w}^{\pi w} A_i(\xi)\, \phi(\xi)\, e^{it\xi}\, d\xi = \mathcal{F}^{-1}(A_i\phi)(t) \quad (1 \leq i \leq N).$$

Hence, $T(H)$ becomes the Paley-Wiener space $PW_{\pi w}$ and then Theorem 3.3 reduces to an asymmetric multi-channel sampling handled in [7].

If $\{\sum_{i=1}^{N} \alpha_{i,n}^{j} k_i(t_{i,n}) \,|\, 1 \leq j \leq M$ and $n \in \mathrm{Z}\}$ is a frame of H in Theorem 3.3, then the sampling series expansion (3.2) still holds.

As in the single channel case, if $\sup_{i,j,n} \|\alpha_{i,n}^{j} x_n^{j*}\| < \infty$, then the multi-channel sampling expansion (3.2) is also stable in the following sense.

Definition 3.4. *(cf. Rawn [11] and Yao and Thomas [13]) We say that* $\{t_{i,n} \,|\, 1 \leq i \leq N \text{ and} n \in \mathrm{Z}\}$ *is a set of stable sampling for* $T(H)$ *if there exists* $A > 0$ *which is independent of* $f_x \in T(H)$ *such that*

$$\|f_x\|_{T(H)}^2 \leq A \sum_{n\in\mathrm{Z}} \sum_{i=1}^{N} |f_x^i(t_{i,n})|^2,\ f_x \in T(H).$$

Let $B > 0$ be the upper Riesz bound for the Riesz basis $\{\frac{1}{\|S_{j,n}\|} S_{j,n}\}_{j=1,n\in\mathrm{Z}}^{M}$ of $T(H)$. Then

$$\|f_x\|_{T(H)}^2 \leq B \sum_{n\in\mathrm{Z}} \sum_{j=1}^{M} \sum_{i=1}^{N} |\overline{\alpha_{i,n}^{j}} f_x^i(t_{i,n})|^2 \|S_{j,n}\|^2$$
$$\leq (\sup_{i,j,n} \|\alpha_{i,n}^{j} S_{j,n}\|)^2 BM \sum_{n\in\mathrm{Z}} \sum_{i=1}^{N} |f_x^i(t_{i,n})|^2$$

so that (3.2) is a stable sampling expansion with respect to $\{t_{i,n}\}$ when $\sup\limits_{i,j,n} \|\alpha_{i,n}^{j}$ $x_n^{j*}\| < \infty$.

We now discuss several examples in which we always take $H = L^2[-\pi, \pi], \Omega = \mathrm{R}$, and $k(t) = \frac{1}{\sqrt{2\pi}}\, e^{-it\xi}$ so that $T = \mathcal{F}^{-1}$ is the inverse Fourier transform and $T(H) = PW_\pi$.

Example 3.5. *(Sampling with Hilbert transform)*

Take $\tilde{k}(t) = i\, sgn(\xi)\, k(t)$ *so that* $\widetilde{T}(f)(t) = \tilde{f}(t)$ *is the Hilbert transform of* $f(t)$ *in* PW_π*. Choosing* $\{t_n\}_{n\in Z} = \{n\}_{n\in Z}$, $\{x_n\}_{n\in Z} = \{i\, sgn\xi\, \frac{e^{-in\xi}}{\sqrt{2\pi}}\}_{n\in Z}$ *is an orthonormal basis of* $L^2[-\pi,\pi]$ *so that* $\{x_n^*\}_{n\in Z} = \{x_n\}_{n\in Z}$*. We then have*

$$S_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} i\, sgn(\xi)\, \frac{e^{-in\xi}}{\sqrt{2\pi}}\, e^{it\xi}\, d\xi = -\mathrm{sinc}\frac{1}{2}(t-n) \sin\frac{\pi}{2}(t-n)$$

where $\mathrm{sinct} := \frac{\sin \pi t}{\pi t}$*. Hence, we have*

$$f(t) = -\sum_{n\in Z} \tilde{f}(n)\; \mathrm{sinc}\frac{1}{2}(t-n)\; \sin\frac{\pi}{2}(t-n), \quad f(t) \in PW_\pi.$$

Using the operational relation $\tilde{\tilde{f}} = -f$ *([5, Appendix B]) and the fact that if* $f \in PW_\pi$*, then so does* $\tilde{f}$*, we also have*

$$\tilde{f}(t) = \sum_{n\in Z} f(n)\; \mathrm{sinc}\frac{1}{2}(t-n)\; \sin\frac{\pi}{2}(t-n), \quad f(t) \in PW_\pi.$$

Example 3.6. *Here, we derive asymmetric derivative sampling formula on* PW_π*, in which we take samples from* $f(t)$ *and* $f'(t)$ *with ratio 2:1.*

Take $k_1(t) = k(t) = \frac{1}{\sqrt{2\pi}} e^{-it\xi}$ *and* $k_2(t) = -i\xi\, k(t) = k'(t)$ *so that* $f^1(t) = f(t)$ *and* $f^2(t) = f'(t)$ *for* $f(t) \in PW_\pi$*. Now, take the set of sampling points* $\{t_{1,n} = \frac{3n}{2}\}_{n\in Z}$ *for* $f_x^1(t)$ *and* $\{t_{2,n} = 3n\}_{n\in Z}$ *for* $f_x^2(t)$*. With* $\alpha_{1,n}^1 = \sqrt{\frac{3}{2}}$, $\alpha_{2,n}^1 = \alpha_{1,n}^2 = 0$ *, and* $\alpha_{2,n}^2 = -\sqrt{3}$, $\{\alpha_{1,n}^1\, k_1(t_{1,n}) + \alpha_{2,n}^1\, k_2(t_{2,n})\}_{n\in Z} \cup \{\alpha_{1,n}^2\, k_1(t_{1,n}) + \alpha_{2,n}^2\, k_2(t_{2,n})\}_{n\in Z} = \{\sqrt{\frac{3}{4\pi}}\, e^{-i3n\xi/2}\}_{n\in Z} \cup \{\sqrt{\frac{3}{2\pi}}\, i\xi\, e^{-i3n\xi}\}_{n\in Z}$ *is a Riesz basis of* $L^2[-\pi,\pi]$*, of which the dual (cf. [6]) is*

$$\left\{\sqrt{\frac{3}{4\pi}}\, \mu_{1,n}(\xi)\, e^{-i3n\xi/2}\right\} \cup \left\{\sqrt{\frac{3}{2\pi}}\, \mu_2(\xi)\, e^{-i3n\xi}\right\}, \quad n \in Z \tag{3.3}$$

where

$$\mu_{1,n}(\xi) = \begin{cases} 3(\xi+\pi)/2\pi, & \xi \in [-\pi, -\pi/3); \\ 1, & \xi \in [-\pi/3, \pi/3); \\ -3(\xi-\pi)/2\pi, & \xi \in [\pi/3, \pi] \end{cases}$$

if n *is even,*

$$\mu_{1,n}(\xi) = \begin{cases} 1/2, & \xi \in [-\pi, -\pi/3); \\ 1, & \xi \in [-\pi/3, \pi/3); \\ 1/2, & \xi \in [\pi/3, \pi] \end{cases}$$

if n is odd, and

$$\mu_2(\xi) = \begin{cases} -3i/4\pi, & \xi \in [-\pi, -\pi/3); \\ 0, & \xi \in [-\pi/3, \pi/3); \\ 3i/4\pi, & \xi \in [\pi/3, \pi]. \end{cases}$$

Taking inverse Fourier transform on (3.3), we have a Riesz basis $\{S_{1,n}\}_n \cup \{S_{2,n}\}_n$ *of* PW_π *where*

$$S_{1,n}(t) = \begin{cases} \sqrt{\frac{2}{3}}\,\mathrm{sinc}\frac{1}{3}\left(t - \frac{3n}{2}\right)\,\mathrm{sinc}\frac{2}{3}\left(t - \frac{3n}{2}\right) & \text{if } n \text{ is even}, \\ \sqrt{\frac{3}{8}}\left\{\frac{1}{3}\,\mathrm{sinc}\frac{1}{3}\left(t - \frac{3n}{2}\right) + \mathrm{sinc}\left(t - \frac{3n}{2}\right)\right\} & \text{if } n \text{ is odd}, \end{cases}$$

$$S_{2,n}(t) = -\frac{\sqrt{3}}{2\pi}\,\mathrm{sinc}\frac{1}{3}(t-3n)\,\sin\frac{2\pi}{3}(t-3n).$$

With these setting we have the nonsymmetric derivative sampling formula:

$$f(t) = \sum_{n\in\mathbb{Z}} \sqrt{\frac{3}{2}} f\left(\frac{3n}{2}\right) S_{1,n}(t) - \sqrt{3} f'(3n) S_{2,n}(t), \quad f(t) \in PW_\pi.$$

Example 3.7. *We now take* $k_1(t) = k(t) = \frac{1}{\sqrt{2\pi}} e^{-it\xi}$ *and* $k_2(t) = e^{i\xi}\,k(t)$ *so that* $f^1(t) = f(t)$ *and* $f^2(t) = f(t-1)$*. We want to express* $f \in PW_\pi$ *via samples from* $f(t)$ *and* $f(t-1)$ *with ratio* $3:2$*. Note that* $\{\sqrt{\frac{5}{6\pi}}\, e^{-i5n\xi/3}\}_{n\in\mathbb{Z}} \cup \{\sqrt{\frac{5}{4\pi}}\, e^{i\xi}\, e^{-i5n\xi/2}\}_{n\in\mathbb{Z}}$ *forms a Riesz basis of* $L^2[-\pi,\pi]$ *with the dual*

$$\left\{\sqrt{\frac{5}{6\pi}}\,\frac{1}{e^{-i4\pi/5} - e^{-i2\pi/5}}\,\mu_{1,n}(\xi)\,e^{-i5n\xi/3}\right\}_{n\in\mathbb{Z}} \cup \left\{\sqrt{\frac{5}{4\pi}}\,\frac{1}{e^{-i4\pi/5} - e^{-i2\pi/5}}\,\mu_{2,n}(\xi)\,e^{i\xi}\,e^{-i5n\xi/2}\right\}_{n\in\mathbb{Z}} \tag{3.4}$$

where

$$\mu_{1,n}(\xi) = \begin{cases} e^{-i\frac{4}{5}\pi} + e^{-i2n\pi/3} + e^{-i\frac{6}{5}\pi}\,e^{i2n\pi/3}, & -\pi \le \xi < -\pi/5; \\ e^{-i\frac{4}{5}\pi} - e^{-i\frac{2}{5}\pi}, & -\pi/5 \le \xi < \pi/5; \\ -e^{-i\frac{2}{5}\pi} - e^{-i2n\pi/3} - e^{-i\frac{6}{5}\pi}\,e^{i2n\pi/3}, & \pi/5 \le \xi \le \pi \end{cases}$$

and

$$\mu_{2,n}(\xi) = \begin{cases} -(-1)^n e^{-i\frac{8}{5}\pi} - e^{-i\frac{2}{5}\pi}, & -\pi \le \xi < -\pi/5; \\ 0, & -\pi/5 \le \xi < \pi/5; \\ e^{-i\frac{4}{5}\pi} + (-1)^n e^{-i\frac{8}{5}\pi}, & \pi/5 \le \xi \le \pi. \end{cases}$$

Then, we can obtain the sampling series

$$f(t) = \sum_{n\in Z} \sqrt{\frac{5}{3}} f\left(\frac{5n}{3}\right) S_{1,n}(t) + \sqrt{\frac{5}{2}} f\left(\frac{5n}{2} - 1\right) S_{2,n}(t), \quad f(t) \in PW_\pi,$$

where $\{S_{1,n}(t)\} \cup \{S_{2,n}(t)\}$ *are the inverse Fourier transforms of functions in (3.4).*

4 Oversampling and recovery of missing samples

We now develop an oversampling expansion, which extends the one in Kramer [2]. Again, let k and $\tilde{k} : \Omega \longrightarrow H$ be H-valued functions. Assume that there exists $\{t_n\}_{n\in Z} \subset \Omega$ such that $\{x_n := \tilde{k}(t_n)\}_{n\in Z}$ is a basis of H with the dual basis $\{x_n^*\}_{n\in Z}$. Define linear operators T and $\widetilde{T}$ on H by $T(x) = \langle x, k(t)\rangle_H := f_x$ and $\widetilde{T}(x) = \langle x, \tilde{k}\rangle_H := \tilde{f}_x$, respectively, and assume $KerT \subseteq Ker\widetilde{T}$. Then, both T and $\widetilde{T}$ are one-to-one, and so $T(H)$ and $\widetilde{T}(H)$ become RKHS's.

Now, let G be a proper closed subspace of H and $P : H \longrightarrow G$ the orthogonal projection onto G. Then, for any $x \in G$ we have

$$x = \sum_{n\in Z} \langle x, \tilde{k}(t_n)\rangle_H x_n^*$$

so that

$$x = P(x) = \sum_{n\in Z} \langle x, \tilde{k}(t_n)\rangle_H P(x_n^*) = \sum_{n\in Z} \tilde{f}_x(t_n) P(x_n^*). \tag{4.1}$$

Theorem 4.1. *Under the above setting there is a sequence of sampling functions* $\{T_n\}_{n\in Z}$ *in* $T(G)$ *such that for any* $x \in G$

$$f_x(t) = \sum_{n\in Z} \tilde{f}_x(t_n) T_n(t) \tag{4.2}$$

which converges in $T(H)$ *and uniformly on any subset of* Ω *over which* $\|k(t)\|_H$ *is bounded. Moreover, if* $\{x_n\}$ *is a Riesz basis of* H*, then* $\{T_n\}_{n\in Z}$ *is a frame of* $T(G)$*.*

Proof. Applying T on both sides of (4.1) gives

$$\begin{aligned} f_x(t) = T(x)(t) &= \sum_{n\in Z} \tilde{f}_x(t_n) T(P(x_n^*))(t) \\ &= \sum_{n\in Z} \tilde{f}_x(t_n) T_n(t), \end{aligned}$$

where $T_n(t) = T(P(x_n^*))(t)$. Since

$$\begin{aligned} |f_x(t) - \sum_{|n|\le N} \tilde{f}_x(t_n)T_n(t)| &= |T(x) - \sum_{|n|\le N} \tilde{f}_x(t_n)T(P(x_n^*))| \\ &= |\langle x - \sum_{|n|\le N} \tilde{f}_x(t_n)P(x_n^*), k(t)\rangle_H| \\ &\le \|x - \sum_{|n|\le N} \tilde{f}_x(t_n)P(x_n^*)\|_H \|k(t)\|_H, \end{aligned}$$

the series (4.2) converges uniformly on any subset over which $\|k(t)\|_H$ is bounded. Finally, if $\{x_n\}$ is a Riesz basis of H, then $\{x_n^*\}$ is also a Riesz basis of H so that $\{P(x_n^*)\}$ is a frame of G since G is a closed subspace of H [1, Proposition 5.3.5]. Hence $\{T_n(t) = T(P(x_n^*))(t)\}$ is a frame of $T(G)$. □

Note that the sample set $\{t_n\}_{n\in \mathrm{Z}}$ oversamples functions in $T(G)$ in the sense that $\{t_n\}_{n\in \mathrm{Z}}$ leads to a basis $\{S_n\}_{n\in \mathrm{Z}}$ of $T(H)$ (see Theorem 3.2), which properly contains $T(G)$ but $\{T_n\}_{n\in \mathrm{Z}}$ in Theorem 4.1 may be overcomplete in $T(G)$. Hence, we may call (4.2) an oversampling expansion of f_x in $T(G)$ for $x \in G$.

Now assume that finitely many sample values $\{\tilde{f}_x(t_n) \,|\, n \in X = \{n_1, n_2, \cdots, n_N\}\}$ are missing. Applying $\widetilde{T}$ on both sides of (4.1) gives

$$\tilde{f}_x(t) = \widetilde{T}(x)(t) = \sum_n \tilde{f}_x(t_n)\widetilde{T}(P(x_n^*))(t) \tag{4.3}$$

which converges not only in $\widetilde{T}(H)$ but also pointwisely in Ω since $\widetilde{T}(H)$ is an RKHS. Setting $t = t_{n_j}$ in (4.3), we have

$$\begin{aligned} \tilde{f}_x(t_{n_j}) &= \sum_n \tilde{f}_x(t_n)\widetilde{T}(P(x_n^*))(t_{n_j}) \quad \text{for } 1 \le j \le N \\ &= \sum_{k=1}^N \tilde{f}_x(t_{n_k})\widetilde{T}(P(x_{n_k}^*))(t_{n_j}) + \sum_{n\notin X} \tilde{f}_x(t_n)\widetilde{T}(P(x_n^*))(t_{n_j}),\ 1 \le j \le N, \end{aligned}$$

which can be rewritten in the matrix form as

$$(\mathbf{I} - \mathbf{T})\,\mathbf{f} = \mathbf{h}$$

where $\mathbf{f} = (\tilde{f}_x(t_{n_1}), \cdots, \tilde{f}_x(t_{n_N}))^T$ is the column vector consisting of missing samples, $\mathbf{h} = (h_1, \cdots, h_N)^T$, where

$$h_j = \sum_{n\notin X} \tilde{f}(t_n)\widetilde{T}(P(x_n^*))(t_{n_j})$$

and $\mathbf{T}$ is the $N \times N$ matrix with entries

$$T_{ij} = \widetilde{T}(P(x_{n_j}^*))(t_{n_i}) = \langle P(x_{n_j}^*), x_{n_i}\rangle_H = \langle P(x_{n_j}^*), P(x_{n_i})\rangle_H.$$

Note that if $\mathbf{I}-\mathbf{T}$ is invertible, the missing samples $\mathbf{f}$ can be recovered uniquely. In particular, if $\langle \mathbf{T}\mathbf{v}, \mathbf{v}\rangle < \|\mathbf{v}\|^2$ for any $\mathbf{v} \in \mathrm{C}^N \setminus \{0\}$, then $\mathbf{I}-\mathbf{T}$ is invertible. We have:

Theorem 4.2. *Under the same hypotheses as in Theorem 4.1, we assume further that $\{x_n\}_n$ is a Riesz basis of H such that $x_n = U(e_n)$ where $\{e_n\}_n$ is an orthonormal basis of H and U is an automorphism of H. If $PU = UP$ and*

$$span\{e_{n_i} \,|\, 1 \le i \le N\} \cap G = \{0\}, \tag{4.4}$$

then any finitely many missing samples $\{\tilde{f}_x(t_{n_i}) \,|\, 1 \le i \le N\}$ in the oversampling expansion (4.2) can be uniquely recovered.

Proof. Note first that $x_n^* = (U^*)^{-1}(e_n)$ where $\{x_n^*\}_n$ is the dual of $\{x_n\}_n$. Hence, we have for any $\mathbf{v} = (v_1, \cdots, v_N)^T \in \mathrm{C}^N \setminus \{0\}$,

$$\begin{aligned}
\langle \mathbf{T}\mathbf{v}, \mathbf{v}\rangle &= \sum_{i,j=1}^{N} \langle P(x_{n_j}^*), P(x_{n_i})\rangle_H v_j \overline{v_i} \\
&= \langle P(U^*)^{-1}(\sum_{j=1}^{N} v_j e_{n_j}), PU(\sum_{i=1}^{N} v_i e_{n_i})\rangle_H \\
&= \langle \sum_{j=1}^{N} v_j e_{n_j}, U^{-1}PU(\sum_{i=1}^{N} v_i e_{n_i})\rangle_H \\
&= \langle \sum_{j=1}^{N} v_j e_{n_j}, P(\sum_{i=1}^{N} v_i e_{n_i})\rangle_H \\
&= \|P(\sum_{j=1}^{N} v_j e_{n_j})\|_H^2 \\
&< \|\sum_{j=1}^{N} v_j e_{n_j}\|_H^2 = \sum_{j=1}^{N} |v_j|^2 = \|\mathbf{v}\|^2
\end{aligned}$$

since $\sum_{j=1}^{N} v_j e_{n_j} \notin G$ and $\{e_n\}_n$ is an orthonormal basis of H. Hence, $\mathbf{I}-\mathbf{T}$ is invertible. □

If, moreover, $\{x_n\}_n$ is an orthonormal basis of H in Theorem 4.2, then any finitely many missing samples $\{\tilde{f}_x(t_{n_i}) \,|\, 1 \le i \le N\}$ can be uniquely recovered when the condition (4.4) holds.

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